CK-12 Foundation

Algebra I

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Chapter 1

Equations and Functions

1.1 Variable Expressions

Learning Objectives

- Evaluate algebraic expressions.
- Evaluate algebraic expressions with exponents.

Introduction – The Language of Algebra

Do you like to do the same problem over and over again? No? Well, you are not alone. **Algebra** was invented by mathematicians so that they could solve a problem once and then use that solution to solve a group of similar problems. The big idea of algebra is that once you have solved one problem you can **generalize** that solution to solve other similar problems.

In this course, we'll assume that you can already do the basic operations of arithmetic. In arithmetic, only numbers and their arithmetical operations (such as $+, -, \times, \div$) occur. In algebra, numbers (and sometimes processes) are denoted by symbols (such as $x, y, a, b, c, \ldots$). These symbols are called **variables.**

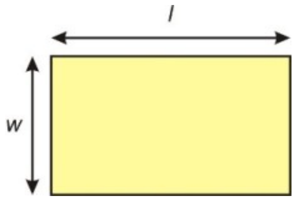
The letter x, for example, will often be used to represent some number. The value of x, however, is not fixed from problem to problem. The letter x will be used to represent a number which may be unknown (and for which we may have to solve) or it may even represent a quantity which varies within that problem.

Using variables offers advantages over solving each problem "from scratch":

- It allows the general formulation of arithmetical laws such as $a + b = b + a$ for all real numbers a and b.
- It allows the reference to "unknown" numbers, for instance: Find a number x such that $3x + 1 = 10$.
- It allows short-hand writing about functional relationships such as, "If you sell x tickets, then your profit will be $3x - 10$ dollars, or $f(x) = 3x - 10$," where "f" is the profit function, and x is the input (i.e. how many tickets you sell).

Example 1

Write an algebraic expression for the perimeter and area of the rectangle as follows.



To find the perimeter, we add the lengths of all 4 sides. We can start at the top-left and work clockwise. The perimeter, P, is therefore:

$$P = l + w + l + w$$

We are adding 2 l's and 2 w's. Would say that:

$$P = 2 \times \text{ l} + 2 \times \text{ w}$$

You are probably familiar with using $\cdot$ instead of $\times$ for multiplication, so you may prefer to write:

$$P = 2 \cdot l + 2 \cdot w$$

It's customary in algebra to omit multiplication symbols whenever possible. For example, $11x$ means the same thing as $11 \cdot x$ or $11 \times x$. We can therefore write the expression for P as:

$$P = 2\text{l} + 2\text{w}$$

Area is *length multiplied by width.* In algebraic terms we get the expression:

$$A = \text{l} \times \text{w} \qquad \rightarrow \qquad A = \text{l} \cdot \text{w} \qquad \rightarrow \qquad A = \text{lw}$$

Note: An example of a **variable expression** is $2\text{l} + 2\text{w}$; an example of an **equation** is $P = 2\text{l} + 2\text{w}$. The main difference between equations and expressions is the presence of an equals sign (=).

In the above example, there is no simpler form for these equations for the perimeter and area. They are, however, perfectly general forms for the perimeter and area of a rectangle. They work whatever the numerical values of the length and width of some particular rectangle are. We would simply substitute values for the length and width of a **real** rectangle into our equation for perimeter and area. This is often referred to as substituting (or **plugging in**) values. In this chapter we will be using the process of substitution to evaluate expressions when we have numerical values for the variables involved.

Evaluate Algebraic Expressions

When we are given an algebraic expression, one of the most common things we will have to do with it is **evaluate** it for some given value of the variable. The following example illustrates this process.

Example 2

Let $x = 12$. *Find the value of* $2x - 7$.

To find the solution, substitute 12 for x in the given expression. Every time we see x we will replace it with 12. **Note:** At this stage we place the value in parentheses:

$$\begin{aligned} 2x - 7 &= 2(12) - 7 \\ &= 24 - 7 \\ &= 17 \end{aligned}$$

The reason we place the substituted value in parentheses is twofold:

1. It will make worked examples easier for you to follow.
2. It avoids any confusion that would arise from dropping a multiplication sign: $2 \cdot 12 = 2(12) \neq 212$.

Example 3

Let $x = -1$. Find the value of $-9x + 2$.

Solution

$$\begin{aligned} -9(-1) + 2 &= 9 + 2 \\ &= 11 \end{aligned}$$

Example 4

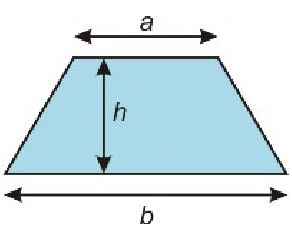
Let $y = -2$. Find the value of $\frac{7}{y} - 11\text{y} + 2$.

Solution

$$\begin{aligned} \frac{7}{(-2)} - 11(-2) + 2 &= -3\frac{1}{2} + 22 + 2 \\ &= 24 - 3\frac{1}{2} \\ &= 20\frac{1}{2} \end{aligned}$$

Many expressions have more than one variable in them. For example, the formula for the perimeter of a rectangle in the introduction has two variables: length (l) and width (w). In these cases be careful to substitute the appropriate value in the appropriate place.

Example 5



The area of a trapezoid is given by the equation $A = \frac{h}{2}(a + b)$. Find the area of a trapezoid with bases $a = 10$ cm, $b = 15$ cm and height $h = 8$ cm.

To find the solution to this problem we simply take the values given for the variables, a, b and h, and *plug them in* to the expression for A:

$$A = \frac{h}{2}(a + b) \qquad \text{Substitute 10 for a, 15 for b and 8 for h.}$$

$$A = \frac{8}{2}(10 + 15) \qquad \text{Evaluate piece by piece. } (10 + 15) = 25; \frac{8}{2} = 4$$

$$A = 4(25) = 100$$

Solution: *The area of the trapezoid is* 100 square *centimeters.*

Example 6

Find the value of $\frac{1}{9}(5x + 3y + z)$ when $x = 7$, $y = -2$ and $z = 11$.

Let's plug in values for x, y and z and then evaluate the resulting expression.

$$\frac{1}{9}(5(7)+3(-2)+(11))$$ Evaluate the individual terms inside the parentheses.

$$\frac{1}{9}(35+(-6)+11)$$ Combine terms inside parentheses.

$$\frac{1}{9}(40)=\frac{40}{9}\approx 4.44$$

Solution ≈ 4.44(rounded to the nearest hundredth) **Example 7**

The total resistance of two electronics components wired in parallel is given by

$$\frac{R_1 R_2}{R_1+R_2}$$

where R_1 *and* R_2 *are the individual resistances (in ohms) of the two components. Find the combined resistance of two such wired components if their individual resistances are* 30 ohms *and* 15 ohms.

Solution

$$\frac{R_1 R_2}{R_1+R_2}$$ Substitute the values $R_1=30$ and $R_2=15$.

$$\frac{(30)(15)}{30+15}=\frac{450}{45}=10 \text{ ohms}$$

The combined resistance is 10 ohms.

Evaluate Algebraic Expressions with Exponents

Many formulas and equations in mathematics contain exponents. Exponents are used as a short-hand notation for repeated multiplication. For example:

$$2\cdot 2=2^2$$
$$2\cdot 2\cdot 2=2^3$$

The exponent stands for how many times the number is used as a factor (multiplied). When we deal with integers, it is usually easiest to simplify the expression. We simplify:

$$2^2=4$$

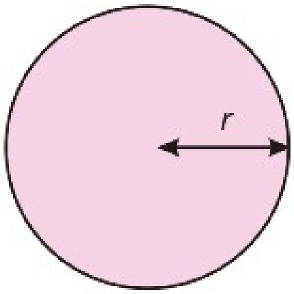
and

$$2^3=8$$

However, we need exponents when we work with variables, because it is much easier to write x^8 than $x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x$.

To evaluate expressions with exponents, substitute the values you are given for each variable and simplify. It is especially important in this case to substitute using parentheses in order to make sure that the simplification is done correctly.

Example 8



The area of a circle is given by the formula $A = \pi r^2$. *Find the area of a circle with radius* $r = 17$ inches.

Substitute values into the equation.

$$\begin{aligned} A &= \pi r^2 && \text{Substitute 17 for r.} \\ A &= \pi(17)^2 && \pi \cdot 17 \cdot 17 = 907.9202\ldots \quad \text{Round to 2 decimal places.} \end{aligned}$$

The area is approximately 907.92 square inches.

Example 9

Find the value of $5x^2 - 4y$ *for* $x = -4$ *and* $y = 5$.

Substitute values in the following:

$$\begin{aligned} 5x^2 - 4y &= 5(-4)^2 - 4(5) && \text{Substitute } x = -4 \text{ and } y = 5. \\ &= 5(16) - 4(5) && \text{Evaluate the exponent } (-4)^2 = 16. \\ &= 80 - 20 \\ &= 60 \end{aligned}$$

Example 10

Find the value of $2x^2 - 3x^2 + 5$, for $x = -5$.

Substitute the value of x in the expression:

$$\begin{aligned} 2x^2 - 3x^2 + 5 &= 2(-5)^3 - 3(-5)^2 + 5 && \text{Substitute } -5 \text{ for x.} \\ &= 2(-125) - 3(25) + 5 && \text{Evaluate exponents } (-5)^3 = (-5)(-5)(-5) = -125 \text{ and } (-5)^2 = (-5)(-5) = 25 \\ &= -250 - 75 + 5 \\ &= -320 \end{aligned}$$

Example 11

Find the value of $\frac{x^2y^3}{x^3+y^2}$, *for* $x = 2$ *and* $y = -4$.

Substitute the values of x and y in the following.

$$\begin{aligned} \frac{x^2y^3}{x^3+y^2} &= \frac{(2)^2(-4)^3}{(2)^3+(-4)^2} && \text{Substitute 2 for x and } -4 \text{ for y.} \\ \frac{4(-64)}{8+16} &= \frac{-256}{24} = \frac{-32}{3} && \text{Evaluate expressions : } (2)^2 = (2)(2) = 4 \text{ and } (2)^3 = (2)(2)(2) = 8. \\ & && (-4)^2 = (-4)(-4) = 16 \text{ and } (-4)^3 = (-4)(-4)(-4) = -64. \end{aligned}$$

Example 12

The height (h) *of a ball in flight is given by the formula:* $h = -32t^2 + 60t + 20$*, where the height is given in feet and the time* (t) *is given in seconds. Find the height of the ball at time* $t = 2$ seconds.

Solution

$$\begin{aligned} h &= -32t^2 + 60t + 20 && \text{Substitute 2 for t.} \\ &= -32(2)^2 + 60(2) + 20 \\ &= -32(4) + 60(2) + 20 \\ &= 12 \text{ feet} \end{aligned}$$

Review Questions

Write the following in a more condensed form by leaving out a multiplication symbol.

1. $2 \times 11x$
2. $1.35 \cdot y$
3. $3 \times \frac{1}{4}$
4. $\frac{1}{4} \cdot z$

Evaluate the following expressions for $a = -3, b = 2, c = 5$ and $d = -4$.

5. $2a + 3b$
6. $4c + d$
7. $5ac - 2b$
8. $\frac{2a}{c-d}$
9. $\frac{3b}{d}$
10. $\frac{a-4b}{3c+2d}$
11. $\frac{1}{a+b}$
12. $\frac{ab}{cd}$

Evaluate the following expressions for $x = -1, y = 2, z = -3$, and $w = 4$.

13. $8x^3$
14. $\frac{5x^2}{6z^3}$
15. $3z^2 - 5w^2$
16. $x^2 - y^2$
17. $\frac{z^3+w^3}{z^3-w^3}$
18. $2x^2 - 3x^2 + 5x - 4$
19. $4w^3 + 3w^2 - w + 2$
20. $3 + \frac{1}{z^2}$
21. The weekly cost C of manufacturing x remote controls is given by the formula $C = 2000 + 3x$, where the cost is given in dollars.
    (a) What is the cost of producing 1000 remote controls?
    (b) What is the cost of producing 2000 remote controls?
22. The volume of a box without a lid is given by the formula: $V = 4x(10 - x)^2$ where x is a length in inches and V is the volume in cubic inches.
    (a) What is the volume when $x = 2$?
    (b) What is the volume when $x = 3$?

Review Answers

1. $22x$
2. $1.35y$
3. $\frac{3}{4}$
4. $\frac{z}{4}$
5. 0
6. 16
7. -79
8. $\frac{-2}{3}$
9. $\frac{-3}{2}$
10. $\frac{-11}{7}$
11. -1
12. $\frac{3}{10}$
13. -8
14. $\frac{-5}{162}$
15. -53
16. -3
17. $\frac{37}{-91}$
18. -14
19. 302
20. $3\frac{1}{9}$
21.
22. (a) \$5000;
    (b) \$8000
23.
24. (a) 512 in^3;
    (b) 588 in^3

1.2 Order of Operations

Learning Objectives

- Evaluate algebraic expressions with grouping symbols.
- Evaluate algebraic expressions with fraction bars.
- Evaluate algebraic expressions with a graphing calculator.

Introduction

Look at and evaluate the following expression:

$$2 + 4 \times 7 - 1 = ?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across, keeping track of the total as you go:

$$2 + 4 = 6$$
$$6 \times 7 = 42$$
$$42 - 1 = 41$$

If you enter the expression into a *non-scientific*, non-graphing calculator you will probably get 41 as the answer. If, on the other hand, you were to enter the expression into a scientific calculator or a graphing calculator you would probably get 29 as an answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition** so we evaluate it first. Let's re-write the expression, but put the multiplication in brackets to indicate that it is to be evaluated first.

$$2 + (4 \times 7) - 1 = ?$$

So we first evaluate the brackets: $4 \times 7 = 28$. Our expression becomes:

$$2 + (28) - 1 = ?$$

When we have only addition and subtraction, we start at the left and keep track of the total as we go:

$$2 + 28 = 30$$
$$30 - 1 = 29$$

Algebra students often use the word **"PEMDAS"** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [ ] and braces { }) first.
2. Evaluate all **E**xponents (squared or cubed terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next – work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction – work from left to right completing **both** addition and subtraction in the order that they appear.

Evaluate Algebraic Expressions with Grouping Symbols

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step. While we will mostly use parentheses ( ) in this book, you may also see square brackets [ ] and curly braces { } and you should include them as part of the first step.

Example 1

Evaluate the following:

a) $4 - 7 - 11 - 2$

b) $4 - (7 - 11) + 2$

c) $4 - [7 - (11 + 2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses. PEMDAS states that we treat addition and subtraction as they appear, starting at the left and working right (it's NOT addition *then* subtraction).

Solution

$$\begin{aligned} 4 - 7 - 11 + 2 &= -3 - 11 + 2 \\ &= -14 + 2 \\ &= -12 \end{aligned}$$

b) This expression has parentheses. We first evaluate $7 - 11 = -4$. Remember that when we subtract a negative it is equivalent to adding a positive:

Solution

$$\begin{aligned} 4 - (7 - 11) + 2 &= 4 - (-4) + 2 \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

c) Brackets are often used to group expressions which already contain parentheses. This expression has both brackets and parentheses. Do the innermost group first, $(11 + 2) = 13$. Then complete the operation in the brackets.

Solution

$$\begin{aligned} 4 - [7 - (11 + 2)] &= 4 - [7 - (13)] \\ &= 4 - [-6] \\ &= 10 \end{aligned}$$

Example 2

Evaluate the following:

a) $3 \times 5 - 7 \div 2$

b) $3 \times (5 - 7) \div 2$

c) $(3 \times 5) - (7 \div 2)$

a) There are no grouping symbols. PEMDAS dictates that we evaluate multiplication and division first, working from left to right: $3 \times 5 = 15$; $7 \div 2 = 3.5$. (NOTE: It's not multiplication *then* addition) Next we perform the subtraction:

Solution

$$\begin{aligned} 3 \times 5 - 7 \div 2 &= 15 - 3.5 \\ &= 11.5 \end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5 - 7 = -2$. Then work from left to right.

Solution

$$
\begin{aligned}
3 \times (5-7) \div 2 &= 3 \times (-2) \div 2 \\
&= (-6) \div 2 \\
&= -3
\end{aligned}
$$

c) First, we evaluate the expressions inside parentheses: $3 \times 5 = 15$, $7 \div 2 = 3.5$. Then work from left to right.

Solution

$$
\begin{aligned}
(3 \times 5) - (7 \div 2) &= 15 - 3.5 \\
&= 11.5
\end{aligned}
$$

Note that in part (c), the result was unchanged by adding parentheses, but the expression does appear easier to read. Parentheses can be used in two distinct ways:

- To alter the order of operations in a given expression
- To clarify the expression to make it easier to understand

Some expressions contain no parentheses, others contain many sets. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Example 3

Use the order of operations to evaluate:

$$8 - [19 - (2 + 5) - 7)]$$

Follow PEMDAS – first parentheses, starting with innermost brackets first:

Solution

$$
\begin{aligned}
8 - (19 - (2 + 5) - 7) &= 8 - (19 - 7 - 7) \\
&= 8 - 5 \\
&= 3
\end{aligned}
$$

In algebra, we use the order of operations when we are substituting values into expressions for variables. In those situations we will be given an expression involving a variable or variables, and also the values to substitute for any variables in that expression.

Example 4

Use the order of operations to evaluate the following:

a) $2 - (3x + 2)$ when $x = 2$

b) $3y^2 + 2y - 1$ when $y = -3$

c) $2 - (t - 7)^2 \times (u^3 - v)$ when $t = 19, u = 4$ and $v = 2$

a) The first step is to substitute in the value for x into the expression. Let's put it in parentheses to clarify the resulting expression.

Solution

$$2 - (3(2) + 2) \qquad 3(2) \text{ is the same as } 3 \times 2$$

Follow PEMDAS – first parentheses. Inside parentheses follow PEMDAS again.

$$\begin{aligned} 2 - (3 \times 2 + 2) = 2 - (6 + 2) & \quad \text{Inside the parentheses, we evaluate the multiplication first.} \\ 2 - 8 = -6 & \quad \text{Now we evaluate the parentheses.} \end{aligned}$$

b) The first step is to substitute in the value for y into the expression.

Solution

$$3 \times (-3)^2 + 2 \times (-3) - 1$$

Follow PEMDAS: we cannot simplify parentheses.

$$\begin{aligned} &= 3 \times (-3)^2 + 2 \times (-3) - 1 && \text{Evaluate exponents} : (-3)^2 = 9 \\ &= 3 \times 9 + 2 \times (-3) - 1 && \text{Evaluate multiplication} : 3 \times 9 = 27; 2 \times -3 = -6 \\ &= 27 + (-6) - 1 && \text{Evaluate addition and subtraction in order from left to right.} \\ &= 27 - 6 - 1 \\ &= 20 \end{aligned}$$

c) The first step is to substitute the values for t, u, and v into the expression.

Solution:

$$2 - (19 - 7)^2 \times (4^3 - 2)$$

Follow **PEMDAS:**

$$\begin{aligned} &= 2 - (19 - 7)^2 \times (4^3 - 2) && \text{Evaluate parentheses} : (19 - 7) = 12; (4^3 - 2) = (64 - 2) = 62 \\ &= 2 - 12^2 \times 62 && \text{Evaluate exponents} : 12^2 = 144 \\ &= 2 - 144 \times 62 && \text{Evaluate the multiplication} : 144 \times 62 = 8928 \\ &= 2 - 8928 && \text{Evaluate the subtraction.} \\ &= -8926 \end{aligned}$$

In parts (b) and (c) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Part (c) in the last example shows another interesting point. When we have an expression inside the parentheses, we use PEMDAS to determine the order in which we evaluate the contents.

Evaluating Algebraic Expressions with Fraction Bars

Fraction bars count as grouping symbols for PEMDAS, and should therefore be evaluated in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses. When **real** parentheses are also present, remember that the innermost grouping symbols should be evaluated first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example 5

Use the order of operations to evaluate the following expressions:

a) $\frac{z+3}{4} - 1$ *When* $z = 2$

b) $\left(\frac{a+2}{b+4} - 1\right) + b$ *When* $a = 3$ and $b = 1$

c) $2 \times \left(\frac{w+(x-2z)}{(y+2)^2} - 1\right)$ *When* $w = 11$, $x = 3$, $y = 1$ and $z = -2$

a) We substitute the value for z into the expression.

Solution:

$$\frac{2+3}{4} - 1$$

Although this expression has no parentheses, we will rewrite it to show the effect of the fraction bar.

$$\frac{(2+3)}{4} - 1$$

Using PEMDAS, we first evaluate the expression on the numerator.

$$\frac{5}{4} - 1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4} = 1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4} - 1 = 1\frac{1}{4} - 1 = \frac{1}{4}$$

b) We substitute the values for a and b into the expression:

Solution:

$$\left(\frac{3+2}{1+4} - 1\right) - 1$$

This expression has nested parentheses (remember the effect of the fraction bar on the numerator and denominator). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator $(3 + 2)$ and denominator $(1 + 4)$ first.

$\left(\frac{5}{5} - 1\right) - 1$	Now we evaluate the inside of the parentheses, starting with division.
$(1 - 1) - 1$	Next the subtraction.
$0 - 1 = -1$	

c) We substitute the values for w, x, y and z into the expression:

Solution:

This complicated expression has several layers of nested parentheses. One method for ensuring that we start with the innermost parentheses is to make use of the other types of brackets. We can rewrite this expression, putting brackets in for the fraction bar. The outermost brackets we will leave as parentheses ( ). Next will be the *invisible brackets* from the fraction bar, these will be written as [ ]. The third level of nested parentheses will be the { }. We will leave negative numbers in round brackets.

$2\left(\frac{[11+\{\ 3-2(-2)\ \}]}{[\{\ 1+2\ \}^2]}-1\right)$ — We start with the innermost grouping sign $\{\}$.

$\{\ 1+2\ \}=3; \{\ 3-2(-2)\ \}=3+4=7$

$2\left(\frac{[11+7]}{[3^2]}-1\right)$ — The next level has two square brackets to evaluate.

$2\left(\frac{18}{9}-1\right)$ — We now evaluate the round brackets, starting with division.

$2(2-1)$ — Finally, we complete the addition and subtraction.

$2(1)=2$

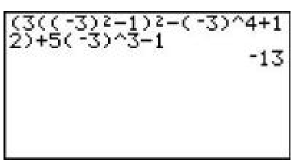
Evaluate Algebraic Expressions with a TI-83/84 Family Graphing Calculator

A graphing calculator is a very useful tool in evaluating algebraic expressions. The graphing calculator follows PEMDAS. In this section we will explain two ways of evaluating expressions with the graphing calculator.

Method 1: Substitute for the variable first. Then evaluate the numerical expression with the calculator.

Example 6

Evaluate $[3(x^2-1)^2-x^4+12]+5x^3-1$ *when* $x=-3$



Solution:

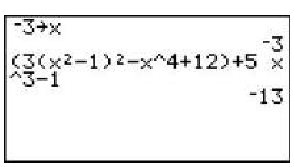
Substitute the value $x=-3$ *into the expression.*

$$[3((-3)^2-1)^2-(-3)^4+12]+5(-3)^3-1$$

Input this in the calculator just as it is and press **[ENTER]**. *(Note, use for exponents)*

The answer is -13.

Method 2: Input the original expression in the calculator first and then evaluate. Let's look at the same example.

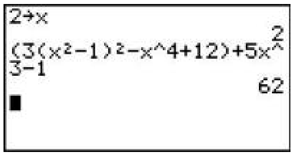


Evaluate $[3(x^2 - 1)^2 - x^4 + 12] + 5x^3 - 1$ *when* $x = -3$

First, store the value $x = -3$ *in the calculator. Type* -3 **[STO]** x *(The letter* x *can be entered using the* x-**[VAR]** *button or* **[ALPHA]** + **[STO]**). *Then type in the expression in the calculator and press* **[ENTER]**.

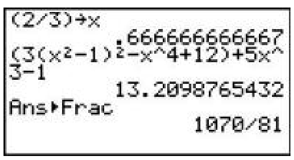
The answer is -13.

The second method is better because you can easily evaluate the same expression for any value you want. For example, let's evaluate the same expression using the values $x = 2$ *and* $x = \frac{2}{3}$.



For $x = 2$, store the value of x in the calculator: 2 **[STO]** x. Press **[2nd]** **[ENTER]** twice to get the previous expression you typed in on the screen without having to enter it again. Press **[ENTER]** to evaluate.

The answer is 62.



For $x = \frac{2}{3}$, store the value of x in the calculator: $\frac{2}{3}$ **[STO]** x. Press **[2nd]** **[ENTER]** twice to get the expression on the screen without having to enter it again. Press **[ENTER]** to evaluate.

The answer is 13.21 or $\frac{1070}{81}$ in fraction form.

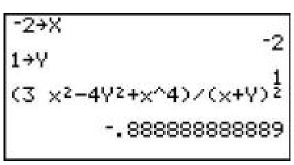
Note: On graphing calculators there is a difference between the minus sign and the negative sign. When we stored the value negative three, we needed to use the negative sign which is to the left of the **[ENTER]** button on the calculator. On the other hand, to perform the subtraction operation in the expression we used the minus sign. The minus sign is right above the plus sign on the right.

You can also use a graphing calculator to evaluate expressions with more than one variable.

Example 7

Evaluate the expression: $\frac{3x^2-4y^2+x^4}{(x+y)^{1/2}}$ *for* $x = -2$, $y = 1$.

Solution



Store the values of x and y. -2 **[STO]** x, 1 **[STO]** y. The letters x and y can be entered using **[ALPHA]** + **[KEY]**. Input the expression in the calculator. When an expression shows the division of two expressions be sure to use parentheses: (numerator) ÷ (denominator)

Press **[ENTER]** to obtain the answer $-.8\bar{8}$ or $-\frac{8}{9}$.

Review Questions

1. Use the order of operations to evaluate the following expressions.
   (a) $8-(19-(2+5)-7)$
   (b) $2+7\times 11-12\div 3$
   (c) $(3+7)\div(7-12)$
   (d) $\frac{2\cdot(3+(2-1))}{4-(6+2)}-(3-5)$
2. Evaluate the following expressions involving variables.
   (a) $\frac{jk}{j+k}$ when $j=6$ and $k=12$.
   (b) $2y^2$ when $x=1$ and $y=5$
   (c) $3x^2+2x+1$ when $x=5$
   (d) $(y^2-x)^2$ when $x=2$ and $y=1$
3. Evaluate the following expressions involving variables.
   (a) $\frac{4x}{9x^2-3x+1}$ when $x=2$
   (b) $\frac{z^2}{x+y}+\frac{x^2}{x-y}$ when $x=1$, $y=-2$, and $z=4$.
   (c) $\frac{4xyz}{y^2-x^2}$ when $x=3$, $y=2$, and $z=5$
   (d) $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x=-1$ and $z=3$
4. Insert parentheses in each expression to make a true equation.
   (a) $5-2\cdot 6-4+2=5$
   (b) $12\div 4+10-3\cdot 3+7=11$
   (c) $22-32-5\cdot 3-6=30$
   (d) $12-8-4\cdot 5=-8$
5. Evaluate each expression using a graphing calculator.
   (a) $x^2+2x-xy$ when $x=250$ and $y=-120$
   (b) $(xy-y^4)^2$ when $x=0.02$ and $y=-0.025$
   (c) $\frac{x+y-z}{xy+yz+xz}$ when $x=\frac{1}{2}$, $y=\frac{3}{2}$, and $z=-1$
   (d) $\frac{(x+y)^2}{4x^2-y^2}$ when $x=3$ and $y=-5d$

Review Answers

1.
2. (a) 3
   (b) 75
   (c) -2
   (d) -2
3.
4. (a) 4
   (b) 300
   (c) 86
   (d) 3
5.
6. (a) $\frac{8}{31}$
   (b) $-\frac{47}{3}$
   (c) -24
   (d) $-\frac{8}{5}$

7.
8. (a) $(5-2)\cdot(6-5)+2=5$
   (b) $(12\div 4)+10-(3\cdot 3)+7=11$
   (c) $(22-32-5)\cdot(3-6)=30$
   (d) $12-(8-4)\cdot 5=-8$
9.
10. (a) 93000
   (b) 0.00000025
   (c) $-\frac{12}{5}$
   (d) $\frac{4}{11}$

1.3 Patterns and Equations

Learning Objectives

- Write an equation.
- Use a verbal model to write an equation.
- Solve problems using equations.

Introduction

In mathematics, and especially in algebra, we look for patterns in the numbers that we see. The tools of algebra assist us in describing these patterns with words and with **equations** (formulas or functions). An equation is a mathematical recipe that gives the value of one variable in terms of the other.

For example, if a theme park charges \$12 admission, then the number of people who enter the park every day and the amount of money taken by the ticket office are related mathematically. We can write a rule to find the amount of money taken by the ticket office.

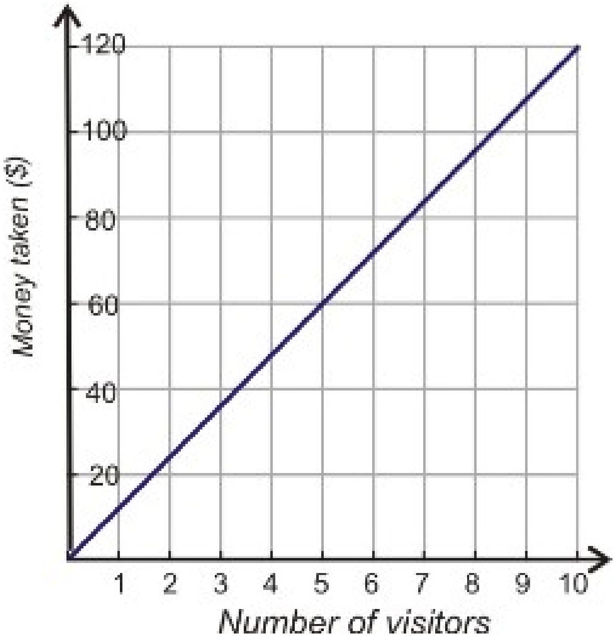
In words, we might say "*The money taken in dollars is (equals) twelve times the number of people who enter the park.*"

We could also make a table. The following table relates the number of people who visit the park and the total money taken by the ticket office.

Number of visitors	1	2	3	4	5	6	7
Money taken(\$)	12	24	36	48	60	72	84

Clearly, we will need a **big** table if we are going to be able to cope with a busy day in the middle of a school vacation!

A third way we might relate the two quantities (visitors and money) is with a graph. If we plot the money taken on the **vertical axis** and the number of visitors on the **horizontal axis**, then we would have a graph that looks like the one shown as follows. Note that this graph shows a smooth line for non-whole number values of x (e.g., $x=2.5$). But, in real life this would not be possible because you cannot have half a person enter the park. This is an issue of domain and range, something we will talk about in the following text.



The method we will examine in detail in this lesson is closer to the first way we chose to describe the relationship. In words we said that "*The money taken in dollars is twelve times the number of people who enter the park.*" In mathematical terms we can describe this sort of relationship with **variables.** A variable is a letter used to represent an unknown quantity. We can see the beginning of a mathematical formula in the words.

The money taken in dollars ***is*** *twelve* ***times*** *the number of people who enter the park.*

This can be translated to:

the money taken in dollars $= 12\times$ (*the number of people who enter the park*)

To make the quantities more visible they have been placed in parentheses. We can now see which quantities can be assigned to **letters**. First we must state which letters (or **variables**) relate to which quantities. We call this **defining the variables:**

Let $x =$ the number of people who enter the theme park.

Let $y =$ the total amount of money taken at the ticket office.

We can now show the fourth way to describe the relationship, with our algebraic equation.

$$y = 12x$$

Writing a mathematical equation using variables is very convenient. You can perform all of the operations necessary to solve this problem without having to write out the known and unknown quantities in long hand over and over again. At the end of the problem, we just need to remember which quantities x and y represent.

Write an Equation

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that relate to real quantities or to numbers that represent values for real quantities. If, for example, we wanted to use the algebraic equation in the example above to find the money taken for a certain number of visitors, we would substitute that value in for x and then solve the resulting equation for y.

Example 1

A theme park charges \$12 *entry to visitors. Find the money taken if* 1296 *people visit the park.*

Let's break the solution to this problem down into a number of steps. This will help us solve all the problems in this lesson.

Step 1 Extract the important information.

$$\text{(money taken in dollars)} = 12 \times \text{(number of visitors)}$$
$$\text{(number of visitors)} = 1296$$

Step 2 Translate into a mathematical equation.

We do this by defining variables and by substituting in known values.

Let $y = \text{(money taken in dollars)}$

$y = 12 \times 1296$ THIS IS OUR EQUATION.

Step 3 Solve the equation.

$y = 15552$ Answer: The money taken is \$15552

Step 4 Check the result.

If \$15552 is taken at the ticket office and tickets are \$12, then we can divide the total amount of money collected by the price per individual ticket.

(number of people) $= \frac{15552}{12} = 1296$

Our answer equals the number of people who entered the park. Therefore, the answer checks out.

Example 2

The following table shows the relationship between two quantities. First, write an equation that describes the relationship. Then, find out the value of b *when* a *is* 750.

a :	0	10	20	30	40	50
b :	20	40	60	80	100	120

Step 1 Extract the important information. We can see from the table that every time a increases by 10, b increases by 20. However, b is not simply twice the value of a. We can see that when $a = 0$, $b = 20$ so this gives a clue as to what rule the pattern follows. Hopefully you should see that the rule linking a and b.

"To find a, double the value of a and add 20."

Step 2 Translate into a mathematical equation:

Table 1.1:

Text	Translates to	Mathematical Expression
"To find b"	$\rightarrow$	$b =$
"double the value of a "	$\rightarrow$	$2a$
"add 20*"*	$\rightarrow$	$+20$

$b = 2a + 20$	THIS IS OUR EQUATION.

Step 3 Solve the equation.

Go back to the original problem. We substitute the values we have for our known variable and rewrite the equation.

"when a is 750"	$\rightarrow$	$b = 2(750) + 20$

Follow the **order of operations** to solve

$$b = 2(750) + 20$$
$$b = 1500 + 20 = 1520$$

Step 4 Check the result.

In some cases you can check the result by plugging it back into the original equation. Other times you must simply double-check your math. Double-checking is **always** advisable. In this case, we can plug our answer for b into the equation, along with the value for a and see what comes out. $1520 = 2(750) + 20$ is TRUE because both sides of the equation are equal and balance. A true statement means that **the answer checks out.**

Use a Verbal Model to Write an Equation

In the last example we developed a **rule**, written in words, as a way to develop an algebraic **equation**. We will develop this further in the next few examples.

Example 3

The following table shows the values of two related quantities. Write an equation that describes the relationship mathematically.

x – value	y – value
−2	10
0	0
2	−10
4	−20
6	−30

Step 1 Extract the important information.

We can see from the table that y is five times bigger than x. The value for y is negative when x is positive, and it is positive when x is negative. Here is the rule that links x and y.

"*y is the negative of five times the value of x*"

Step 2 Translate this statement into a mathematical equation.

Table 1.2:

Text	Translates to	Mathematical Expression
"y is"	$\rightarrow$	$y =$
"negative 5 times the value of x"	$\rightarrow$	$-5x$

$y = -5x$ THIS IS OUR EQUATION.

Step 3 There is nothing in this problem to **solve** for. We can move to Step 4.

Step 4 Check the result.

In this case, the way we would check our answer is to use the equation to generate our own xy pairs. If they match the values in the table, then we know our equation is correct. We will substitute x values of -2, 0, 2, 4, 6 in and solve for y.

$$\begin{array}{lll} x = -2: & y = -5(-2) & y = +10 \\ x = 0: & y = -5(0) & y = 0 \\ x = 2: & y = -5(2) & y = -10 \\ x = 4: & y = -5(4) & y = -20 \\ x = 6: & y = -5(6) & y = -30 \end{array}$$

Each xy pair above exactly matches the corresponding row in the table.

The answer checks out.

Example 4

Zarina has a \$100 *gift card, and she has been spending money on the card in small regular amounts. She checks the balance on the card weekly, and records the balance in the following table.*

Table 1.3:

Week Number	Balance (\$)
1	100
2	78
3	56
4	34

Write an equation for the money remaining on the card in any given week.

Step 1 Extract the important information.

We can see from the table that Zarina spends \$22 every week.

- As the week number **increases** by 1, the balance **decreases** by 22.
- The other information is given by any ***point*** (any week, balance pair). Let's take week 1:
- When $(\text{week number}) = 1, (\text{balance}) = 100$

Step 2 Translate into a mathematical equation.

Define variables:

Let week number = n

Let Balance = b

Table 1.4:

Text	Translates to	Mathematical Expression
As n increases by 1, b decreases by 22	$\rightarrow$	$b = -22n+?$

The ? indicates that we need another term. Without another term the balance would be $-22, -44, -66, \ldots$ We know that the balance in week 1 is 100. Let's substitute that value.

$$100 = -22(1)+?$$

The ? number that gives 100 when 22 is subtracted from it is 122. equation is therefore:

$$b = -22n + 122 \qquad \text{THIS IS OUR EQUATION.}$$

Step 3 All we were asked to **find** was the expression. We weren't asked to solve it, so we can move to Step 4.

Step 4 Check the result.

To check that this equation is correct, we see if it really reproduces the data in the table. To do that we plug in values for n

$$\begin{array}{lllll} n = 1 & \rightarrow & b = -22(1) + 122 & \rightarrow & b = 122 - 22 = 100 \\ n = 2 & \rightarrow & b = -22(2) + 122 & \rightarrow & b = 122 - 44 = 78 \\ n = 3 & \rightarrow & b = -22(3) + 122 & \rightarrow & b = 122 - 66 = 56 \\ n = 4 & \rightarrow & b = -22(4) + 122 & \rightarrow & b = 122 - 88 = 34 \end{array}$$

The equation perfectly reproduces the data in the table.

The answer checks out.

Note: Zarina will run out of money on her gift card (i.e. her balance will be 0) between weeks 5 and 6.

Solve Problems Using Equations

Let's solve the following real-world problem by using the given information to write a mathematical equation that can be solved for a solution.

Example 5

A group of students are in a room. After 25 *students leave, it is found that* $\frac{2}{3}$ *of the original group is left in the room. How many students were in the room at the start?*

Step 1 Extract the important information

We know that 25 students leave the room.

We know that $\frac{2}{3}$ of the original number of students are left in the room.

We need to find how many students were in the room at the start.

Step 2 Translate into a mathematical equation. Initially we have an unknown number of students in the room. We can refer to them as the original number.

Let's define the variable $x =$ the original number of students in the room.

25 students leave the room. The number of students left in the room is:

Table 1.5:

Text	Translates to	Mathematical Expression
the original number of students in the room	$\rightarrow$	x
25 students leave the room	$\rightarrow$	$x - 25$
$\frac{2}{3}$ of the original number is left in the room	$\rightarrow$	$\frac{2}{3}x$

$$x - 25 = \frac{2}{3}x \qquad \text{THIS IS OUR EQUATION.}$$

Step 3 Solve the equation.

Add 25 *to both sides.*

$$\begin{aligned} x - 25 &= \frac{2}{3}x \\ x - 25 + 25 &= \frac{2}{3}x + 25 \\ x &= \frac{2}{3}x + 25 \end{aligned}$$

Subtract $\frac{2}{3}x$ *from both sides.*

$$\begin{aligned} x - \frac{2}{3}x &= \frac{2}{3}x - \frac{2}{3}x + 25 \\ \frac{1}{3}x &= 25 \end{aligned}$$

Multiply both sides by 3.

$$\begin{aligned} 3 \cdot \frac{1}{3}x &= 25 \cdot 3 \\ x &= 75 \end{aligned}$$

Remember that x represents the original number of students in the room. So,

Answer There were 75 students in the room to start with.

Step 4 Check the answer:

If we start with 75 students in the room and 25 of them leave, then there are $75 - 25 = 50$ students left in the room.

$\frac{2}{3}$ of the original number is $\frac{2}{3} \cdot 75 = 50$

This means that the number of students who are left over equals to $\frac{2}{3}$ of the original number.

The answer checks out.

The method of defining variables and writing a mathematical equation is the method you will use the most in an algebra course. This method is often used together with other techniques such as making a table of values, creating a graph, drawing a diagram and looking for a pattern.

Review Questions

Table 1.6:

Day	Profit
1	20
2	40
3	60
4	80
5	100

1.
2. (a) Write a mathematical equation that describes the relationship between the variables in the table:
   (b) what is the profit on day 10?
3.
4. (a) Write a mathematical equation that describes the situation: *A full cookie jar has* 24 *cookies. How many cookies are left in the jar after you have eaten some?*
   (b) How many cookies are in the jar after you have eaten 9 cookies?
5. Write a mathematical equation for the following situations and solve.
   (a) Seven times a number is 35 . What is the number?
   (b) One number is 25 more than 2 times another number. If each number is multiplied by five, their sum would be 350 . What are the numbers?
   (c) The sum of two consecutive integers is 35 . What are the numbers?
   (d) Peter is three times as old as he was six years ago. How old is Peter?
6. How much water should be added to one liter of pure alcohol to make a mixture of 25% alcohol?
7. Mia drove to Javier's house at 40 miles per hour. Javier's house is 20 miles away. Mia arrived at Javier's house at 2:00 pm. What time did she leave?
8. The price of an mp3 player decreased by 20% from last year to this year. This year the price of the Player is $120 . What was the price last year?

Review Answers

1.
2. (a) $P = 20t$; P = profit; t = number of days. P = profit; t = number of days
   (b) Profit = 200
3.
4. (a) $y = 24 - x$; y = number of cookies in the jar; x = number of cookies eaten
   (b) 15 cookies
5.
6. (a) x = the number; $7x = 35$; number = 5

(b) $x =$ another number; $2x + 25 =$ another number; $5x + 5(2x + 25) = 350$; numbers = 15 and 55
(c) $x =$ first integer; $x + 1 =$ second integer; $x + x + 1 = 35$; first integer = 17, second integer = 18
(d) $x =$ Peter's age; $x = 3(x - 6)$; Peter is 9 years old.

7. 3 liters
8. 1:30 pm
9. $150

1.4 Equations and Inequalities

Learning Objectives

- Write equations and inequalities.
- Check solutions to equations.
- Check solutions to inequalities.
- Solve real-world problems using an equation.

Introduction

In algebra, an **equation** is a mathematical expression that contains an equal sign. It tells us that two expressions represent the same number. For example, $y = 12x$ is an equation. An **inequality** is a mathematical expression that contains inequality signs. For example $y \leq 12x$ is an inequality. Inequalities are used to tell us that an expression is either larger or smaller than another expression. Equations and inequalities can contain **variables** and **constants**.

- Variables are usually given a letter and they are used to represent unknown values. These quantities can change because they depend on other numbers in the problem.
- Constants are quantities that remain unchanged.

Equations and inequalities are used as a short hand notation for situations that involve numerical data. They are very useful because most problems require several steps to arrive at a solution, and it becomes tedious to repeatedly write out the situation in words.

Write Equations and Inequalities

Here are some examples of equations.

a) $3x - 2 = 5$

b) $x + 9 = 2x + 5$

c) $\frac{x}{3} = 15$

d) $x^2 + 1 = 10$

To write an inequality, we use the following symbols.

$>$ **greater than**

$\geq$ **greater than or equal to**

$<$ **less than**

$\leq$ **less than or equal to**

$\neq$ not equal to

Here are some examples of inequalities.

a) $3x < 5$

b) $4 - x \leq 2x$

c) $x^2 + 2x - 1 > 0$

d) $\frac{3x}{4} \geq \frac{x}{2} - 3$

The most important skill in algebra is the ability to translate a word problem into the correct equation or inequality so you can find the solution easily. Going from a word problem to the solution involves several steps. Two of the initial steps are **defining the variables** and **translating** the word problem into a mathematical equation.

Defining the variables means that we assign letters to any unknown quantities in the problem.

Translating means that we change the word expression into a mathematical expression containing variables and mathematical operations with an equal sign or an inequality sign.

Example 1

Define the variables and translate the following expressions into equations.

a) *A number plus* 12 is 20.

b) 9 *less than twice a number is* 33.

c) *Five more than four times a number is* 21.

d) $20 *was one quarter of the money spent on the pizza.*

Solution

a)

Define

Let $n =$ the number we are seeking

Translate

A number plus 12 is 20

$$n + 12 = 20$$

Answer

The equation is: $n + 12 = 20$

b)

Define:

Let $n =$ the number we are seeking

Translate

9 less than twice a number is 33

This means that twice a number minus 9 is 33

$$2 \times n - 9 = 33$$

Answer

The equation is: $2n - 9 = 33$

c)

Define

Let $n =$ the number we are seeking

Translate

Five more than four times a number is 21.

This means that four times a number plus five is 21.

$$4 \times n + 5 = 21$$

Answer

The equation is: $4n + 5 = 21$

d)

Define

Let $m =$ the money spent on the pizza

Translate

\$20 was one quarter of the money spent on the pizza.

Translate

$20 = \frac{1}{4} \times \mathrm{m}$

Answer

The equation is: $\frac{1}{4}\mathrm{m} = 20$

Often word problems need to be reworded before you can write an equation.

Example 2

Find the solution to the following problems.

a) *Shyam worked for two hours and packed* 24 *boxes. How much time did he spend on packing one box?*

b) *After a* 20% *discount, a book costs* \$12. *How much was the book before the discount?*

Solution

a)

Define

Let $t =$ time it take to pack one box

Translate

Shyam worked for two hours and packed 24 boxes.

This means that two hours is 24 times the time it takes to pack one book.

$$2 = 24 \times t$$

Solve

$t = \frac{2}{24}$ so $t = \frac{1}{12}$ hours or $t = \frac{1}{12} \times 60$ minutes $= 5$ minutes

Answer

Shyam takes 5 minutes to pack a box.

b)

Define:

Let $p =$ the price of the book before the discount.

Translate

After a 20% discount, a book costs $12.

This means that the price −20% of price is $12

$$p - 0.20p = 12$$

Solve

$0.8p = 12$ so $p = \frac{12}{0.8}$ and $p = 15$

Answer

The price of the book before the discount was $15.

Check

20% discount means: $0.20 \times \$15 = \$3$

Price after discount: $\$18 - \$3 = \$15$

The answer checks out.

Example 3

Define the variables and translate the following expressions into inequalities.

a) The sum of 5 and a number is less than or equal to 2.

b) The distance from San Diego to Los Angeles is less than 150 miles.

c) Diego needs to earn more than an 82 on his test to receive a B in his algebra class.

d) A child needs to be 42 inches or more to go on the roller coaster.

Solution

a)

Define

Let $n =$ the unknown number.

Translate

$$5 + n \leq 2$$

b)

Define

Let $d =$ the distance from San Diego to Los Angeles in miles.

Translate

$$d < 150$$

c)

Define

Let $x =$ Diego's test grade.

Translate

$$x > 82$$

d)

Define

Let $h =$ the height of child in inches.

Translate:

$$h \geq 42$$

Check Solutions to Equations

You will often need to check solutions to equations in order to check your work. In a math class, checking that you arrived at the correct solution is very good practice. We check the solution to an equation by replacing the variable in an equation with the value of the solution. A solution should result in a true statement when plugged into the equation.

Example 4

Check that $x = 5$ *is the solution to the equation* $3x + 2 = -2x + 27$.

Solution

To check that $x = 5$ is the solution to the equation, we "plug in" the value of 5 for the variable, x:

$$\begin{aligned}
3x + 2 &= -2x + 27\\
3 \cdot x + 2 &= -2 \cdot x + 27\\
3 \cdot 5 + 2 &= -2 \cdot 5 + 27\\
15 + 2 &= -10 + 27\\
17 &= 17
\end{aligned}$$

This is a true statement.

This means that $x = 5$ is the solution to equation $3x + 2 = -2x + 27$.

Example 5

Check that the given number is a solution to the corresponding equation.

a) $y = -1; 3y + 5 = -2y$

b) $z = 3; z^2 + 2z = 8$

c) $x = -\frac{1}{2}; 3x + 1 = x$

Solution

Replace the variable in each equation with the given value.

a)

$$3(-1) + 5 = -2(-1)$$
$$-3 + 5 = 2$$
$$2 = 2$$

This is a true statement. This means that $y = -1$ is a solution to $3y + 5 = -2y$.

b)

$$3^2 + 2(3) = 8$$
$$9 + 6 = 8$$
$$15 = 8$$

This is not a true statement. This means that $z = 3$ is **not a solution** to $z^2 + 2z = 8$.

c)

$$3\left(-\frac{1}{2}\right) + 1 = -\frac{1}{2}$$
$$\left(-\frac{3}{2}\right) + 1 = -\frac{1}{2}$$
$$-\frac{1}{2} = -\frac{1}{2}$$

This is a true statement. This means that $x = \frac{1}{2}$ is a solution to $3x + 1 = x$.

Check Solutions to Inequalities

To check the solution to an inequality, we replace the variable in the inequality with the value of the solution. A solution to an inequality produces a true statement when substituted into the inequality.

Example 6

Check that the given number is a solution to the corresponding inequality.

a) $a = 10; 20a \leq 250$

b) $b = -2; \frac{3-b}{b} > -4$

c) $x = \frac{3}{4}; 4x + 5 \leq 8$

d) $z = 25; \frac{z}{5} + 1 < z - 20$

Solution

Replace the variable in each inequality with the given value.

a) $20(10) \leq 250$

$200 \leq 250$

This statement is true. This means that $a = 10$ is a solution to the inequality $20a \leq 250$. Note that $a = 10$ is not the only solution to this inequality. If we divide both sides of the inequality by 20 we can write that

$$a \leq 12.5.$$

So any number equal to or less than 12.5 is going to be a solution to this inequality.

b)

$$\begin{aligned}\frac{3-(-2)}{(-2)} &> -4\\ \frac{3+2}{-2} &> -4\\ -\frac{5}{2} &> -4\\ -2.5 &> -4\end{aligned}$$

This statement is true. This means that $b = -2$ is a solution to the inequality $\frac{3-b}{b} > -4$.

c)

$$\begin{aligned}4\left(\frac{3}{4}\right)+5 &\geq 8\\ 3+5 &\geq 8\\ 8 &\geq 8\end{aligned}$$

This statement is true. It is true because the equal sign is included in the inequality. This means that $x = \frac{3}{4}$ is a solution to the inequality $4x + 5 \geq 8$.

d)

$$\begin{aligned}\frac{25}{5}+1 &< 25-20\\ 5+1 &< 5\\ 6 &< 5\end{aligned}$$

This statement is not true. This means that $z = 25$ is not a solution to $\frac{z}{5} + 1 < z - 20$.

Solve Real-World Problems Using an Equation

Let's use what we have learned about defining variables, writing equations and writing inequalities to solve some real-world problems.

Example 7

Tomatoes cost \$0.50 *each and avocados cost* \$2.00 *each. Anne buys six more tomatoes than avocados. Her total bill is* \$8. *How many tomatoes and how many avocados did Anne buy?*

Solution

Define

Let a = number of avocados Anne buys

Translate

Anne buys six more tomatoes than avocados

This means that $a + 6$ = number of tomatoes

Translate

Tomatoes cost \$0.50 each and avocados cost \$2.00 each. Her total bill is \$8.

This means that \$0.50 times the number of tomatoes plus \$2 times the number of avocados equals \$8

$$\begin{aligned} 0.5 \times (a+6) + 2 \times a &= 8 \\ 0.5a + 0.5 \times 6 + 2a &= 8 && \text{THIS IS OUR EQUATION.} \\ 2.5a + 3 &= 8 && \text{Simplify} \\ 2.5a &= 5 \\ a &= 2 \end{aligned}$$

Remember that $a =$ the number of avocados, so Anne buys two avocados.

We also know that the number of tomatoes is given by $a + 6 = 2 + 6 = 8$

Answer

Anne bought 2 avocados and 8 tomatoes.

Check

If Anne bought two avocados and eight tomatoes, the total cost is:

$2 \times \$2 + 8 \times \$0.50 = \$4 + \$4 = \$8$

The answer checks out.

Example 8

To organize a picnic Peter needs at least two times as many hamburgers as hotdogs. He has 24 hotdogs. What is the possible number of hamburgers Peter has?

Solution

Define

Let $x =$ number of hamburgers

Translate

Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs.

This means that twice the number of hot dogs is less than or equal to the number of hamburgers.

$$2 \times 24 \leq x$$

Simplify

$$48 \leq x$$

Answer

Peter needs at least 48 hamburgers

Check We found $x = 48$. 48 hamburgers is twice the number of hot dogs. So more than 48 hamburgers is more than twice the number of hot dogs.

The answer checks out.

Review Questions

1. Define the variables and translate the following expressions into equations.

(a) Peter's Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
(b) Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
(c) Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
(d) Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.

2. Define the variables and translate the following expressions into inequalities.

(a) A bus can seat 65 passengers or fewer.
(b) The sum of two consecutive integers is less than 54.
(c) An amount of money is invested at 5% annual interest. The interest earned at the end of the year is greater than or equal to \$250.
(d) You buy hamburgers at a fast food restaurant. A hamburger costs \$0.49. You have at most \$3 to spend. Write an inequality for the number of hamburgers you can buy.

3. Check that the given number is a solution to the corresponding equation.

(a) $a = -3$; $4a + 3 = -9$
(b) $x = \frac{4}{3}$; $\frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
(c) $y = 2$; $2.5y - 10.0 = -5.0$
(d) $z = -5$; $2(5 - 2z) = 20 - 2(z - 1)$

4. Check that the given number is a solution to the corresponding inequality.

(a) $x = 12$; $2(x + 6) \leq 8x$
(b) $z = -9$; $1.4z + 5.2 > 0.4z$
(c) $y = 40$; $-\frac{5}{2}y + \frac{1}{2} < -18$
(d) $t = 0.4$; $80 \geq 10(3t + 2)$

5. The cost of a Ford Focus is 27% of the price of a Lexus GS 450h. If the price of the Ford is \$15000, what is the price of the Lexus?
6. On your new job you can be paid in one of two ways. You can either be paid \$1000 per month plus 6% commission of total sales or be paid \$1200 per month plus 5% commission on sales over \$2000. For what amount of sales is the first option better than the second option? Assume there are always sales over \$2000.

Review Answers

1.
2. (a) $x =$ number of square yards of lawn; $25 = 10 + 0.2x$
(b) $p =$ number of people at the party; $324 = 200 + 4p$
(c) $m =$ number of miles; $55 + 0.45\text{m} = 100$
(d) $n =$ number of blocks; $n = 4 + 7$
3.
4. (a) $x =$ number of passengers; $x \leq 65$
(b) $n =$ the first integer; $2\text{n} + 1 < 54$
(c) $P =$ amount of money invested; $0.05P \geq 250$
(d) $n =$ number of hamburgers; $0.49\text{n} \leq 3$
5.
6. (a) $4(-3) + 3 = -9$ so $-12 + 3 = -9$ so $-9 = -9$. This is a true statement.
(b) $\frac{3}{4}\left(\frac{4}{3}\right) + \frac{1}{2} = \frac{3}{2}$ so $1 + \frac{1}{2} = \frac{3}{2}$ so $\frac{3}{2} = \frac{3}{2}$ This is a true statement.
(c) $2.5(2) - 10.0 = -5.0$ so $5.0 - 10.0 = -5.0$ so $-5.0 = -5.0$. This is a true statement.

(d) $2(5-2(-5)) = 20-2((-5)-1)$ so $2(5+10) = 20-2(-6)$ so $2(15) = 20+12$ so $30 = 32$. This is not a true statement.

7.

8. (a) $2(12+6) \leq 8(12)$ so $2(18) \leq 96$ so $36 \leq 96$. This is true statement.
   (b) $1.4(-9)+5.2 > 0.4(-9)$ so $-12.6+5.2 > -3.6$ so $-7.4 > -3.6$. This is not a true statement.
   (c) $-\frac{5}{2}(40) < -18$ so $-100+\frac{1}{2} < -18$ so $-99.5 < -18$. This is a true statement.
   (d) $80 \geq 10(3(0.4)+2)$ so $80 \geq 10(1.2+2)$ so $80 \geq 10(3.2)$ so $80 \geq 32$. This is a true statement.
9. x = price of a Lexus; $0.27x = 15000$; $x = \$55556$
10. x = total sales; $1000+0.06x > 1200+0.05(x-2000)$ so $x > 10000$.

1.5 Functions as Rules and Tables

Learning Objectives

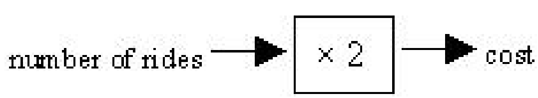
- Identify the domain and range of a function.
- Make a table for a function.
- Write a function rule.
- Represent a real-world situation with a function.

Introduction

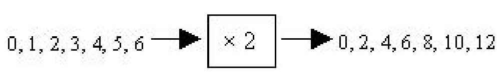
A **function** is a rule for relating two or more variables. For example, the price a person pays for phone service may depend on the number of minutes he/she talks on the phone. We would say that the cost of phone service is a *function* of the number of minutes she talks. Consider the following situation.

Josh goes to an amusement park where he pays $2 *per ride.*

There is a relationship between the number of rides on which Josh goes and the total cost for the day. To figure out the cost you multiply the number of rides by two. A **function** is the rule that takes us from the number of rides to the cost. Functions usually, *but not always* are rules based on mathematical operations. You can think of a function as a box or a machine that contains a mathematical operation.



A set of numbers is fed into the function box. Those numbers are changed by the given operation into a set of numbers that come out from the opposite side of the box. We can input different values for the number of rides and obtain the cost.

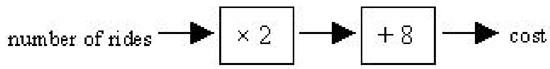


The input is called the **independent variable** because its value can be any possible number. The output results from applying the operation and is called the **dependent variable** because its value depends on the input value.

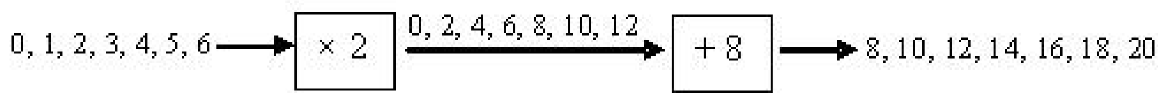
Often functions are more complicated than the one in this example. Functions usually contain more than one mathematical operation. Here is a situation that is slightly more complicated.

Jason goes to an amusement park where he pays $8 *admission and* $2 *per ride.*

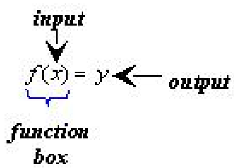
This function represents the total amount Jason pays. The rule for the function is "multiply the number of rides by 2 and add 8."



We input different values for the number of rides and we arrive at different outputs (costs).



These flow diagrams are useful in visualizing what a function is. However, they are cumbersome to use in practice. We use the following short-hand notation instead.



First, we define the variables.

$x =$ the number of rides Josh goes on

$y =$ the total amount of money Jason paid at the amusement park.

So, x represents the input and y represents the output. The notation: $f()$ represents the function or the mathematical operations we use on the input to obtain the output. In the last example, the cost is 2 times the number of rides plus 8. This can be written as a function.

$$f(x) = 2x + 8$$

The output is given by the formula $f(x) = 2x + 8$. The notations y and $f(x)$ are used interchangeably but keep in mind that y represents output value and $f(x)$ represents the mathematical operations that gets us from the input to the output.

Identify the Domain and Range of a Function

In the last example, we saw that we can input the number of rides into the function to give us the total cost for going to the amusement park. The set of all values that are possible for the input is called the **domain** of the function. The set of all values that are possible for the output is called the **range** of function. In many situations the **domain** and **range** of a function is the set of all real numbers, but this is not always the case. Let's look at our amusement park example.

Example 1

Find the domain and range of the function that describes the situation:

Jason goes to an amusement park where he pays \$8 *admission and* \$2 *per ride.*

Solution

Here is the function that describes this situation.

$$f(x) = 2x + 8 = y$$

In this function, x is the number of rides and y is the total cost. To find the domain of the function, we need to determine which values of x make sense as the input.

- The values have to be zero or positive because Jason can't go on a negative number of rides.
- The values have to be integers because, for example, Jason could not go on 2.25 rides.
- Realistically, there must be a maximum number of rides that Jason can go on because the park closes, he runs out of money, etc. However, since we are not given any information about this we must consider that all non-negative integers could be possible regardless of how big they are.

Answer For this function, the domain is the set of all non-negative integers.

To find the range of the function we must determine what the values of y will be when we apply the function to the input values. The domain is the set of all non-negative integers $(0, 1, 2, 3, 4, 5, 6...)$. Next we plug these values into the function for x.

$$f(x) = 2x + 8 = y$$

Then, $y = 8, 10, 12, 14, 16, 18, 20, \ldots$

Answer The range of this function is the set of all even integers greater than or equal to 8.

Example 2

Find the domain and range of the following functions.

a) A ball is dropped from a height and it bounces up to 75% of its original height.

b) $y = x^2$

Solution

a) Let's define the variables:

$x =$ original height

$y =$ bounce height

Here is a function that describes the situation. $y = f(x) = 0.75x$.

The variable x can take any real value greater than zero.

The variable y can also take any real value greater than zero.

Answer The domain is the set of all real numbers greater than zero.

The range is the set of all real numbers greater than zero.

b) Since we don't have a word-problem attached to this equation we can assume that we can use any real number as a value of x.

Since $y = x^2$, the value of y will always be non-negative whether x is positive, negative, or zero.

Answer The domain of this function is all real numbers.

The range of this function is all non-negative real numbers

As we saw, for a function, the variable x is called the **independent variable** because it can be any of the values from the domain. The variable y is called the **dependent variable** because its value depends on x. Any symbols can be used to represent the dependent and independent variables. Here are three different examples.

$$y = f(x) = 3x$$
$$R = f(w) = 3w$$
$$v = f(t) = 3t$$

These expressions all represent the same function. The dependent variable is three times the independent variable. In practice, the symbols used for the independent and dependent variables are based on common usage. For example: t for time, d for distance, v for velocity, etc. The standard symbols to use are y for the dependent variable and x for the independent variable.

A Function:

- Only accepts numbers from the domain.
- For each input, there is exactly one output. All the outputs form the range.

Multimedia Link For another look at the domain of a function, see the following video where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function Khan Academy CA Algebra I Functions (6:34) .

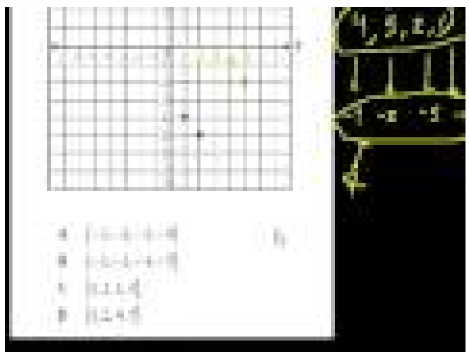


Figure 1.1: 79-80, functions, domain and range (Watch on Youtube)

Make a Table For a Function

A table is a very useful way of arranging the data represented by a function. We can match the input and output values and arrange them as a table. Take the amusement park example again.

Jason goes to an amusement park where he pays \$8 *admission and* \$2 *per ride.*

We saw that to get from the input to the output we perform the operations $2 \times input + 8$. For example, we input the values $0, 1, 2, 3, 4, 5, 6$, and we obtain the output values $8, 10, 12, 14, 16, 18, 20$. Next, we can make the following table of values.

x	y
0	8
1	10
2	12
3	14
4	16
5	18
6	20

A table allows us organize out data in a compact manner. It also provides an easy reference for looking up data, and it gives us a set of coordinate points that we can plot to create a graphical representation of the function.

Example 3

Make a table of values for the following functions.

a) $f(x) = 5x - 9$ Use the following numbers for input values: $-4, -3, -2, -1, 0, 1, 2, 3, 4$.

b) $f(x) = \frac{1}{x}$ Use the following numbers for input values: $-1, -0.5, -0.2, -0.1, -0.01, 0.01, 0.1, 0.2, 0.5, 1$.

Solution

Make a table of values by filling the first column with the input values and the second column with the output values calculated using the given function.

a)

x	$f(x) = 5x - 9 = y$
-4	$5(-4) - 9 = -29$
-3	$5(-3) - 9 = -24$
-2	$5(-2) - 9 = -19$
-1	$5(-1) - 9 = -14$
0	$5(0) - 9 = -9$
1	$5(1) - 9 = -4$
2	$5(2) - 9 = 1$
3	$5(3) - 9 = 6$
4	$5(4) - 9 = 11$

b)

x	$f(x) = \frac{1}{x} = y$
-1	$\frac{1}{-1} = -1$
-0.5	$\frac{1}{-0.5} = -2$
-0.2	$\frac{1}{-0.2} = -5$
-0.1	$\frac{1}{-0.1} = -10$
-0.01	$\frac{1}{-0.01} = -100$
0.01	$\frac{1}{0.01} = 100$
0.1	$\frac{1}{0.1} = 10$
0.2	$\frac{1}{0.2} = 5$
0.5	$\frac{1}{0.5} = 2$
1.0	$\frac{1}{1} = 1$

You are not usually given the input values of a function. These are picked based on the particular function or circumstance. We will discuss how we pick the input values for the table of values throughout this book.

Write a Function Rule

In many situations, we collect data by conducting a survey or an experiment. Then we organize the data in a table of values. Most often, we would like to find the function rule or formula that fits the set of values in the table. This way we can use the rule to predict what could happen for values that are not in the table.

Example 4

Write a function rule for the table.

Number of CDs	2	4	6	8	10
Cost($)	24	48	72	86	120

Solution

You pay \$24 for 2 CDs, \$48 for 4 CDs, \$120 for 10 CDs. That means that each CD costs \$12.

We can write the function rule.

Cost = \$12× number of CDs or $f(x) = 12x$

Example 5

Write a function rule for the table.

x	−3	−2	−1	0	1	2	3
y	3	2	1	0	1	2	3

Solution

You can see that a negative number turns in the same number but a positive and a non-negative number stays the same. This means that the output values are obtained by applying the absolute value function to the input values: $f(x) = |x|$.

Writing a functional rule is probably the hardest thing to do in mathematics. In this book, you will write functional rules mostly for linear relationships which are the simplest type of function.

Represent a Real-World Situation with a Function

Let's look at a few real-world situations that can be represented by a function.

Example 5

Maya has an internet service that currently has a monthly access fee of \$11.95 *and a connection fee of* \$0.50 *per hour. Represent her monthly cost as a function of connection time.*

Solution

Define Let $x =$ the number of hours Maya spends on the internet in one month

Let $y =$ Maya's monthly cost

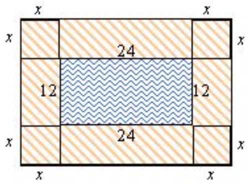
Translate There are two types of cost flat fee of \$11.95 and charge per hour of \$0.50

The total cost = flat fee + hourly fee × number of hours

Answer The function is $y = f(x) = 11.95 + 0.50x$

Example 6

Alfredo wants a deck build around his pool. The dimensions of the pool are $12\ feet \times 24\ feet$. He does not want to spend more than a set amount and the decking costs $3 per square foot. Write the cost of the deck as a function of the width of the deck.



Solution

Define Let $x =$ width of the deck

Let $y =$ cost of the deck

Make a sketch and label it

Translate You can look at the decking as being formed by several rectangles and squares. We can find the areas of all the separate pieces and add them together:

Area of deck $= 12x + 12x + 24x + 24x + x^2 + x^2 + x^2 + x^2 + 72x + 4x^2$

To find the toal cost we multiply the area by the cost per square foot.

Answer $f(x) = 3(72x + 4x^2) = 216x + 12x^2$

Example 7

A cell phone company sells two million phones in their first year of business. The number of phones they sell doubles each year. Write a function that gives the number of phones that are sold per year as a function of how old the company is.

Solution

Define Let $x =$ age of company in years

Let $y =$ number of phones that are sold per year

Make a table

Age (years)	1	2	3	4	5	6	7
Number of phones (millions)	2	4	8	16	32	64	128

Write a function rule

The number of phones sold per year doubles every year. We start with one million the first year:

$$\begin{aligned} &\text{Year1}: && 2 \text{ million} \\ &\text{Year2}: && 2 \times 2 = 4 \text{ million} \\ &\text{Year3}: && 2 \times 2 \times 2 = 8 \text{ million} \\ &\text{Year4}: && 2 \times 2 \times 2 \times 2 = 16 \text{ million} \end{aligned}$$

We can keep multiplying by two to find the number of phones sold in the next years. You might remember that when we multiply a number by itself several times we can use exponential notation.

$$\begin{aligned} 2 &= 2^1 \\ 2 \times 2 &= 2^2 \\ 2 \times 2 \times 2 &= 2^3 \end{aligned}$$

In this problem, the exponent represents the age of the company.

Answer $y = f(x) = 2^x$

Review Questions

Identify the domain and range of the following functions.

1. Dustin charges \$10 per hour for mowing lawns.
2. Maria charges \$25 per hour for tutoring math, with a minimum charge of \$15.
3. $f(x) = 15x - 12$
4. $f(x)2x^2 + 5$
5. $f(x) = \frac{1}{x}$
6. What is the range of the function $y = x^2 - 5$ when the domain is $-2, -1, 0, 1, 2$?
7. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is $-2.5, 1.5, 5$?
8. Angie makes \$6.50 per hour working as a cashier at the grocery store. Make a table of values that shows her earning for input values $5, 10, 15, 20, 25, 30$.
9. The area of a triangle is given by: $A = \frac{1}{2}bh$. If the height of the triangle is 8 centimeters, make a table of values that shows the area of the triangle for heights $1, 2, 3, 4, 5$, and 6 centimeters.
10. Make a table of values for the function $f(x) = \sqrt{2x + 3}$ for input values $-1, 0, 1, 2, 3, 4, 5$.
11. Write a function rule for the table

x	3	4	5	6
y	9	16	25	36

12. Write a function rule for the table

hours	0	1	2	3
cost	15	20	25	30

13. Write a function rule for the table

x	0	1	2	3
y	24	12	6	3

14. Write a function that represents the number of cuts you need to cut a ribbon in x number of pieces.
15. Solomon charges a \$40 flat rate and \$25 per hour to repair a leaky pipe. Write a function that represents the total fee charge as a function of hours worked. How much does Solomon earn for a 3 hour job?
16. Rochelle has invested \$2500 in a jewelry making kit. She makes bracelets that she can sell for \$12.50 each. How many bracelets does Rochelle need to make before she breaks even?

Review Answers

1. Domain: non-negative rational numbers; Range: non-negative rational numbers.
2. Domain: non-negative rational numbers; Range: rational numbers greater than 15.
3. Domain: all real numbers; Range: all real numbers.
4. Domain: all real numbers; Range: real number greater than or equal to 5.
5. Domain: all real numbers except 0; Range: all real numbers except 0.
6. $-1, -4, -5$
7. $-2, 0, \frac{7}{4}$
8.

hours	5	10	15	20	25	30
earnings	\$32.50	\$65	\$97.50	\$130	\$162.50	\$195

9.

height (cm)	1	2	3	4	5	6
Area	4	8	12	16	20	24

10.

x	-1	0	1	2	3	4	5
y	1	1.73	2.24	2.65	3	3.32	3.6

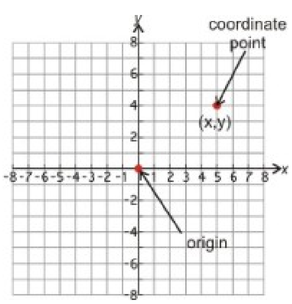
11. $y = x^2$
12. $y = 15 + 5x$
13. $y = \frac{24}{2^x}$
14. $f(x) = x - 1$
15. $y = 40 + 25x$; \$115
16. 200 bracelets

1.6 Functions as Graphs

Learning Objectives

- Graph a function from a rule or table.
- Write a function rule from a graph.
- Analyze the graph of a real world situation.
- Determine whether a relation is a function.

Introduction



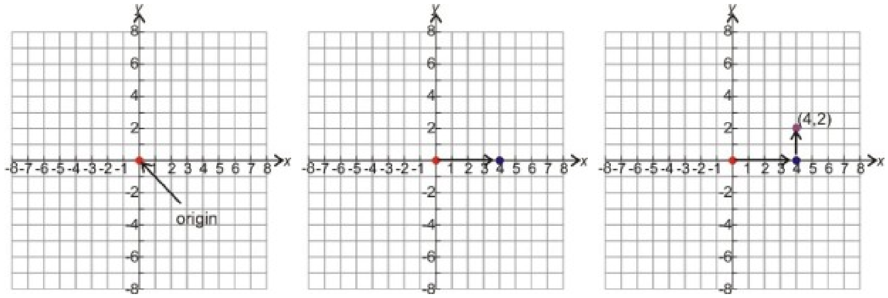
We represent functions graphically by plotting points on a **Coordinate Plane** (this is also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical number line that cross at a point called the **origin.** The origin is point $(0, 0)$ and it is the "starting" location. In order to plot points on the grid, you are told how many units you go right or left and how many units you go up or down from the origin. The horizontal line is called the $x-$**axis** and the vertical line is called the $y-$**axis**. The arrows at the end of each axis indicate that the plane continues past the end of the drawing.

From a function, we can gather information in terms of pairs of points. For each value of the independent variable in the domain, the function is used to calculate the value of the dependent variable. We call these pairs of points **coordinate points** or x, y values and they are written as (x, y).

To graph a coordinate point such as $(4, 2)$ we start at the origin.

Then we move 4 units to the right.

And then we move 2 units up from the last location.



Example 1

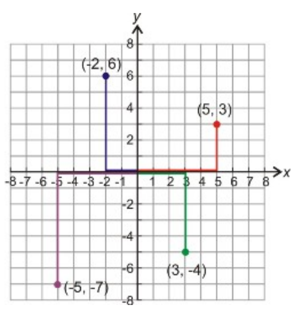
Plot the following coordinate points on the Cartesian plane.

(a) $(5, 3)$

(b) $(-2, 6)$

(c) $(3, -4)$

(d) $(-5, -7)$



Solution

We show all the coordinate points on the same plot.

Notice that:

For a positive x value we move to the right.

For a negative x value we move to the left.

For a positive y value we move up.

For a negative y value we move down.

The $x-$axis and $y-$axis divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right. The plotted point for (a) is in the **First** quadrant, (b) is in the **Second** quadrant, (c) is in the **Fourth** quadrant, and (d) is in the **Third** quadrant.

Graph a Function From a Rule or Table

Once a rule is known or if we have a table of values that describes a function, we can draw a graph of the function. A table of values gives us coordinate points that can be plotted on the Cartesian plane.

Example 2

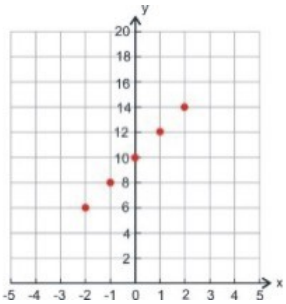
Graph the function that has the following table of values.

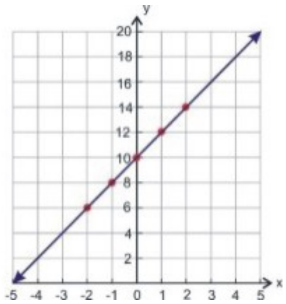
x	-2	-1	0	1	2
y	6	8	10	12	14

Solution

The table gives us five sets of coordinate points $(-2,6), (-1,8), (0,10), (1,12), (2,14)$.

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function or the context where it appears we can assume that the domain is the set of all real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth line. Also, we must realize that the line continues infinitely in both directions.





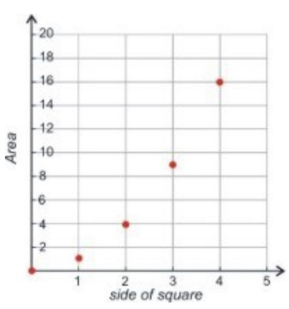
Example 3

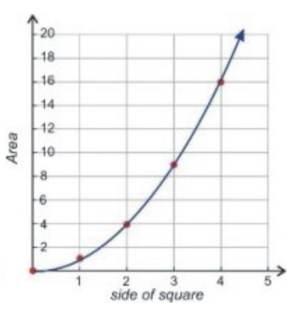
Graph the function that has the following table of values.

Side of the Square	0	1	2	3	4
Area of the Square	0	1	4	9	16

The table gives us five sets of coordinate points: $(0,0), (1,1), (2,4), (3,9), (4,16)$.

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function, we can assume that the domain is the set of all non-negative real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth curve. The curve does not make sense for negative values of the independent variable so it stops at $x = 0$ but it continues infinitely in the positive direction.

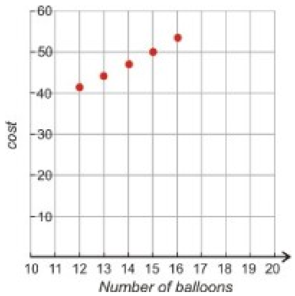




Example 4

Graph the function that has the following table of values.

Number of Balloons	12	13	14	15	16
Cost	41	44	47	50	53



This function represents the total cost of the balloons delivered to your house. Each balloon is \$3 and the store delivers if you buy a dozen balloons or more. The delivery charge is a \$5 flat fee.

Solution

The table gives us five sets of coordinate points $(12, 41), (13, 44), (14, 47), (15, 50), (16, 53)$.

To graph the function, we plot all the coordinate points. Since the $x-$values represent the number of balloons for 12 balloons or more, the domain of this function is all integers greater than or equal to 12. In this problem, the points are not connected by a line or curve because it does not make sense to have non-integer values of balloons.

In order to draw a graph of a function given the function rule, we must first make a table of values. This will give us a set of coordinate points that we can plot on the Cartesian plane. Choosing the values of the independent variables for the table of values is a skill you will develop throughout this course. When you pick values here are some of the things you should keep in mind.

- Pick only values from the domain of the function.
- If the domain is the set of real numbers or a subset of the real numbers, the graph will be a continuous curve.
- If the domain is the set of integers of a subset of the integers, the graph will be a set of points not connected by a curve.
- Picking integers is best because it makes calculations easier, but sometimes we need to pick other values to capture all the details of the function.
- Often we start with a set of values. Then after drawing the graph, we realize that we need to pick different values and redraw the graph.

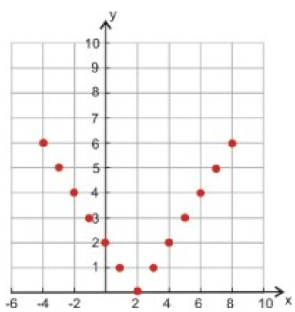
Example 5

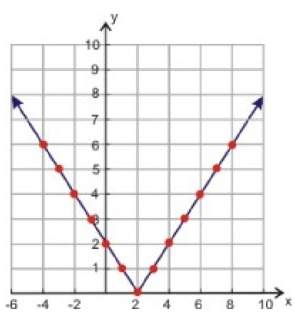
Graph the following function $f(x) = |x - 2|$

Solution

Make a table of values. Pick a variety of negative and positive integer values for the independent variable. Use the function rule to find the value of the dependent variable for each value of the independent variable. Then, graph each of the coordinate points.

x	$y = f(x) = \|x-2\|$
-4	$\|-4-2\| = \|-6\| = 6$
-3	$\|-3-2\| = \|-5\| = 5$
-2	$\|-2-2\| = \|-4\| = 4$
-1	$\|-1-2\| = \|-3\| = 3$
0	$\|0-2\| = \|-2\| = 2$
1	$\|1-2\| = \|-1\| = 1$
2	$\|2-2\| = \|0\| = 0$
3	$\|3-2\| = \|1\| = 1$
4	$\|4-2\| = \|2\| = 2$
5	$\|5-2\| = \|3\| = 3$
6	$\|6-2\| = \|4\| = 4$
7	$\|7-2\| = \|5\| = 5$
8	$\|8-2\| = \|6\| = 6$





It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will find that you will only need a few points in the table of values to create an accurate graph.

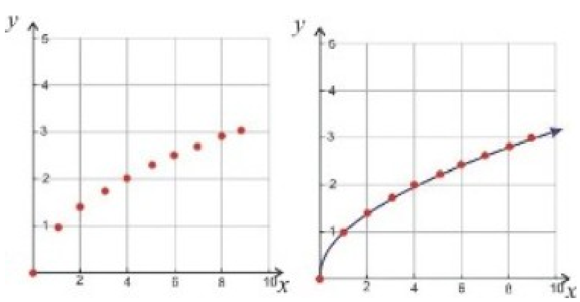
Example 6

Graph the following function: $f(x) = \sqrt{x}$

Solution

Make a table of values. We cannot use negative numbers for the independent variable because we can't take the square root of a negative number. The square root doesn't give real answers for negative inputs. The domain is all positive real numbers, so we pick a variety of positive integer values for the independent variable. Use the function rule to find the value of the dependent variable for each value of the independent variable.

x	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$



Note that the range is all positive real numbers.

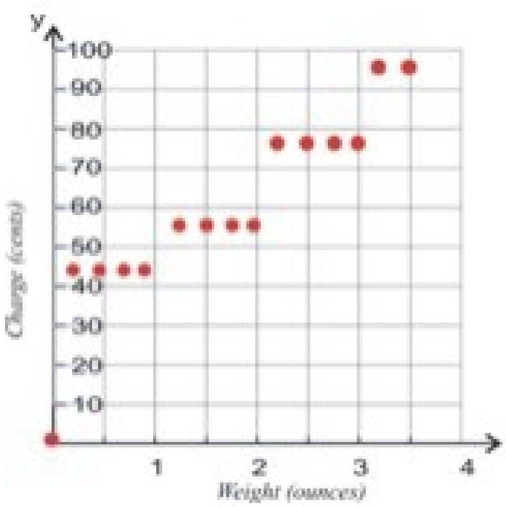
Example 7

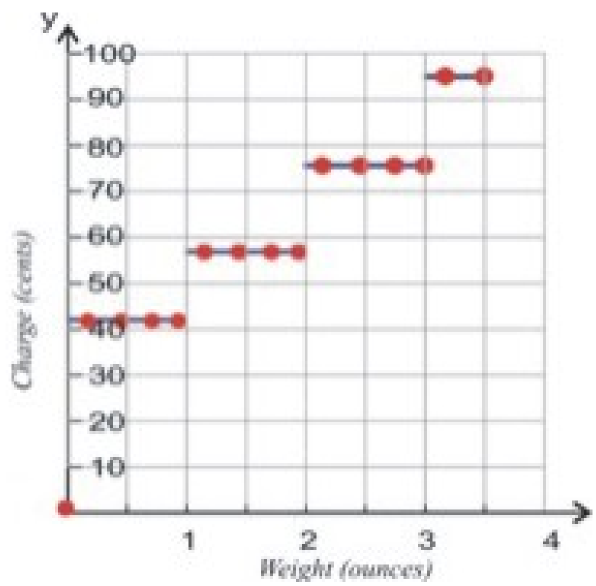
The post office charges 41 *cents to send a letter that is one ounce or less and an extra* 17 *cents for any amount up to and including an additional ounce. This rate applies to letters up to* 3.5 *ounces.*

Solution

Make a table of values. We cannot use negative numbers for the independent variable because it does not make sense to have negative weight. We pick a variety of positive integer values for the independent variable but we also need to pick some decimal values because prices can be decimals too. This will give us a clear picture of the function. Use the function rule to find the value of the dependent variable for each value of the independent variable.

x	y
0	0
0.2	41
0.5	41
0.8	41
1	41
1.2	58
1.5	58
1.8	58
2	58
2.2	75
2.5	75
2.8	75
3.0	75
3.2	92
3.5	92



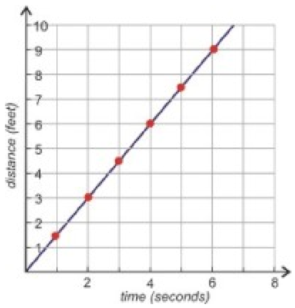


Write a Function Rule from a Graph

Sometimes you will need to find the equation or rule of the function by looking at the graph of the function. From a graph, you can read pairs of coordinate points that are on the curve of the function. The coordinate points give values of dependent and independent variables that are related to each other by the rule. However, we must make sure that this rule works for all the points on the curve. In this course you will learn to recognize different kinds of functions. There will be specific methods that you can use for each type of function that will help you find the function rule. For now we will look at some simple examples and find patterns that will help us figure out how the dependent and independent variables are related.

Example 8

The graph to the right shows the distance that an ant covers over time. Find the function rule that shows how distance and time are related to each other.



Solution

We make table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

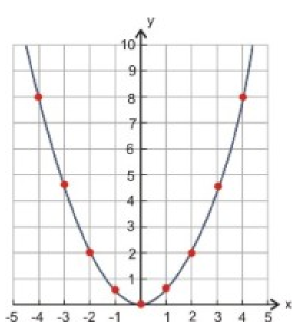
We can see that for every second the distance increases by 1.5 feet. We can write the function rule as:

$$\text{Distance} = 1.5 \times \text{time}$$

The equation of the function is $f(x) = 1.5x$

Example 9

Find the function rule that describes the function shown in the graph.



Solution:

We make a table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

x	-4	-3	-2	-1	0	1	2	3	4
y	8	4.5	2	.5	0	.5	2	4.5	8

We notice that the values of y are half of perfect squares. Re-write the table of values as:

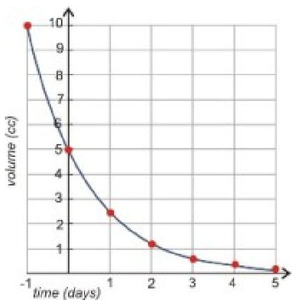
x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{16}{2}$	$\frac{9}{2}$	$\frac{4}{2}$	$\frac{1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{4}{2}$	$\frac{9}{2}$	$\frac{16}{2}$

We can see that to obtain y, we square x and divide by 2.

The function rule is $y = \frac{1}{2}x^2$ and the equation of the function is $f(x) = \frac{1}{2}x^2$.

Example 10

Find the function rule that shows what is the volume of a balloon at different times.



Solution

We make table of values of several coordinate points to see if we can identify a pattern of how they are related to each other.

Time	-1	0	1	2	3	4	5
Volume	10	5	2.5	1.2	0.6	0.3	0.15

We can see that for every day the volume of the balloon is cut in half. Notice that the graph shows negative time. The negative time can represent what happened on days before you started measuring the volume.

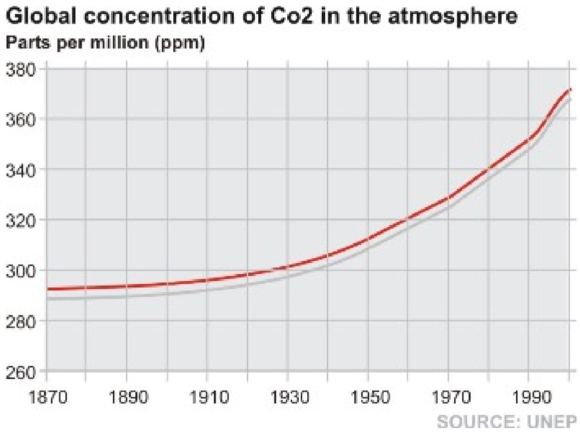
$$\begin{aligned}
&\text{Day0 : Volume} = 5\\
&\text{Day1 : Volume} = 5 \cdot \frac{1}{2}\\
&\text{Day2 : Volume} = 5 \cdot \frac{1}{2} \cdot \frac{1}{2}\\
&\text{Day3 : Volume} = 5 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}
\end{aligned}$$

The equation of the function is $f(x) = 5\left(\frac{1}{2}\right)^x$

Analyze the Graph of a Real-World Situation

Graphs are used to represent data in all areas of life. You can find graphs in newspapers, political campaigns, science journals and business presentations.

Here is an example of a graph you might see reported in the news. Most mainstream scientists believe that increased emissions of greenhouse cases, particularly carbon dioxide, are contributing to the warming of the planet. This graph shows how carbon dioxide levels have increased as the world has industrialized.



From this graph, we can find the concentration of carbon dioxide found in the atmosphere in different years.

1900	285 part per million
1930	300 part per million
1950	310 parts per million
1990	350 parts per million

We can find approximate function rules for these types of graphs using methods that you learn in more advanced math classes. The function $f(x) = 0.0066x^2 - 24.9x + 23765$ approximates this graph very well.

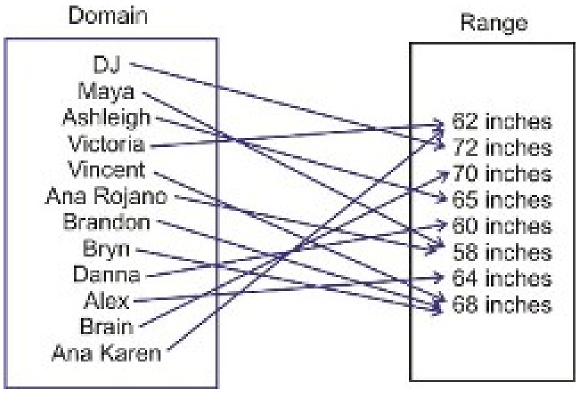
Determine Whether a Relation is a Function

You saw that a function is a relation between the independent and the dependent variables. It is a rule that uses the values of the independent variable to give the values of the dependent variable. A function rule can be expressed in words, as an equation, as a table of values and as a graph. All representations are useful and necessary in understanding the relation between the variables. Mathematically, a function is a special kind of relation.

In a function, for each input there is exactly one output.

This usually means that each $x-$value has only one $y-$value assigned to it. But, not all functions involve x and y.

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. Each person in the class cannot be more than one height at the same time. This relation is a function because for each person there is exactly one height that belongs to him or her.



Notice that in a function, a value in the range can belong to more than one element in the domain, so more than one person in the class can have the same height. The opposite is not possible, one person cannot have multiple heights.

Example 11

Determine if the relation is a function.

a) $(1,3),(-1,-2),(3,5),(2,5),(3,4)$

b) $(-3,20),(-5,25),(-1,5),(7,12),(9,2)$

c)

x	2	4	6	8	10
y	41	44	47	50	53

d)

x	2	1	0	1	2
y	12	10	8	6	4

Solution

The easiest way to figure out if a relation is a function is to look at all the $x-$values in the list or the table. If a value of x appears more than once and the $y-$values are different then the relation is not a function.

a) $(1,3),(-1,-2),(3,5),(2,5),(3,4)$

You can see that in this relation there are two different $y-$values that belong to the $x-$value of 3. This means that this relation in ***not*** a function.

b) $(-3,20),(-5,25),(-1,5),(7,12),(9,2)$

Each value of x has exactly one $y-$value. The relation is a function.

c)

x	2	4	6	8	10
y	4	4	4	4	4

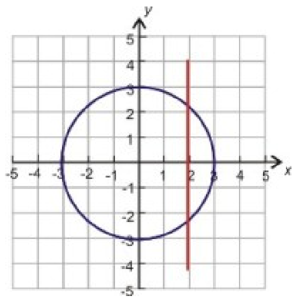
Each value of x appears only once. The relation is a function.

d)

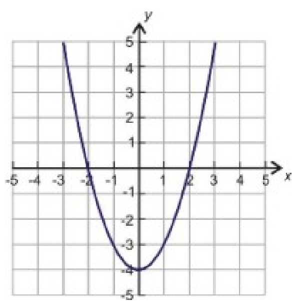
x	2	1	0	1	2
y	12	10	8	6	4

In this relation there are two $y-$values that belong to the $x-$value of 2 and two $y-$values that belong to the $x-$value of 1. The relation is ***not*** a function.

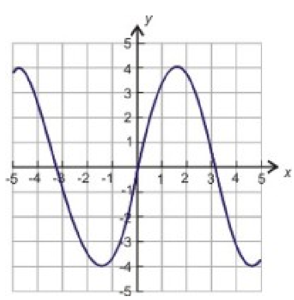
When a relation is represented graphically, we can determine if it is a function by using the **vertical line test.** If you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. Here are some examples.



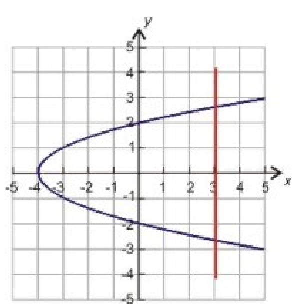
Not a function. It fails the vertical line test.



A function. No vertical line will cross more than one point on the graph.



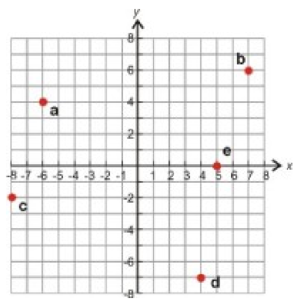
A function. No vertical line will cross more than one point on the graph.



Not a function. It fails the vertical line test.

Review Questions

1. Plot the coordinate points on the Cartesian plane.
   (a) $(4, -4)$
   (b) $(2, 7)$
   (c) $(-3, -5)$
   (d) $(6, 3)$
   (e) $(-4, 3)$
2. Give the coordinates for each point in the Cartesian plane.



3. Graph the function that has the following table of values.
   (a)

x	-10	-5	0	5	10
y	-3	-0.5	2	4.5	7

   (b)

side of cube (in)	0	1	2	3
volume(in^3)	0	1	8	27

   (c)

time (hours)	-2	-1	0	1	2
distance from town center (miles)	50	25	0	25	50

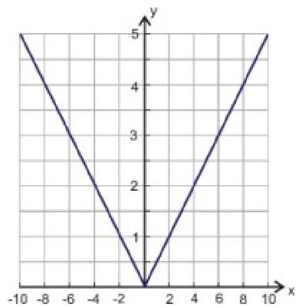
4. Graph the following functions.
   (a) Brandon is a member of a movie club. He pays a \$50 annual membership and \$8 per movie.
   (b) $f(x) = (x-2)^2$
   (c) $f(x) = 3.2^x$
5. Determine whether each relation is a function:
   (a) $(1, 7), (2, 7), (3, 8), (4, 8), (5, 9)$
   (b) $(1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3)$
   (c)

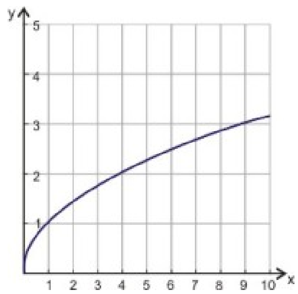
x	-4	-3	-2	-1	0
y	16	9	4	1	0

   (d)

Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

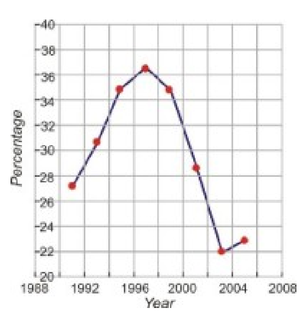
6. Write the function rule for each graph.

(a) 

(b) 

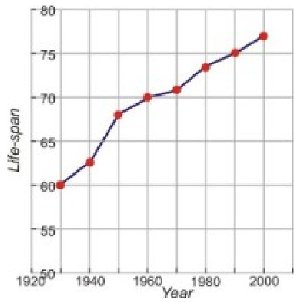
7. The students at a local high school took The Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. A person qualifies as a current smoker if he/she has smoked one or more cigarettes in the past 30 days. What percentage of high-school students were current smokers in the following years?

(a) 1991
(b) 1996
(c) 2004
(d) 2005



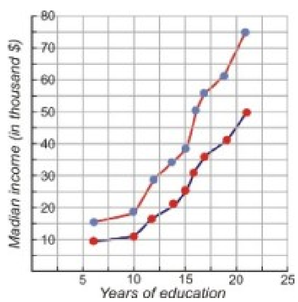
8. The graph below shows the average life-span of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control. What is the average life-span of a person born in the following years?

(a) 1940
(b) 1955
(c) 1980
(d) 1995



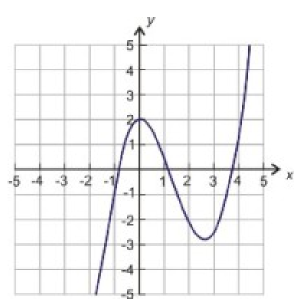
9. The graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females. (Source: US Census, 2003.) What is the median income of a male that has the following years of education?

   (a) 10 years of education
   (b) 17 years of education
   (c) What is the median income of a female that has the same years of education?
   (d) 10 years of education
   (e) 17 years of education
   (f)

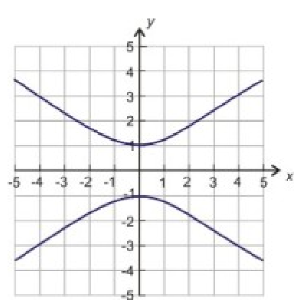


10. Use the vertical line test to determine whether each relation is a function.

   (a)

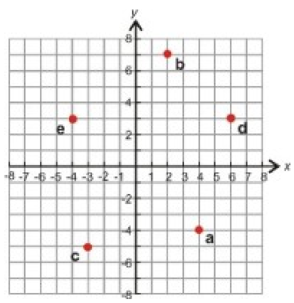


   (b)



Review Answers

1.

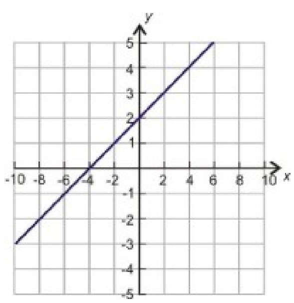


2.

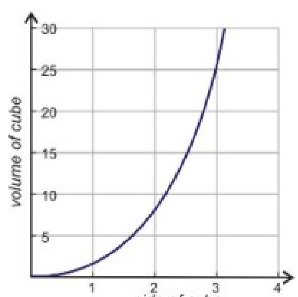
3. (a) $(-6, 4)$;
   (b) $(7, 6)$;
   (c) $(-8, -2)$;
   (d) $(4, -7)$;
   (e) $(5, 0)$

4.

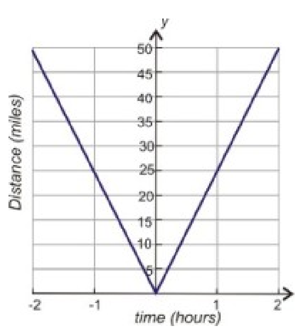
5. (a)



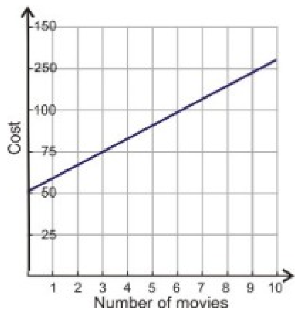
(b)

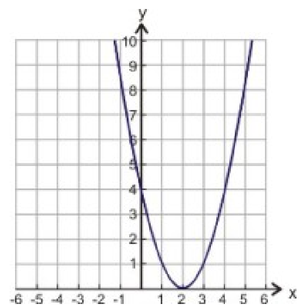


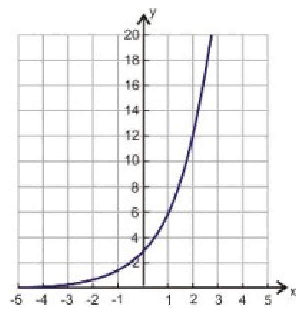
(c)



6.

7. (a) 

(b) 

(c) 

8.
9. (a) function
   (b) not a function
   (c) function
   (d) not a function
10.
11. (a) $f(x) = \frac{1}{2}|x|$
    (b) $f(x) = \sqrt{x}$
12.
13. (a) 27.5%
    (b) 35.6%
    (c) 22.2%
    (d) 23%
14.
15. (a) 63 years
    (b) 69 years
    (c) 74 years
    (d) 76 years
16.
17. (a) $19,500
    (b) $56,000
    (c) $10,000
    (d) $35,000

18.
19. (a) function
    (b) not a function

1.7 Problem-Solving Plan

Learning Objectives

- Read and understand given problem situations.
- Make a plan to solve the problem.
- Solve the problem and check the results.
- Compare alternative approaches to solving the problem.
- Solve real-world problems using a plan.

Introduction

We always think of mathematics as the subject in school where we solve lots of problems. Throughout your experience with mathematics you have solved many problems and you will certainly encounter many more. Problem solving is necessary in all aspects of life. Buying a house, renting a car, figuring out which is the better sale are just a few examples where people use problem solving techniques. In this book, you will use a systematic plan to solve real-world problem and learn different strategies and approaches to solving problems. In this section, we will introduce a problem-solving plan that will be useful throughout this book.

Read and Understand a Given Problem Situation

The first step to solving a word problem is to **read and understand** the problem. Here are a few questions that you should be asking yourself.

What am I trying to find out?
What information have I been given?
Have I ever solved a similar problem?

This is also a good time to define any variables. When you identify your **knowns** and **unknowns**, it is often useful to assign them a letter to make notation and calculations easier.

Make a Plan to Solve the Problem

The next step in the problem-solving plan is to **make a plan** or **develop a strategy.** How can the information you know assist you in figuring out the unknowns?

Here are some common strategies that you will learn.

- Drawing a diagram.
- Making a table.
- Looking for a pattern.
- Using guess and check.

- Working backwards.
- Using a formula.
- Reading and making graphs.
- Writing equations.
- Using linear models.
- Using dimensional analysis.
- Using the right type of function for the situation.

In most problems, you will use a combination of strategies. For example, drawing a diagram and looking for patterns are good strategies for most problems. Also, making a table and drawing a graph are often used together. The "writing an equation" strategy is the one you will work with the most in your study of algebra.

Solve the Problem and Check the Results

Once you develop a plan, you can implement it and **solve the problem.** That means using tables, graph and carrying out all operations to arrive at the answer you are seeking.

The last step in solving any problem should always be to **check and interpret** the answer. Here are some questions to help you to do that.

Does the answer make sense?

If you plug the solution back into the problem do all the numbers work out?

Can you use another method to arrive at the same answer?

Compare Alternative Approaches to Solving the Problem

Sometimes a certain problem is best solved by using a specific method. Most of the time, however, it can be solved by using several different strategies. When you are familiar with all of the problem-solving strategies, it is up to you to choose the methods that you are most comfortable with and that make sense to you. In this book, we will often use more than one method to solve a problem. This way we can demonstrate the strengths and weakness of different strategies when applied to different types of problems.

Regardless of the strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

Step 1

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solve your problem solving plan.

Step 3

Carry out the plan – Solve

This is where you solve the equation you developed in Step 2.

Step 4

Look – Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

Solve Real-World Problems Using a Plan

Let's now apply this problem solving plan to a problem.

Example 1

A coffee maker is on sale at 50% *off the regular ticket price. On the "Sunday Super Sale" the same coffee maker is on sale at an* ***additional*** 40% *off. If the final price is* \$21, *what was the original price of the coffee maker?*

Solution:

Step 1

Understand

We know: A coffee maker is discounted 50% and then 40%

The final price is \$21.

We want: The original price of the coffee maker.

Step 2

Strategy

Let's look at the given information and try to find the relationship between the information we know and the information we are trying to find.

50% off the original price means that the sale price is half of the original or 0.5× original price

So, the first sale price = 0.5× original price

A savings of 40% off the new price means you pay 60% of the new price 0.6× new price = 0.6 × (0.5× original price) = 0.3× original price

So, the price after the second sale = 0.3× original price

We know that after two sales, the final price is \$21

0.3× original price = \$21

Step 3

Solve

Since 0.3× original price = \$21

We can find the original price by dividing \$21 by 0.3.

Original price = \$21 ÷ 0.3 = \$70

Answer The original price of the coffee maker was \$70.

Step 4

Check

We found that the original price of the coffee maker is \$70.

To check that this is correct let's apply the discounts.

50% of \$70 $= .5 \times$ \$70 = \$35 savings.

So, after the first sale you pay: original price – savings = \$70 – \$35 = \$35.

40% of \$35 $= .4 \times$ \$35 = \$14 savings.

So, after the second sale you pay: \$35 – \$14 = \$21.

The answer checks out.

Review Questions

1. A sweatshirt costs \$35. Find the total cost if the sales tax is 7.75%.
2. This year you got a 5% raise. If your new salary is \$45,000, what was your salary before the raise?
3. It costs \$250 to carpet a room that is 14 ft $\times$ 18 ft. How much does it cost to carpet a room that is 9 ft $\times$ 10 ft?
4. A department store has a 15% discount for employees. Suppose an employee has a coupon worth \$10 off any item and she wants to buy a \$65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
5. To host a dance at a hotel you must pay \$250 plus \$20 per guest. How much money would you have to pay for 25 guests?
6. It costs \$12 to get into the San Diego County Fair and \$1.50 per ride. If Rena spent \$24 in total, how many rides did she go on?
7. An ice cream shop sells a small cone for \$2.95, a medium cone for \$3.50 and a large cone for \$4.25. Last Saturday, the shop sold 22 small cones, 26 medium cones and 15 large cones. How much money did the store earn?
8. The sum of angles in a triangle is 180 degrees. If the second angle is twice the size of the first angle and the third angle is three times the size of the first angle, what are the measures of the angles in the triangle?

Review Answers

1. \$37.71
2. \$42857
3. \$89.29
4. \$45.25
5. \$750
6. 8 rides
7. \$219.65
8. 30°, 60°, 90°

1.8 Problem-Solving Strategies: Make a Table and Look for a Pattern

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy: make a table.

- Develop and use the strategy: look for a pattern.
- Plan and compare alternative approaches to solving the problem.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this section, we will apply the problem-solving plan you learned about in the last section to solve several real-world problems. You will learn how to develop and use the methods **make a table** and **look for a pattern.** Let's review our problem-solving plan.

Step 1

Understand the problem Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3

Carry out the plan – Solve

This is where you solve the equation you developed in Step 2.

Step 4

Look – Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

Read and Understand Given Problem Situations

The most difficult parts of problem-solving are most often the first two steps in our problem-solving plan. You need to read the problem and make sure you understand what you are being asked. If you do not understand the question, then you can not solve the problem. Once you understand the problem, you can devise a strategy that uses the information you have been given to arrive at a result.

Let's apply the first two steps to the following problem.

Example 1:

Six friends are buying pizza together and they are planning to split the check equally. After the pizza was ordered, one of the friends had to leave suddenly, before the pizza arrived. Everyone left had to pay \$1 *extra as a result. How much was the total bill?*

Step 1

Understand

We want to find how much the pizza cost.

We know that five people had to pay an extra \$1 each when one of the original six friends had to leave.

Step 2

Strategy

We can start by making a list of possible amounts for the total bill.

We divide the amount by six and then by five. The total divided by five should equal $1 more than the total divided by six.

Look for any patterns in the numbers that might lead you to the correct answer.

In the rest of this section you will learn how to **make a table** or **look for a pattern** to figure out a solution for this type of problem. After you finish reading the rest of the section, you can finish solving this problem for homework.

Develop and Use the Strategy: Make a Table

The method "Make a Table" is helpful when solving problems involving numerical relationships. When data is organized in a table, it is easier to recognize patterns and relationships between numbers. Let's apply this strategy to the following example.

Example 2

Josie takes up jogging. On the first week she jogs for 10 minutes *per day, on the second week she jogs for* 12 minutes *per day. Each week, she wants to increase her jogging time by* 2 minutes *per day. If she jogs six days per week each week, what will be her total jogging time on the sixth week?*

Solution

Step 1

Understand

We know in the first week Josie jogs 10 minutes per day for six days.

We know in the second week Josie jogs 12 minutes per day for six days.

Each week, she increases her jogging time by 2 minutes per day and she jogs 6 days per week.

We want to find her total jogging time in week six.

Step 2

Strategy

A good strategy is to list the data we have been given in a table and use the information we have been given to find new information. We can make a table with the following headings.

Table 1.7:

Week	Minutes per Day	Minutes per Week

We are told that Josie jogs 10 minutes per day for six days in the first week and 12 minutes per day for six days in the second week. We can enter this information in our table:

Table 1.8:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72

You are told that each week Josie increases her jogging time by 2 minutes per day and jogs 6 times per week. We can use this information to continue filling in the table until we get to week six.

Table 1.9:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72
3	14	84
4	16	96
5	18	108
6	20	120

Step 3

Apply strategy/solve

To get the answer we read the entry for week six.

Answer In week six Josie jogs a total of 120 minutes .

Step 4

Check

Josie increases her jogging time by two minutes per day. She jogs six days per week.

This means that she increases her jogging time by 12 minutes per week.

Josie starts at 60 minutes per week and she increases by 12 minutes per week for five weeks.

That means the total jogging time $= 60 + 12 \times 5 = 120$ minutes

The answer checks out.

You can see that by making a table we were able to organize and clarify the information we were given. It also helped guide us in the next steps of the problem. This problem was solved solely by making a table. In many situations, this strategy would be used together with others to arrive at the solution.

Develop and Use the Strategy: Look for a Pattern

Look for a pattern is a strategy that you can use to look for patterns in the data in order to solve problems. The goal is to look for items or numbers that are repeated or a series of events that repeat. The following problem can be solved by finding a pattern.

Example 3

You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 layers?

Solution

Step 1

Understand

We know that we arrange tennis balls in triangles as shown.

We want to know how many balls there are in a triangle that has 8 layers.

Step 2

Strategy

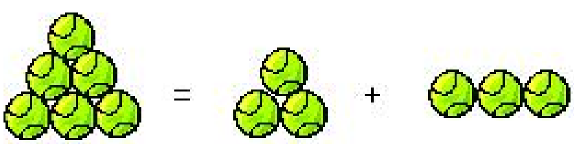
A good strategy is to make a table and list how many balls are in triangles of different layers.

One layer It is simple to see that a triangle with one layer has only one ball.

Two layers For a triangle with two layers we add the balls from the top layer to the balls of the bottom layer. It is useful to make a sketch of the different layers in the triangle.

$= 2 + 1 = 3$

Three layers we add the balls from the top triangle to the balls from the bottom layer.



$= 3 + 3 = 6$

We can fill the first three rows of the table.

Table 1.10:

Number of Layers	Number of Balls
1	1
2	3
3	6

We can see a pattern.

To create the next triangle, we add a new bottom row to the existing triangle.

The new bottom row has the same number of balls as there are layers.

- A triangle with 3 layers has 3 balls in the bottom layer.

To get the total balls for the new triangle, we add the number of balls in the old triangle to the number of rows in the new bottom layer.

Step 3

Apply strategy/solve:

We can complete the table by following the pattern we discovered.

Number of balls = number of balls in previous triangle + number of layers in the new triangle

Table 1.11:

Number of Layers	Number of Balls
1	1
2	3
3	6
4	$6 + 4 = 10$
5	$10 + 5 = 15$
6	$15 + 6 = 21$
7	$21 + 7 = 28$
8	$28 + 8 = 36$

Answer There are 36 balls in a triangle arrangement with 8 layers.

Step 4

Check

Each layer of the triangle has one more ball than the previous one. In a triangle with 8 layers,

layer 1 has 1 ball, layer 2 has 2 balls, layer 3 has 3 balls, layer 4 has 4 balls, layer 5 has 5 balls, layer 6 has 6balls, layer 7 has 7 balls, layer 8 has 8 balls.

When we add these we get: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ balls

The answer checks out.

Notice that in this example we made tables and drew diagrams to help us organize our information and find a pattern. Using several methods together is a very common practice and is very useful in solving word problems.

Plan and Compare Alternative Approaches to Solving Problems

In this section, we will compare the methods of "Making a Table" and "Looking for a Pattern" by using each method in turn to solve a problem.

Example 4

Andrew cashes a $180 *check and wants the money in* $10 *and* $20 *bills. The bank teller gives him* 12 *bills. How many of each kind of bill does he receive?*

Solution

Method 1: Making a Table

Step 1

Understand

Andrew gives the bank teller a $180 check.

The bank teller gives Andrew 12 bills. These bills are mixed $10 bills and $20 bills.

We want to know how many of each kind of bill Andrew receives.

Step 2

Strategy

Let's start by making a table of the different ways Andrew can have twelve $10 bills and $20 bills.

Andrew could have twelve $10 bills and zero $20 bills or eleven $10 bills and one $20 bills, so on.

We can calculate the total amount of money for each case.

Step 3

Apply strategy/solve

Table 1.12:

$10 bills	$20 bills	Total amount
12	0	$\$10(12) + \$20(0) = \$120$
11	1	$\$10(11) + \$20(1) = \$130$
10	2	$\$10(10) + \$20(2) = \$140$
9	3	$\$10(9) + \$20(3) = \$150$
8	4	$\$10(8) + \$20(4) = \$160$
7	5	$\$10(7) + \$20(5) = \$170$
6	6	$\$10(6) + \$20(6) = \$180$
5	7	$\$10(5) + \$20(7) = \$190$
4	8	$\$10(4) + \$20(8) = \$200$
3	9	$\$10(3) + \$20(9) = \$210$
2	10	$\$10(2) + \$20(10) = \$220$
1	11	$\$10(1) + \$20(11) = \$230$
0	12	$\$10(0) + \$20(12) = \$240$

In the table we listed all the possible ways you can get twelve $10 bills and $20 bills and the total amount of money for each possibility. The correct amount is given when Andrew has six $10 bills and six $20 bills.

Answer: Andrew gets six $10 bills and six $20 bills.

Step 4

Check

Six $10 bills and six $20 bills $= 6(\$10) + 6(\$20) = \$60 + \$120 = \$180$.

The answer checks out.

Let's solve the same problem using the method "Look for a Pattern."

Method 2: Looking for a Pattern

Step 1

Understand

Andrew gives the bank teller a $180 check.

The bank teller gives Andrew 12 bills. These bills are mixed $10 bills and $20 bills.

We want to know how many of each kind of bill Andrew receives.

Step 2

Strategy

Let's start by making a table of the different ways Andrew can have twelve \$10 bills and \$20 bills.

Andrew could have twelve \$10 bills and zero \$20 bills or eleven \$10 bills and one \$20 bill, so on.

We can calculate the total amount of money for each case.

Look for patterns appearing in the table that can be used to find the solution.

Step 3

Apply strategy/solve

Let's fill the rows of the table until we see a pattern.

Table 1.13:

\$10 bills	\$20 bills	Total amount
12	0	$\$10(12) + \$20(0) = \$120$
11	1	$\$10(11) + \$20(1) = \$130$
10	2	$\$10(10) + \$20(2) = \$140$

We see that every time we reduce the number of \$10 bills by one and increase the number of \$20 bills by one, the total amount increased by \$10. The last entry in the table gives a total amount of \$140 so we have \$40 to go until we reach our goal. This means that we should reduce the number of \$10 bills by four and increase the number of \$20 bills by four. We have

Six \$10 bills and six \$20 bills

$$6(\$10) + 6(\$20) = \$180$$

Answer: Andrew gets six \$10 bills and six \$20 bills

Step 4

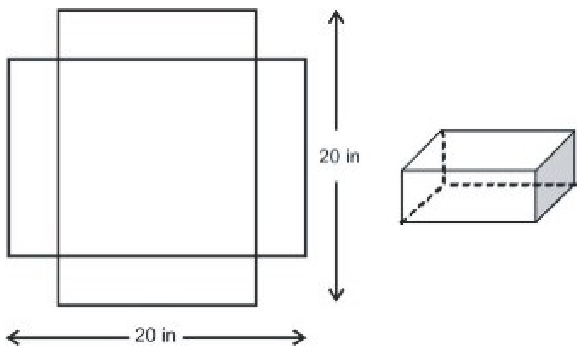
Check

Six \$10 bills and six \$20 bills $= 6(\$10) + 6(\$20) = \$60 + \$120 = \$180$.

The answer checks out.

You can see that the second method we used for solving the problem was less tedious. In the first method, we listed all the possible options and found the answer we were seeking. In the second method, we started with listing the options but we looked for a pattern that helped us find the solution faster. The methods of "Making a Table" and "Look for a Pattern" are both more powerful if used alongside other problem-solving methods.

Solve Real-World Problems Using Selected Strategies as Part of a Plan



Example 5:

Anne is making a box without a lid. She starts with a 20 in × 20 in *square piece of cardboard and cuts out four equal squares from each corner of the cardboard as shown. She then folds the sides of the box and glues the edges together. How big does she need to cut the corner squares in order to make the box with the biggest volume?*

Solution

Step 1

Understand

Anne makes a box out a 20 in × 20 in piece of cardboard.

She cuts out four equal squares from the corners of the cardboard.

She folds the sides and glues them to make a box.

How big should the cut out squares be to make the box with the biggest volume?

Step 2

Strategy

We need to remember the formula for the volume of a box.

Volume = Area of base × height

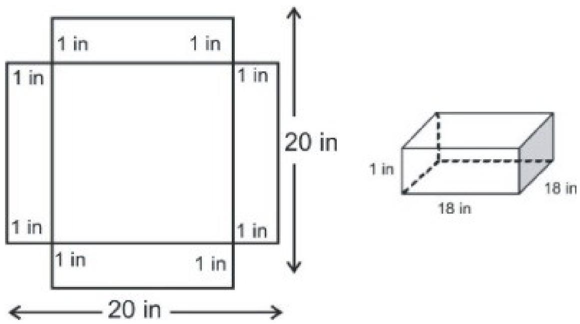
Volume = width × length × height

Make a table of values by picking different values for the side of the squares that we are cutting out and calculate the volume.

Step 3

Apply strategy/solve

Let's "make" a box by cutting out four corner squares with sides equal to 1 inch. The diagram will look like this:



You see that when we fold the sides over to make the box, the height becomes 1 inch, the width becomes 18 inches and the length becomes 18 inches.

Volume = width × length × height

Volume $= 18 \times 18 \times 1 = 324\ in^3$

Let's make a table that shows the value of the box for different square sizes:

Table 1.14:

Side of Square	Box Height	Box Width	Box Length	Volume
1	1	18	18	$18 \times 18 \times 1 = 324$
2	2	16	16	$16 \times 16 \times 2 = 512$
3	3	14	14	$14 \times 14 \times 3 = 588$
4	4	12	12	$12 \times 12 \times 4 = 576$
5	5	10	10	$10 \times 10 \times 5 = 500$
6	6	8	8	$8 \times 8 \times 6 = 384$
7	7	6	6	$6 \times 6 \times 7 = 252$
8	8	4	4	$4 \times 4 \times 8 = 128$
9	9	2	2	$2 \times 2 \times 9 = 36$
10	10	0	0	$0 \times 0 \times 10 = 0$

We stop at a square of 10 inches because at this point we have cut out all of the cardboard and we cannot make a box anymore. From the table we see that we can make the biggest box if we cut out squares with a side length of three inches. This gives us a volume of $588\ in^3$.

Answer The box of greatest volume is made if we cut out squares with a side length of three inches.

Step 4 Check

We see that $588\ in^3$ is the largest volume appearing in the table. We picked integer values for the sides of the squares that we are cut out. Is it possible to get a larger value for the volume if we pick non-integer values? Since we get the largest volume for the side length equal to three inches, let's make another table with values close to three inches that is split into smaller increments:

Table 1.15:

Side of Square	Box Height	Box Width	Box Length	Volume
2.5	2.5	15	15	$15 \times 15 \times 2.5 = 562.5$
2.6	2.6	14.8	14.8	$14.8 \times 14.8 \times 2.6 = 569.5$

Table 1.15: (continued)

Side of Square	Box Height	Box Width	Box Length	Volume
2.7	2.7	14.6	14.6	$14.6 \times 14.6 \times 2.7 = 575.5$
2.8	2.8	14.4	14.4	$14.4 \times 14.4 \times 2.8 = 580.6$
2.9	2.9	14.2	14.2	$14.2 \times 14.2 \times 2.9 = 584.8$
3	3	14	14	$14 \times 14 \times 3 = 588$
3.1	3.1	13.8	13.8	$13.8 \times 13.8 \times 3.1 = 590.4$
3.2	3.2	13.6	13.6	$13.6 \times 13.6 \times 3.2 = 591.9$
3.3	3.3	13.4	13.4	$13.4 \times 13.4 \times 3.3 = 592.5$
3.4	3.4	13.2	13.2	$13.2 \times 13.2 \times 3.4 = 592.4$
3.5	3.5	13	13	13×13×3.5 = 591.5

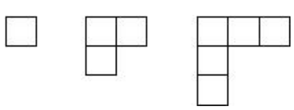
Notice that the largest volume is not when the side of the square is three inches, but rather when the side of the square is 3.3 inches .

Our original answer was not incorrect but it was obviously not as accurate as it could be. You can get an even more accurate answer if we take even smaller increments of the side length of the square. We can choose measurements that are smaller and larger than 3.3 inches .

The answer checks out if we want it rounded to zero decimal places but, **A more accurate answer is** 3.3 inches .

Review Questions

1. Go back and find the solution to the problem in Example 1.
2. Britt has $2.25 in nickels and dimes. If she has 40 coins in total how many of each coin does she have?
3. A pattern of squares is out together as shown. How many squares are in the 12^{th} diagram?



4. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts 24 cups the first week, cuts down to 21 cups the second week, and 18 cups the third week, how many weeks will it take him to reach his goal?
5. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
6. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
7. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long would it take him to catch up with Grace?
8. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?

Review Answers

1. \$30
2. 5 dimes and 35 nickels
3. 23 squares
4. 7 weeks
5. 50 cents
6. 5.5 hours
7. 5 hours
8. 3 ft in^3

1.9 Additional Resources

- For more on problem solving, see this George Pólya Wikipedia http://en.wikipedia.org/wiki/George_P%C3%B3lya Wikipedia entry

Chapter 2

Real Numbers

2.1 Integers and Rational Numbers

Learning Objectives

- Graph and compare integers.
- Classify and order rational numbers.
- Find opposites of numbers.
- Find absolute values.
- Compare fractions to determine which is bigger.

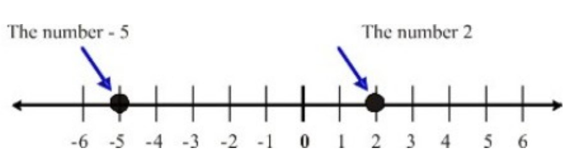
Graph and Compare Integers

Integers are the counting numbers $(1, 2, 3 \ldots)$, the negative counting numbers $(-1, -2, -3 \ldots)$, and zero. There are an infinite number of integers. Examples of integers are $0, 3, 76, -2, -11, 995, \ldots$ and you may know them by the name **whole numbers**. When we represent integers on the number line they fall exactly on the whole numbers.

Example 1

Compare the numbers 2 *and* -5

First, we will plot the two numbers on a number line.



We can compare integers by noting which is the **greatest** and which is the **least**. The **greatest** number is farthest to the right, and the **least** is farthest to the left.

In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that 2 is greater than -5. We use the symbol ">" to mean "greater than".

Solution

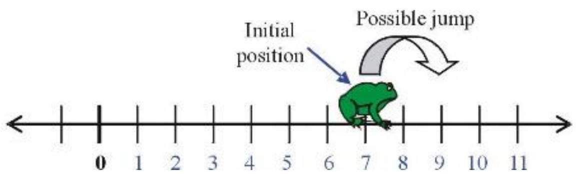
$2 > -5$

Example 2

A frog is sitting perfectly on top of number 7 *on a number line. The frog jumps randomly to the left or right, but always jumps a distance of exactly* 2. *Describe the set of numbers that the frog may land on, and list all the possibilities for the frog's position after exactly* 5 *jumps.*

Solution

We will graph the frog's position, and also indicate what a jump of 2 looks like. We see that one possibility is that the frog lands on 5. Another possibility is that it lands on 9. It is clear that the frog will always land on an **odd number**.



After one jump the frog could be on either the 9 or the 5 (but not on the 7). After two jumps the frog could be on 11, 7 or 3. By counting the number of times the frog jumps to the right or left, we may determine where the frog lands. After five jumps, there are many possible locations for the frog. There is a systematic way to determine the possible locations by how many times the frog jumped right, and by how many times the frog jumped left.

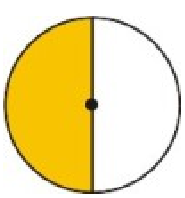
$$\begin{aligned} &\text{RRRRR} = 5 \text{ jumps right} && \text{location} = 7 + (5 \cdot 2) = 17 \\ &\text{RRRRL} = 4 \text{ jumps right, 1 jump left} && \text{location} = 7 + (3 \cdot 2) = 13 \\ &\text{RRRLL} = 3 \text{ jumps right, 2 jumps left} && \text{location} = 7 + (1 \cdot 2) = 9 \\ &\text{RRLLL} = 2 \text{ jumps right, 3 jumps left} && \text{location} = 7 - (1 \cdot 2) = 5 \\ &\text{RLLLL} = 1 \text{ jump right, 3 jumps left} && \text{location} = 7 - (3 \cdot 2) = 1 \\ &\text{LLLLL} = 5 \text{ jumps left} && \text{location} = 7 - (5 \cdot 2) = -3 \end{aligned}$$

These are the possible locations of the frog after exactly five jumps. Notice that the order does not matter: three jumps right, one left and one right is the same as four jumps to the right and one to the left.

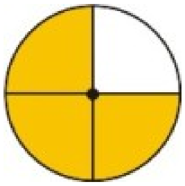
Classifying Rational Numbers

When we divide an integer by another integer (*not* zero) we get what we call a **rational number.** It is called this because it is the **ratio** of one number to another. For example, if we divide one integer a by a second integer b the rational number we get is $\frac{a}{b}$, provided that b is not zero. When we write a rational number like this, the top number is called the **numerator**. The bottom number is called the **denominator**. You can think of the rational number as a fraction of a cake. If you cut the cake into b slices, your share is a of those slices.

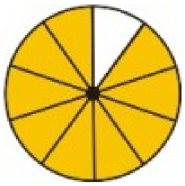
For example, when we see the rational number $\frac{1}{2}$, we imagine cutting the cake into two parts. Our share is one of those parts. Visually, the rational number $\frac{1}{2}$ looks like this.



With the rational number $\frac{3}{4}$, we cut the cake into four parts and our share is three of those parts. Visually, the rational number $\frac{3}{4}$ looks like this.



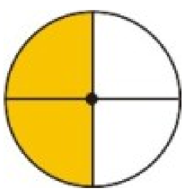
The rational number $\frac{9}{10}$ represents nine slices of a cake that has been cut into ten pieces. Visually, the rational number $\frac{9}{10}$ looks like this.



Proper fractions are rational numbers where the numerator (the number on the top) is less than the denominator (the number on the bottom). A proper fraction represents a number less than one. With a proper fraction you always end up with less than a whole cake!

Improper fractions are rational numbers where the numerator is greater than the denominator. Improper fractions can be rewritten as a mixed number – an integer plus a proper fraction. An improper fraction represents a number greater than one.

Equivalent fractions are two fractions that give the same numerical value when evaluated. For example, look at a visual representation of the rational number $\frac{2}{4}$.



You can see that the shaded region is identical in size to that of the rational number one-half $\frac{1}{2}$. We can write out the prime factors of both the numerator and the denominator and cancel matching factors that appear in both the numerator **and** denominator.

$$\left(\frac{2}{4}\right) = \left(\frac{\not{2} \cdot 1}{\not{2} \cdot 2 \cdot 1}\right) \qquad \text{We then re-multiply the remaining factors.} \qquad \left(\frac{2}{4}\right) = \left(\frac{1}{2}\right)$$

This process is called **reducing** the fraction, or writing the fraction in lowest terms. Reducing a fraction does not change the value of the fraction. It just simplifies the way we write it. When we have canceled all common factors, we have a fraction in its **simplest form**.

Example 3

Classify and simplify the following rational numbers

a) $\left(\frac{3}{7}\right)$

b) $\left(\frac{9}{3}\right)$

c) $\left(\frac{50}{60}\right)$

a) 3 and 7 are both prime – there is no simpler form for this rational number so...

Solution

$\frac{3}{7}$ is already in its simplest form.

b) $9 = 3 \cdot 3$ and 3 is prime. We rewrite the fraction as: $\left(\frac{9}{3}\right) = \left(\frac{\cancel{3} \cdot 3 \cdot 1}{\cancel{3} \cdot 1}\right)$. $9 > 3$ so...

Solution

$\frac{9}{3}$ is an improper fraction and simplifies to $\frac{3}{1}$ or simply 3.

c) $50 = 5 \cdot 5 \cdot 2$ and $60 = 5 \cdot 3 \cdot 2 \cdot 2$. We rewrite the fraction thus: $\frac{50}{60} = \left(\frac{\cancel{5} \cdot 5 \cdot \cancel{2} \cdot 1}{\cancel{5} \cdot 5 \cdot \cancel{2} \cdot 2 \cdot 1}\right)$. $50 < 60$ so...

Solution

$\frac{50}{60}$ is a proper fraction and simplifies to $\frac{5}{6}$.

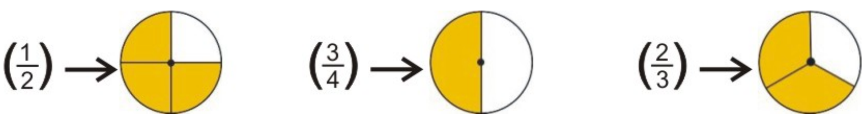
Order Rational Numbers

Ordering rational numbers is simply a case of arranging numbers in order of increasing value. We write the numbers with the least (most negative) first and the greatest (most positive) last.

Example 4

Put the following fractions in order from least to greatest: $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}$

Let's draw out a representation of each fraction.



We can see visually that the largest number is $\frac{3}{4}$ and the smallest is $\frac{1}{2}$:

Solution

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$$

With simple fractions, it is easy to order them. Think of the example above. We know that one-half is greater than one quarter, and we know that two thirds is bigger than one-half. With more complex fractions, however we need to find a better way to compare.

Example 5

Which is greater, $\frac{3}{7}$ *or* $\frac{4}{9}$*?*

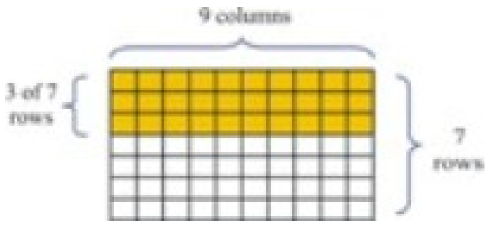
In order to determine this we need to find a way to rewrite the fractions so that we can better compare them. We know that we can write equivalent fractions for both of these. If we make the denominators in our equivalent fractions the same, then we can compare them directly. We are looking for the lowest common multiple of each of the denominators. This is called finding the **lowest common denominator** (LCD).

The lowest common multiple of 7 and 9 is 63. Our fraction will be represented by a shape divided into 63 sections. This time we will use a rectangle cut into 9 by $7 = 63$ pieces:

7 divides into 63 nine times so:

$$\left(\frac{3}{7}\right) = \frac{9}{9}\left(\frac{3}{7}\right) = \left(\frac{27}{63}\right)$$

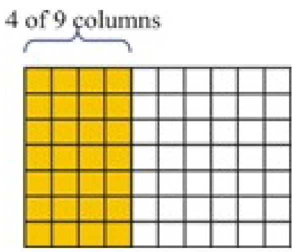
Note that multiplying by $\frac{9}{9}$ is the same as multiplying by 1. Therefore, $\frac{27}{63}$ is an equivalent fraction to $\frac{3}{7}$. Here it is shown visually.



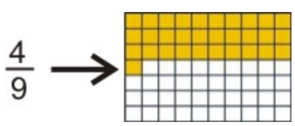
9 divides into 63 seven times so:

$$\left(\frac{4}{9}\right) = \frac{7}{7}\left(\frac{4}{9}\right) = \left(\frac{28}{63}\right)$$

$\frac{28}{63}$ is an equivalent fraction to $\frac{4}{9}$. Here it is shown visually.



By writing the fractions over a **common denominator** of 63, you can easily compare them. Here we take the 28 shaded boxes out of 63 (from our image of $\frac{4}{9}$ above) and arrange them in a way that makes it easy to compare with our representation of $\frac{3}{7}$. Notice there is one little square "left over".



Solution

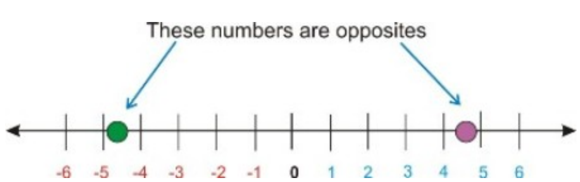
Since $\frac{28}{63}$ is greater than $\frac{27}{63}$, $\frac{4}{9}$ is greater than $\frac{3}{7}$.

Remember

To compare rational numbers re-write them with a **common denominator**.

Find the Opposites of Numbers

Every number has an opposite. On the number line, a number and its opposite are *opposite* each other. In other words, they are the same distance from zero, but they are on opposite sides of the number line.



By definition, the opposite of zero is zero.

Example 6

Find the value of each of the following.

a) $3 + (-3)$

b) $5 + (-5)$

c) $(-11.5) + (11.5)$

d) $\frac{3}{7} + \frac{-3}{7}$

Each of the pairs of numbers in the above example are **opposites.** The opposite of 3 is (-3), the opposite of 5 is (-5), the opposite of (-11.5) is 11.5 and the opposite of $\frac{3}{7}$ is $\frac{3}{7}$.

Solution

The value of each and every sum in this problem is 0.

Example 7

Find the opposite of each of the following:

a) 19.6

b) $-\frac{4}{9}$

c) x

d) xy^2

e) $(x - 3)$

Since we know that opposite numbers are on opposite sides of zero, we can simply multiply each expression by -1. This changes the sign of the number to its opposite.

a) **Solution**

The opposite of 19.6 is -19.6.

b) **Solution**

The opposite of is $-\frac{4}{9}$ is $\frac{4}{9}$.

c) **Solution**

The opposite of x is $-x$.

d) **Solution**

The opposite of xy^2 is $-xy^2$.

e) **Solution**

The opposite of $(x - 3)$ is $-(x - 3) = 3 - x$.

Note: With the last example you must multiply the **entire expression** by -1. A **common mistake** in this example is to assume that the opposite of $(x - 3)$ is $(x + 3)$. DO NOT MAKE THIS MISTAKE!

Find absolute values

When we talk about absolute value, we are talking about distances on the number line. For example, the number 7 is 7 units away from zero. The number -7 is also 7 units away from zero. The absolute value of a number is the distance it is from zero, so the absolute value of 7 and the absolute value of -7 are both 7.

We **write** the absolute value of -7 like this $|-7|$
We **read** the expression $|x|$ like this "the absolute value of x."

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols evaluate that operation first.

- The absolute value of a number or an expression is **always** positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, and not the direction.

Example 8

Evaluate the following absolute value expressions.

a) $|5 + 4|$

b) $3 - |4 - 9|$

c) $|-5 - 11|$

d) $-|7 - 22|$

Remember to treat any expressions inside the absolute value sign as if they were inside parentheses, and evaluate them first.

Solution

a)

$$\begin{aligned}|5 + 4| &= |9| \\ &= 9\end{aligned}$$

b)

$$\begin{aligned}3 - |4 - 9| &= 3 - |-5| \\ &= 3 - 5 \\ &= -2\end{aligned}$$

c)

$$\begin{aligned}|-5 - 11| &= |-16| \\ &= 16\end{aligned}$$

d)

$$\begin{aligned}-|7 - 22| &= -|-15| \\ &= -(15) \\ &= -15\end{aligned}$$

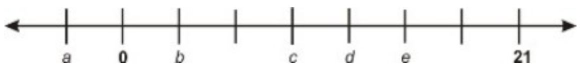
Lesson Summary

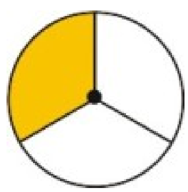
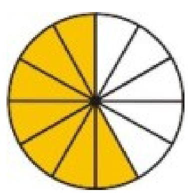
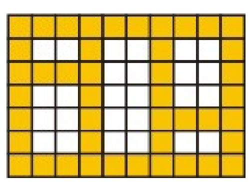
- **Integers** (or **whole numbers**) are the counting numbers $(1, 2, 3 \ldots)$, the negative counting numbers $(-1, -2, -3 \ldots)$, and zero.
- A **rational number** is the **ratio** of one integer to another, like $\frac{a}{b}$ or $\frac{3}{5}$. The top number is called the **numerator** and the bottom number (which can not be zero) is called the **denominator**.
- **Proper fractions** are rational numbers where the numerator is less than the denominator.
- **Improper fractions** are rational numbers where the numerator is greater than the denominator.
- Equivalent fractions are two fractions that give the same numerical value when evaluated.
- To **reduce** a fraction (write it in **simplest form**) write out all prime factors of the numerator and denominator, cancel common factors, then recombine.

- To compare two fractions it helps to write them with a **common denominator**: the same integer on the bottom of each fraction.
- The **absolute value** of a number is the distance it is from zero on the number line. The absolute value of a number or expression will always be positive or zero.
- Two numbers are **opposites** if they are the same distance from zero on the number line and on opposite sides of zero. The opposite of an expression can be found by multiplying **the entire expression** by -1.

Review Questions

1. The tick-marks on the number line represent evenly spaced integers. Find the values of a, b, c, d and e.



2. Determine what fraction of the whole each shaded region represents.
   (a)
   
   (b)
   
   (c)
   
3. Place the following sets of rational numbers in order, from least to greatest.
   (a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
   (b) $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
   (c) $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$
   (d) $\frac{7}{11}, \frac{8}{13}, \frac{12}{19}$
4. Find the simplest form of the following rational numbers.
   (a) $\frac{22}{44}$
   (b) $\frac{9}{27}$
   (c) $\frac{12}{18}$
   (d) $\frac{315}{420}$
5. Find the opposite of each of the following.
   (a) 1.001
   (b) $(5-11)$
   (c) $(x+y)$
   (d) $(x-y)$

6. Simplify the following absolute value expressions.
   (a) $11 - |-4|$
   (b) $|4 - 9| - |-5|$
   (c) $|-5 - 11|$
   (d) $7 - |22 - 15 - 19|$
   (e) $-|-7|$
   (f) $|-2 - 88| - |88 + 2|$

Review Answers

1. $a = -3$; $b = 3$; $c = 9$; $d = 12$; $e = 15$
2. $a = \frac{1}{3}$; $b = \frac{7}{12}$; $c = \frac{22}{35}$
3.
4. (a) $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$
   (b) $\frac{11}{12} < \frac{12}{11} < \frac{13}{10}$
   (c) $\frac{59}{100} < \frac{49}{80} < \frac{39}{60}$
   (d) $\frac{8}{13} < \frac{12}{19} < \frac{7}{11}$
5.
6. (a) $\frac{1}{2}$
   (b) $\frac{1}{3}$
   (c) $\frac{2}{3}$
   (d) $\frac{3}{4}$
7.
8. (a) -1.001
   (b) $6 - (x + y)$
   (c) $(y - x)$
9.
10. (a) 7
    (b) 0
    (c) 16
    (d) -5
    (e) -7
    (f) 0

2.2 Addition of Rational Numbers

Learning Objectives

- Add using a number line.
- Add rational numbers.
- Identify and apply properties of addition.
- Solve real-world problems using addition of fractions.

Add Using a Number Line

In Lesson one, we learned how to represent numbers on a number line. When we perform addition on a number line, we start at the position of the first number, and then move to the right by the number of units shown in the sum.

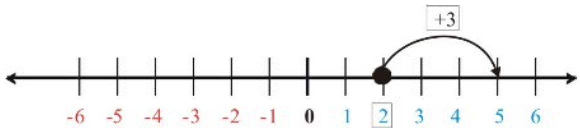
Example 1

Represent the sum $2 + 3$ *on a number line.*

We start at the number 2, and then move 3 to the right. We end at the number 5.

Solution

$$2 + 3 = 5$$



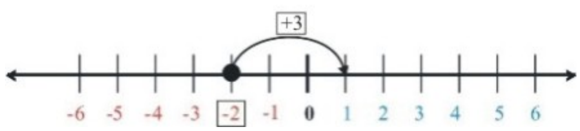
Example 2

Represent the sum $-2 + 3$ *on a number line.*

We start at the number -2, and then move 3 to the right. We thus end at $+1$.

Solution

$$-2 + 3 = 1$$



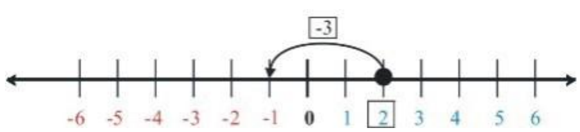
Example 3

Represent the sum $2 - 3$ *on a number line.*

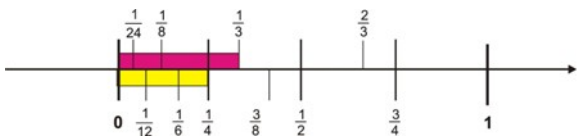
We are now faced with a subtraction. When subtracting a number, an equivalent action is **adding a negative number**. Either way, we think of it, we are moving to the left. We start at the number 2, and then move 3 to the left. We end at -1.

Solution

$$2 - 3 = -1$$



We can use the number line as a rudimentary way of adding fractions. The enlarged number line below has a number of common fractions marked. The markings on a ruler or a tape measure follow the same pattern. The two shaded bars represent the lengths $\frac{1}{3}$ and $\frac{1}{4}$.



To find the difference between the two fractions look at the difference between the two lengths. You can see the red is $\frac{1}{12}$ longer than the yellow. You could use this as an estimate of the difference.

$$\text{equation} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

To find the sum of the two fractions, we can lay them end to end. You can see that the sum $\frac{1}{3} + \frac{1}{4}$ is a little over one half.

Adding Rational Numbers

We have already seen the method for writing rational numbers over a common denominator. When we add two fractions we need to ensure that the denominators match before we can determine the sum.

Example 4

Simplify $\frac{3}{5} + \frac{1}{6}$

To combine these fractions, we need to rewrite them over a common denominator. We are looking for the **lowest common denominator** (LCD). We need to identify the **lowest common multiple** or **least common multiple** (LCM) of 5 and 6. That is the smallest number that both 5 and 6 divide into without remainder.

- The lowest number that 5 and 6 both divide into without remainder is 30. The LCM of 5 and 6 is 30, so the lowest common denominator for our fractions is also 30.

We need to rewrite our fractions as new **equivalent fractions** so that the denominator in each case is 30.

If you think back to our idea of a cake cut into a number of slices, $\frac{3}{5}$ means 3 slices of a cake that has been cut into 5 pieces. You can see that if we cut the same cake into 30 pieces (6 times as many) we would need 18 slices to have an equivalent share, since $18 = 3 \times 6$.

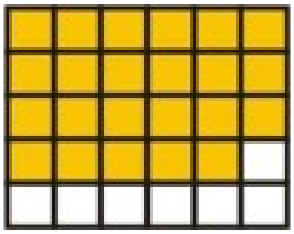
$\frac{3}{5}$ is equivalent to $\frac{18}{30}$

By a similar argument, we can rewrite the fraction $\frac{1}{6}$ as a share of a cake that has been cut into 30 pieces. If we cut it into 5 times as many pieces we require 5 times as many slices.

$\frac{1}{6}$ is equivalent to $\frac{5}{30}$

Now that both fractions have the same common denominator, we can add the fractions. If we add our 18 smaller pieces of cake to the additional 5 pieces you see that we get a total of 23 pieces. 23 pieces of a cake that has been cut into 30 pieces means that our answer is.

Solution



$$\frac{3}{5}+\frac{1}{6}=\frac{18}{30}+\frac{5}{30}=\frac{23}{30}$$

You should see that when we have fractions with a common denominator, we **add the numerators** but we **leave the denominators alone**. Here is this information in algebraic terms.

When adding fractions: $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

Example 5

Simplify $\frac{14}{11}+\frac{1}{9}$

The lowest common denominator in this case is 99. This is because the lowest common multiple of 9 and 11 is 99. So we write equivalent fractions for both $\frac{14}{11}$ and $\frac{1}{9}$ with denominators of 99.

11 divides into 99 nine times so $\frac{14}{11}$ is equivalent to $\frac{14\cdot 9}{11\cdot 9}=\frac{126}{99}$

We can multiply the numerator and denominator by 9 (or by any number) since $9/9=1$ and 1 is the multiplicative identity.

9 divides into 99 eleven times so $\frac{1}{9}$ is equivalent to $\frac{1\cdot 11}{9\cdot 11}=\frac{11}{99}$.

Now we simply add the numerators.

Solution

$$\frac{14}{11}+\frac{1}{9}=\frac{126}{99}+\frac{11}{99}=\frac{137}{99}$$

Example 6

Simplify $\frac{1}{12}+\frac{2}{9}$

The least common denominator in this case is 36. This is because the LCM of 12 and 9 is 36. We now proceed to write the equivalent fractions with denominators of 36.

12 divides into 35 three times so $\frac{1}{12}$ is equivalent to $\frac{1\cdot 3}{12\cdot 3}=\frac{3}{36}$.

9 divides into 36 four times so $\frac{2}{9}$ is equivalent to $\frac{2\cdot 4}{9\cdot 4}=\frac{8}{36}$.

Solution

$$\frac{1}{12}+\frac{2}{9}=\frac{11}{36}$$

You can see that we quickly arrive at an equivalent fraction by multiplying the numerator and the denominator by the same non-zero number.

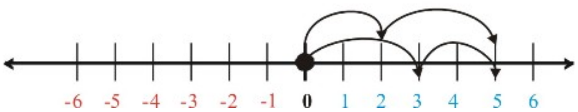
The fractions $\frac{a}{b}$ *and* $\left(\frac{a\cdot c}{b\cdot c}\right)$ *are equivalent when* $c\neq 0$

Identify and Apply Properties of Addition

The three mathematical properties which involve addition are the **commutative, associative,** and the **additive identity properties**.

- **Commutative property** When two numbers are added, the sum is the same even if the order of the items being added changes.

Example $3+2=2+3$



On a number line this means move 3 units *to the right then* 2 units *to the right. The commutative property says this is equivalent of moving* 2 units *to the right then* 3 units *to the right. You can see that they are both the same, as they both end at* 5.

- **Associative Property** When three or more numbers are added, the sum is the same regardless of how they are grouped.

Example $(2+3)+4=2+(3+4)$

- **Additive Identity Property** The sum of any number and zero is the original number.

Example $5+0=5$

Example 7

Nadia and Peter are building sand castles on the beach. Nadia built a castle two feet tall, stopped for ice-cream and then added one more foot to her castle. Peter built a castle one foot tall before stopping for a sandwich. After his sandwich, he built up his castle by two more feet. Whose castle is the taller?

Solution

Nadia's castle is $(2+1)$ feet tall. Peter's castle is $(1+2)$ feet tall. According to the **Commutative Property of Addition,** the two castles are the same height.

Example 8

Nadia and Peter each take candy from the candy jar. Peter reaches in first and grabs one handful. He gets seven pieces of candy. Nadia grabs with both hands and gets seven pieces in one hand and five in the other. The following day Peter gets to go first. He grabs with both hands and gets five pieces in one hand and six in the other. Nadia, grabs all the remaining candy, six pieces, in one hand. In total, who got the most candy?

Solution

On day one, Peter gets 7 candies, and on day two he gets $(5+6)$ pieces. His total is $7+(5+6)$. On day one, Nadia gets $(7+5)$ pieces. On day two, she gets 6. Nadia's total is therefore $(7+5)+6$. According to the **Associative Property of Addition** they both received exactly the same amount.

Solve Real-World Problems Using Addition

Example 9

Peter is hoping to travel on a school trip to Europe. The ticket costs \$2400. *Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?*

The first thing we need to do is extract the relevant information. Here is what Peter can count on.

$\left(\frac{1}{2}\right)$	From parents
$\left(\frac{1}{6}\right)$	From grandma
$\left(\frac{1}{4}\right)$	From grandparents in Florida

Here is out problem. $\frac{1}{2}+\frac{1}{6}+\frac{1}{4}$

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12. This is our LCD.

2 divides into 12 six times :	$\frac{1}{2}=\frac{6 \cdot 1}{6 \cdot 2}=\frac{6}{12}$
6 divides into 12 two times :	$\frac{1}{6}=\frac{2 \cdot 1}{2 \cdot 6}=\frac{2}{12}$
4 divides into 12 six times :	$\frac{1}{4}=\frac{3 \cdot 1}{3 \cdot 4}=\frac{3}{12}$
So an equivalent sum for our problem is	$\frac{6}{12}+\frac{2}{12}+\frac{3}{12}=\frac{(6+2+3)}{12}=\frac{11}{12}$

Solution

Peter can count on eleven-twelfths of the cost of the trip ($\$2,200$ out of $\$2,400$).

Lesson Summary

- To add fractions, rewrite them over the **lowest common denominator (LCD)**. The lowest common denominator is the **lowest** (or **least**) **common multiple (LCM)** of the two denominators.

- When **adding fractions**: $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

- The fractions $\frac{a}{b}$ and $\frac{a \cdot c}{b \cdot c}$ are **equivalent** when $c \neq 0$

- The **additive properties** are:

- **Commutative property** the sum of two numbers is the same even if the order of the items to be added changes.

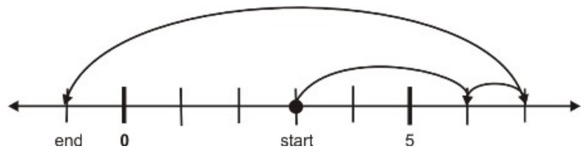
Ex: $2+3=3+2$

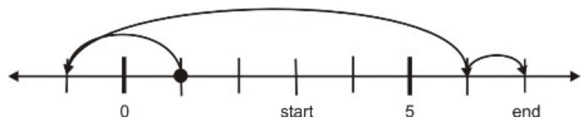
- **Associative Property** When three or more numbers are added, the sum is the same regardless of how they are grouped.

Ex: $(2+3)+4=2+(3+4)$

- **Additive Identity Property** The sum of any number and zero is the original number.

Review Question

1. Write the sum that the following moves on a number line represent.
   (a)
   

   (b)
   

2. Add the following rational numbers and write the answer in its **simplest form.**
   (a) $\frac{3}{7}+\frac{2}{7}$
   (b) $\frac{3}{10}+\frac{1}{5}$
   (c) $\frac{5}{16}+\frac{5}{12}$
   (d) $\frac{3}{8}+\frac{9}{16}$
   (e) $\frac{8}{25}+\frac{7}{10}$
   (f) $\frac{1}{6}+\frac{1}{4}$
   (g) $\frac{7}{15}+\frac{2}{9}$
   (h) $\frac{5}{19}+\frac{2}{27}$
3. Which property of addition does each situation involve?
   (a) Whichever order your groceries are scanned at the store, the total will be the same.
   (b) However many shovel-loads it takes to move 1 ton of gravel the number of rocks moved is the same.
4. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?

Review Answers

1.
2. (a) $3+3+1-8=-1$
   (b) $1-2+7+1=7$
3.
4. (a) $\frac{5}{7}$
   (b) $\frac{1}{2}$
   (c) $\frac{35}{48}$
   (d) $\frac{15}{16}$
   (e) $\frac{51}{50}$
   (f) $\frac{5}{12}$
   (g) $\frac{31}{45}$
   (h) $\frac{173}{513}$
5.
6. (a) Commutative and Associative

(b) Associative

7. $\frac{1}{12}$ is added as tax.

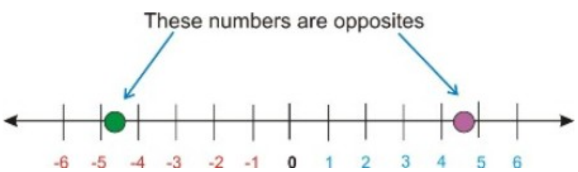
2.3 Subtraction of Rational Numbers

Learning objectives

- Find additive inverses.
- Subtract rational numbers.
- Evaluate change using a variable expression.
- Solve real world problems using fractions.

Find Additive Inverses

The **additive inverse** of a number is simply **opposite** of the number. (see section 2.1.4). Here are opposites on a number line.



When we think of additive inverses we are really talking about the **opposite process** (or **inverse process**) of **addition**. In other words, the process of **subtracting** a number is the same as adding the **additive inverse** of that number. When we add a number to its additive inverse, we get zero as an answer.

$(6) + (-6) = 0$ — -6 is the additive inverse of 6.

$(279) + (-279) = 0$ — -279 is the additive inverse of 279.

$(x) + (-x) = 0$ — $-x$ is the additive inverse of x.

Subtract Rational Numbers

The method for subtracting fractions (as you should have assumed) is just the same as addition. We can use the idea of an additive inverse to relate the two processes. Just like in addition, we are going to need to write each of the rational numbers over a common denominator.

Example 2

Simplify $\frac{1}{3} - \frac{1}{9}$

The lowest common multiple of 9 and 3 is 9. Our common denominator will be nine. We will not alter the second fraction because the denominator is already nine.

3 divides into 9 three times $\frac{1}{3} = \frac{1 \cdot 3}{3 \cdot 3} = \frac{3}{9}$. In other words $\frac{3}{9}$ is an **equivalent fraction** to $\frac{1}{3}$.

Our sum becomes $\frac{3}{9} - \frac{1}{9}$

Remember that when we add fractions with a common denominator, we **add** the **numerators** and the **denominator is unchanged**. A similar relationship holds for subtraction, only that we subtract the numerators.

When subtracting fractions $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Solution

$$\frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

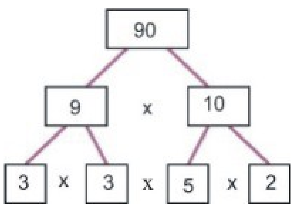
Two-ninths is the simplest form for our answer. So far we have only dealt with examples where it is easy to find the least common multiple of the denominators. With larger numbers, it is not so easy to be certain that we have the **least common denominator** (LCD). We need a more systematic method. In the next example, we will use the method of **prime factors** to find the least common denominator.

Example 3

Simplify $\frac{29}{90} - \frac{13}{126}$

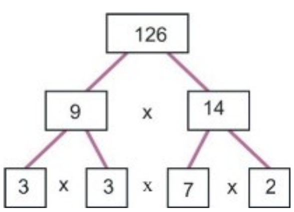
This time we need to find the lowest common multiple (LCM) of 90 and 126. To find the LCM, we first find the prime factors of 90 and 126. We do this by continually dividing the number by factors until we cannot divide any further. You may have seen a factor tree before:

The factor tree for 90 looks like this:



$$\begin{aligned} 90 &= 9 \cdot 10 \\ 9 &= 3 \cdot 3 \\ 10 &= 5 \cdot 2 \\ 90 &= 3 \cdot 3 \cdot 5 \cdot 2 \end{aligned}$$

The factor tree for 126 looks like this:



$$\begin{aligned} 126 &= 9 \cdot 14 \\ 9 &= 3 \cdot 3 \\ 14 &= 7 \cdot 2 \\ 126 &= 3 \cdot 3 \cdot 7 \cdot 2 \end{aligned}$$

The LCM for 90 and 126 is made from the **smallest possible collection of primes** that enables us to construct either of the two numbers. We take only enough of each prime to make the number with the highest number of factors of that prime in its factor tree.

Table 2.1:

Prime	Factors in 90	Factors in 126	We Take
2	1	1	1

Table 2.1: (continued)

Prime	Factors in 90	Factors in 126	We Take
3	2	2	2
5	1	0	1
7	0	1	1

So we need: one 2, two 3′s, one 5 and one 7. In other words: $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$

- The lowest common multiple of 90 and 126 is 630. The LCD for our calculation is 630.

90 divides into 630 seven times (notice that 7 is the only factor in 630 that is missing from 90) $\frac{29}{90} = \frac{7 \cdot 29}{7 \cdot 90} = \frac{203}{630}$

126 divides into 630 five times (notice that 5 is the only factor in 630 that is missing from 126) $\frac{13}{126} = \frac{5 \cdot 13}{5 \cdot 126} = \frac{65}{630}$

Now we complete the problem.

$$\frac{29}{90} - \frac{13}{126} = \frac{203}{630} - \frac{65}{630} = \frac{(203-65)}{630} = \frac{138}{630} \left\{ \text{remember, } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \right\}$$

This fraction **simplifies**. To be sure of finding the **simplest form** for $\frac{138}{630}$ we write out the numerator and denominator as **prime factors**. We already know the prime factors of 630. The prime factors of 138 are $138 = 2 \cdot 3 \cdot 23$.

$$\frac{138}{630} = \frac{2 \cdot 3 \cdot 23}{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7} \qquad \text{one factor of 2 and one factor of 3 cancels}$$

Solution

$$\frac{27}{90} - \frac{13}{126} = \frac{23}{105}$$

Example 4

A property management firm is acquiring parcels of land in order to build a small community of condominiums. It has bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. Here are the relevant fractions.

$\frac{4}{5}, \frac{5}{12}$ and $\frac{19}{20}$ The plots of land that the firm has acquired.

$\frac{1}{6}$ The amount of land that the firm has to give up.

This sum will determine the amount of land available for development.

$$\frac{4}{5} + \frac{5}{12} + \frac{19}{20} - \frac{1}{6} \qquad \text{We need to find the LCM of } 5, 12, 20 \text{ and } 6.$$

$5 = 5$ one 5

$12 = 2 \cdot 2 \cdot 3$ two 2′s, one 3

$20 = 2 \cdot 2 \cdot 5$ two 2′s, one 5

$6 = 2 \cdot 3$ one 3

The smallest set of primes that encompasses all of these is $2, 2, 3, 5$. Our LCD is thus $2 \cdot 2 \cdot 3 \cdot 5 = 60$

Now we can convert all fractions to a common denominator of 60. To do this, we multiply by the factors of 60 that are missing in the denominator we are converting. For example, 5 is missing two $2'$s and a 3. This results in $2 \cdot 2 \cdot 3 = 12$.

$$\frac{4}{5} = \frac{12 \cdot 4}{12 \cdot 5} = \frac{48}{60}$$
$$\frac{5}{12} = \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60}$$
$$\frac{19}{20} = \frac{3 \cdot 19}{10 \cdot 6} = \frac{57}{60}$$
$$\frac{1}{6} = \frac{10 \cdot 1}{10 \cdot 6} = \frac{10}{60}$$

Our converted sum can be rewritten as: $\frac{48}{60} + \frac{25}{60} + \frac{57}{60} - \frac{10}{60} = \frac{(48+25+57-10)}{60} = \frac{120}{60}$

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator. This fraction reduces to $\frac{2}{1}$ or simply two. One is sometimes called the ***invisible denominator***, as every whole number can be thought of as a rational number whose denominator is one.

Solution

The property firm has two acres available for development.

Evaluate Change Using a Variable Expression

When we write algebraic expressions to represent a real quantity, the difference between two values is the **change** in that quantity.

Example 5

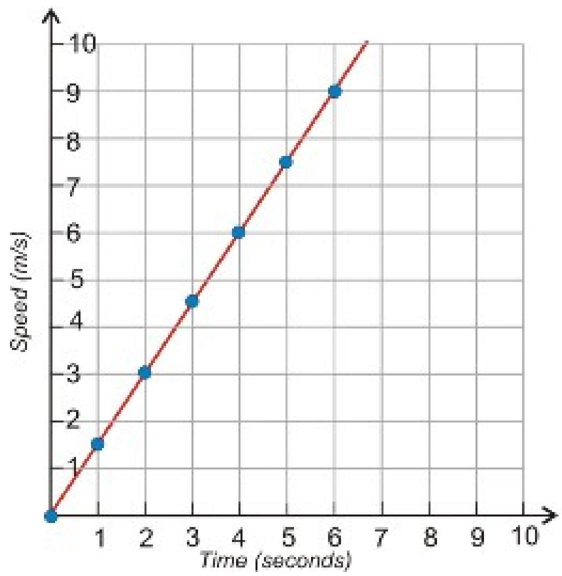
The speed of a train increases according to the expression speed $= 1.5t$ where speed is measured in meters per second, and "t" is the time measured in seconds. Find the change in the speed between $t = 2$ seconds and $t = 6$ seconds.

This function represents a train that is stopped when time equals zero (speed $= 0 \times 0.25$). As the stopwatch ticks, the train's speed increases in a linear pattern. We can make a table of what the train's speed is at every second.

Table 2.2:

Time (seconds)	Speed (m/s)
0	0
1	1.5
2	3
3	4.5
4	6
5	7.5
6	9

We can even graph this function. The graph of speed vs. time is shown here.



We wish to find the change in speed between $t = 2$ seconds and $t = 6$ seconds. There are several ways to do this. We could look at the table, and read off the speeds at 2 seconds (3 m/s) and 6 seconds (9 m/s). Or we could determine the speeds at those times by using the graph.

Another way to find the change would be to substitute the two values for t into our expression for speed. First, we will substitute $t = 2$ into our expression. To indicate that the speed we get is the speed at time = 2 seconds, we denote it as speed(2).

$$\text{speed}(2) = 1.5(2) = 3$$

Next, we will substitute $t = 6$ into our expression. This is the speed at 6 seconds, so we denote it as speed(6)

$$\text{speed}(6) = 1.5(6) = 9$$

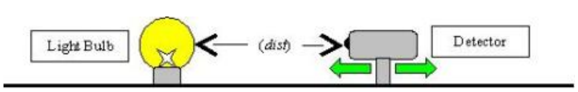
The change between $t = 2$and $t = 6$ is speed(6) − speed(2) = 9 − 3 = 6 m/s.

The speed change is **positive**, so the change is an **increase**.

Solution

Between the two and six seconds the train's speed increases by 6 m/s.

Example 6



The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation.

$$\text{Intensity} = 3/(\text{dist})^2$$

Where (dist) is the distance measured in **meters**, *and intensity is measured in* **lumens**. *Calculate the change in intensity when the detector is moved from two meters to three meters away.*

We first find the values of the intensity at distances of two and three meters.

$$\text{Intensity}(2) = \frac{3}{(2)^2} = \frac{3}{4}$$
$$\text{Intensity}(3) = \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}$$

The **difference** in the two values will give the **change** in the intensity. We move **from** two meters **to** three meters away.

$$\text{Change=Intensity}(3) - \text{Intensity}(2) = \frac{1}{3} - \frac{3}{4}$$

To find the answer, we will need to write these fractions over a common denominator.

The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

$$\frac{1}{3} = \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12}$$
$$\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}$$

Our change is given by this equation.

$$\frac{4}{12} - \frac{9}{12} = \frac{(4-9)}{12} = -\frac{5}{12}$$ A negative indicates that the intensity is reduced.

Solution

When moving the detector from two meters to three meters the intensity falls by $\frac{5}{12}$ lumens.

Lesson Summary

- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- When **subtracting fractions**: $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

Review Questions

1. Subtract the following rational numbers. Be sure that your answer is in the **simplest form**.
   (a) $\frac{5}{12} - \frac{9}{18}$
   (b) $\frac{2}{3} - \frac{1}{4}$
   (c) $\frac{3}{4} - \frac{1}{3}$
   (d) $\frac{15}{11} - \frac{9}{7}$
   (e) $\frac{2}{13} - \frac{1}{11}$
   (f) $\frac{7}{27} - \frac{9}{39}$
   (g) $\frac{6}{11} - \frac{3}{22}$

(h) $\frac{13}{64} - \frac{7}{40}$
(i) $\frac{11}{70} - \frac{11}{30}$

2. Consider the equation $y = 3x + 2$. Determine the change in y between $x = 3$ and $x = 7$.
3. Consider the equation $y = \frac{2}{3}x + \frac{1}{2}$. Determine the change in y between $x = 1$ and $x = 2$.
4. The time taken to commute from San Diego to Los Angeles is given by the equation time $= \frac{120}{speed}$ where *time* is measured in **hours** and *speed* is measured in **miles per hour** (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to train. The bus averages 40 mph to a new high speed train which averages 90 mph.

Review Answers

1.
2. (a) $\frac{-1}{12}$
   (b) $\frac{5}{12}$
   (c) $\frac{5}{12}$
   (d) $\frac{6}{77}$
   (e) $\frac{9}{143}$
   (f) $\frac{10}{351}$
   (g) $\frac{9}{22}$
   (h) $\frac{9}{320}$
   (i) $\frac{-22}{105}$
3. Change $= +12$
4. Change $= -\frac{1}{3}$
5. The journey time would decrease by $1\frac{2}{3}$ hours.

2.4 Multiplication of Rational Numbers

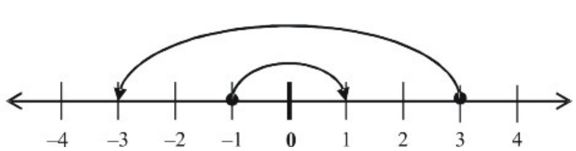
Learning Objectives

- Multiply by negative one.
- Multiply rational numbers.
- Identify and apply properties of multiplication.
- Solve real-world problems using multiplication.

Multiplying Numbers by Negative One

Whenever we multiply a number by negative one we change the sign of the number. In more mathematical words, multiplying by negative one maps a number onto its opposite. The number line below shows the process of multiplying negative one by the numbers three and negative one.

$$3 \cdot -1 = -3$$
$$-1 \cdot -1 = 1$$



- When we multiply a number by negative one the absolute value of the new number is the same as the absolute value of the old number. Both numbers are the same distance from zero.
- The product of a number, x, and negative one is $-x$. This does not mean that $-x$ is necessarily less than zero. If x itself is negative then $-x$ is a positive quantity because a negative times a negative is a positive.
- When we multiply an expression by negative one remember to multiply the **entire expression by** negative one.

Example 1

Multiply the following by negative one.

a) 79.5

b) π

c) $(x+1)$

d) $|x|$

a) **Solution**

$$79.5 \cdot (-1) = -79.5$$

b) **Solution**

$$\pi \cdot (-1) = -\pi$$

c) **Solution**

$$(x+1) \cdot (-1) = -(x+1) = -x-1$$

d) **Solution**

$$|x| \cdot (-1) = -|x|$$

Note that in the last case the negative sign does **not** distribute into the absolute value. Multiplying the **argument** of an absolute value equation (the term between the absolute value symbol) does not change the absolute value. $|x|$ is always positive. $|-x|$ is always positive. $-|x|$ is always negative.

Whenever you are working with expressions, you can check your answers by substituting in numbers for the variables. For example you could check part d of example one by letting $x = -3$.

$|-3| \neq -|3|$ since $|-3| = 3$ and $-|3| = -3$.

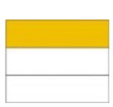
Multiply Rational Numbers

Example 2

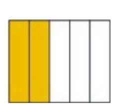
Simplify $\frac{1}{3} \cdot \frac{2}{5}$

One way to solve this is to think of money. For example, we know that *one third of sixty dollars* is written as $\frac{1}{3} \cdot \$60$. We can read the above problem as *one-third of two-fifths*. Here is a visual picture of the fractions ***one-third*** and ***two-fifths.***

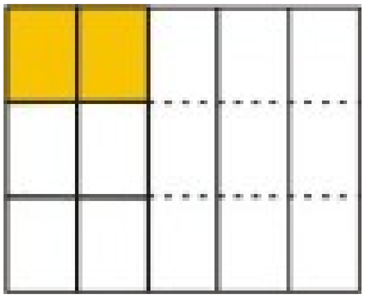
$\frac{1}{3}$



$\frac{2}{5}$



Notice that *one-third of two-fifths* looks like the *one-third* of the shaded region in the next figure.



Here is the intersection of the two shaded regions. The whole has been divided into five pieces width-wise and three pieces height-wise. We get two pieces out of a total of fifteen pieces.

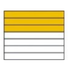
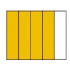
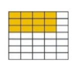
Solution

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$$

Example 3

Simplify $\frac{3}{7} \cdot \frac{4}{5}$

We will again go with a visual representation.

$\frac{3}{7}$  $\frac{4}{5}$  $\frac{3}{7} \cdot \frac{4}{5}$ 

We see that the whole has been divided into a total of $7 \cdot 5$ pieces. We get $3 \cdot 4$ of those pieces.

Solution

$$\frac{3}{70} \cdot \frac{4}{5} = \frac{12}{35}$$

When multiplying rational numbers, the numerators multiply together and the denominators multiply together.

When multiplying fractions $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Even though we have shown this result for the product of two fractions, this rule holds true when multiplying multiple fractions together.

Example 4

Multiply the following rational numbers

a) $\frac{1}{2} \cdot \frac{3}{4}$

b) $\frac{2}{5} \cdot \frac{5}{9}$

c) $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$

d) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$

a) **Solution**

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$$

b) **Solution** With this problem, we can cancel the fives.

$$\frac{2}{5} \cdot \frac{5}{9} = \frac{2 \cdot 5}{5 \cdot 9} = \frac{2}{9}$$

c) **Solution** With this problem, multiply **all the numerators** and **all the denominators**.

$$\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5} = \frac{1 \cdot 2 \cdot 2}{3 \cdot 7 \cdot 5} = \frac{4}{105}$$

d) **Solution** With this problem, we can cancel any factor that appears as both a numerator **and** a denominator since any number divided by itself is one, according to the Multiplicative Identity Property.

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{5} = \frac{1}{5}$$

With multiplication of fractions, we can either simplify before we multiply or after. The next example uses factors to help simplify before we multiply.

Example 5

Evaluate and simplify $\frac{12}{25} \cdot \frac{35}{42}$

We can see that 12 and 42 are both multiples of six, and that 25 and 35 are both factors of five. We write the product again, but put in these factors so that we can cancel them prior to multiplying.

$$\frac{12}{25} \cdot \frac{35}{42} = \frac{6 \cdot 2}{25} \cdot \frac{35}{6 \cdot 7} = \frac{6 \cdot 2 \cdot 5 \cdot 7}{5 \cdot 5 \cdot 6 \cdot 7} = \frac{2}{5}$$

Solution

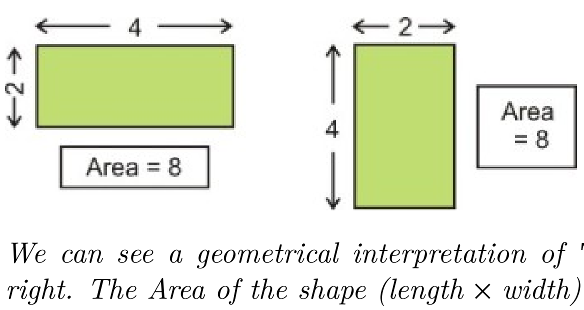
$$\frac{12}{25} \cdot \frac{35}{42} = \frac{2}{5}$$

Identify and Apply Properties of Multiplication

The four mathematical properties which involve multiplication are the **Commutative, Associative, Multiplicative Identity** and **Distributive Properties.**

- **Commutative property** When two numbers are multiplied together, the product is the same regardless of the order in which they are written:

Example$4 \cdot 2 = 2 \cdot 4$



We can see a geometrical interpretation of **The Commutative Property of Multiplication** *to the right. The Area of the shape (length* $\times$ *width) is the same no matter which way we draw it.*

- **Associative Property** When three or more numbers are multiplied, the product is the same regardless of their grouping

Example $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

- **Multiplicative Identity Property:** The product of one and any number is that number.

Example $5 \cdot 1 = 5$.

- **Distributive property** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Example: $4(6+3) = 4 \cdot 6 + 4 \cdot 3$

Example 6

Nadia and Peter are raising money by washing cars. Nadia is charging \$3 *per car, and she washes five cars in the first morning. Peter charges* \$5 *per car (including a wax). In the first morning, he washes and waxes three cars. Who has raised the most money?*

Solution

Nadia raised $5 \cdot \$3$. Peter raised $3 \cdot \$5$. According to **The Commutative Property of Multiplication,** they both raised the **same amount** of money.

Example 7

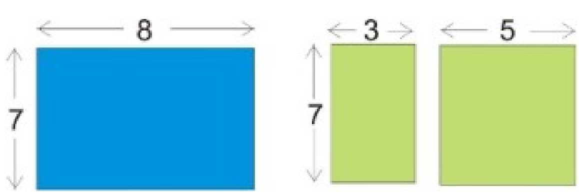
Andrew is counting his money. He puts all his money into \$10 *piles. He has one pile. How much money does Andrew have?*

Solution

The amount of money in each pile is \$10. The number of piles is one. The total amount of money is $\$10 \cdot 1$. According to **The Multiplicative Identity Property,** Andrew has a total of \$10.

Example 8

A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8×7 meter *plot, or two smaller plots of* 3×7 meters *and* 5×7 meter *. Which option gives him the largest area for his potatoes?*



Solution

In the first option, the gardener has a total area of (8×7).

Since $8 = (3+5)$ we have $(3+5) \cdot 7$ squaremeter, which equals $(3 \cdot 7) + (5 \cdot 7)$.

In the second option, the total area is $(3 \cdot 7) + (5 \cdot 7)$ squaremeters.

According to **The Distributive Property** both options give the gardener the same area to plant potatoes

Solve Real-World Problems Using Multiplication

Example 9

In the chemistry lab there is a bottle with two liters of a 15% solution of hydrogen peroxide (H_2O_2). John removes one-fifth of what is in the bottle, and puts it in a beaker. He measures the amount of H_2O_2 and adds twice that amount of water to the beaker. Calculate the following measurements.'

a) The amount of H_2O_2 left in the bottle.

b) The amount of diluted H_2O_2 in the beaker.

c) The concentration of the H_2O_2 in the beaker.

a) To determine the amount of H_2O_2 left in the bottle, we first determine the amount that was removed. That amount was $\frac{1}{5}$ of the amount in the bottle (2 liters).

$$\text{Amount removed} = \frac{1}{5} \cdot 2 \text{ liters} = \frac{2}{5} \text{ liter (or 0.4 liters)}$$
$$\text{Amount remaining} = 2 - \frac{2}{5} = \frac{10}{5} - \frac{2}{5} = \frac{8}{5} \text{ liter (or1.6 liters)}$$

Solution

There is 1.6 liters left in the bottle.

b) We determined that the amount of the 15% H_2O_2 solution removed was $\frac{2}{5}$ liter. The amount of water added was twice this amount.

$$\text{Amount of water} = 2 \cdot \frac{2}{5} = \frac{4}{5} \text{ liter.}$$
$$\text{Total amount} = \frac{4}{5} + \frac{2}{5} = \frac{6}{5} \text{ liter (or1.2 liters)}$$

Solution

There are 1.2 liters of diluted H_2O_2 in the beaker.

c) The new concentration of H_2O_2 can be calculated.

Initially, with $\frac{2}{5}$ of undiluted H_2O_2 there is 15% of $\frac{2}{5}$ liters of pure H_2O_2:

Amount of pure $H_2O_2 = 0.15 \cdot \frac{2}{5} = 0.15 \cdot 0.4 = 0.06$ liter of **pure** H_2O_2.

After dilution, this H_2O_2 is dispersed into 1.2 liters of solution. The concentration $= \frac{0.06}{1.2} = 0.05$.

To convert to a percent we multiply this number by 100.

Solution

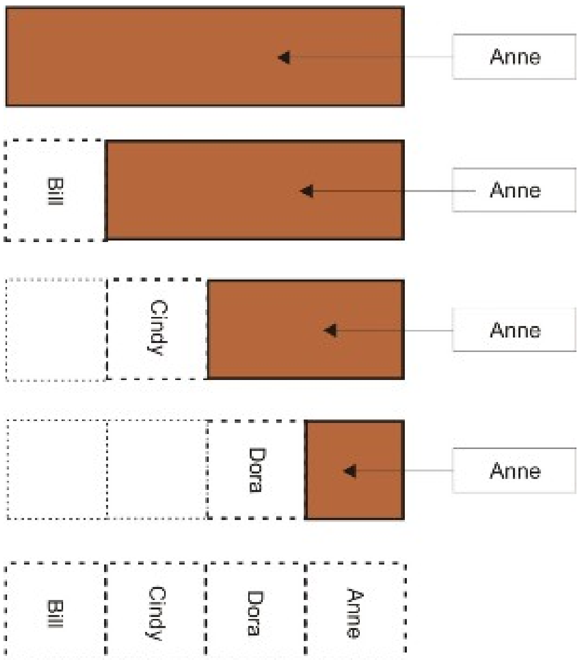
The final of diluted H_2O_2in the bottle is 5%.

Example 10

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off $\frac{1}{4}$ of the bar and eats it.

Another friend, Cindy, takes $\frac{1}{3}$ of what was left. She splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?

First, let's look at this problem visually.



Anne starts with a full candy bar.

Bill breaks of $\frac{1}{4}$ of the bar.

Cindy takes $\frac{1}{3}$ of what was left.

Dora gets half of the remaining candy bar.

We can see that the candy bar ends up being split four ways. The sum of each piece is equal to one.

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

We can also examine this problem using rational numbers. We keep a running total of what fraction of the bar remains. Remember, when we read a fraction followed by *of* in the problem, it means we multiply by that fraction.

We start with one full bar of chocolate	The total we begin with is 1.
"Bill breaks off of the bar"	We multiply the amount of bar(1)by $\frac{1}{4}$
Bill removes $\frac{1}{4} \cdot 1 - \frac{1}{4}$ of the whole bar.	The bar remaining is $1 - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$
"Cindy takes $\frac{1}{3}$ of what is left"	We multiply the amount of bar $\left(\frac{3}{4}\right)$ by $\frac{1}{3}$
Cindy removes $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of a whole bar.	The bar remaining is $\frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$
Anne and Dora get two "equal pieces"	Dora gets $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of a whole bar.
	Anne gets the remaining $\frac{1}{2} - \frac{1}{4} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

Extension: If each piece that is left is 3oz, how much did the original candy bar weigh?

Lesson Summary

- When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.

- To multiply fractions $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

- The **multiplicative properties** are:

- **Commutative property** the product of two numbers is the same whichever order the items to be multiplied are written.

Ex:$2 \cdot 3 = 3 \cdot 2$

- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped.

Ex: $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

- **Multiplicative Identity Property** The product of any number and one is the original number.

Ex: $2 \cdot 1 = 2$

- **Distributive property** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Ex: $4(2+3) = 4(2) + 4(3)$

Review Questions

1. Multiply the following by negative one.
   (a) 25
   (b) -105
   (c) x^2
   (d) $(3+x)$
   (e) $(3-x)$
2. Multiply the following rational numbers, write your answer in the **simplest form**.
   (a) $\frac{5}{12} \times \frac{9}{10}$
   (b) $\frac{2}{3} \times \frac{1}{4}$
   (c) $\frac{3}{4} \times \frac{1}{3}$
   (d) $\frac{15}{11} \times \frac{9}{7}$

(e) $\frac{1}{13} \times \frac{1}{11}$
(f) $\frac{7}{27} \times \frac{9}{14}$
(g) $\left(\frac{3}{5}\right)^2$
(h) $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
(i) $\frac{12}{15} \times \frac{35}{13} \times \frac{10}{2} \times \frac{26}{36}$

3. Three monkeys spend a day gathering coconuts together. When they have finished, they are very tired and fall asleep. The following morning, the first monkey wakes up. Not wishing to disturb his friends, he decides to divide the coconuts into three equal piles. There is one left over, so he throws this odd one away, helps himself to his share, and goes home.
A few minutes later, the second monkey awakes. Not realizing that the first has already gone, he too divides the coconuts into three equal heaps. He finds one left over, throws the odd one away, helps himself to his fair share, and goes home.
In the morning, the third monkey wakes to find that he is alone. He spots the two discarded coconuts, and puts them with the pile, giving him a total of twelve coconuts. How many coconuts did the first and second monkey take? [**Extension:** solve by working backward]

Review Answers

1.
2. (a) -25
(b) 105
(c) $-x^2$
(d) $-(x+3)$ or $-x-3$
(e) $(x-3)$
3.
4. (a) $\frac{3}{8}$
(b) $\frac{1}{6}$
(c) $\frac{1}{4}$
(d) $\frac{135}{77}$
(e) $\frac{1}{143}$
(f) $\frac{1}{6}$
(g) $\frac{27}{125}$
(h) $\frac{1}{15}$
(i) $\frac{70}{9}$
5. The first monkey takes eight coconuts. The second monkey takes five coconuts.

2.5 The Distributive Property

Learning Objectives

- Apply the distributive property.
- Identify parts of an expression.
- Solve real-world problems using the distributive property.

Introduction

At the end of the school year, an elementary school teacher makes a little gift bag for each of his students. Each bag contains one class photograph, two party favors and five pieces of candy. The teacher will distribute the bags among his 28 students. How many of each item does the teacher need?

Apply the Distributive Property

When we have a problem like the one posed in the introduction, **The Distributive Property** can help us solve it. To begin, we can write an expression for the contents of each bag.

Contents $=$ (photo $+ 2$ favor $+ 5$ candy)

Contents $= (p + 2f + 5c)$ We may even use single letter variables to write an expression.

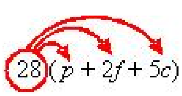
We know that the teacher has 28 students, therefore we can write the following expression for the number of items that the teacher will need.

$$\text{Items} = 28 \cdot (p + 2f + 5c) \qquad \text{28 times the individual contents of each bag.}$$

We generally omit any multiplication signs that are not strictly necessary.

$$\text{Items} = 28(p + 2f + 5c)$$

The Distributive Property of Multiplication means that when faced with a term multiplying other terms inside parentheses, the outside term multiplies with each of the terms inside the parentheses.



$$= 28(p + 2f + 5c) = 28(p) + 28(2f) + 28(5c) = 28p + 56f + 140c$$

So the teacher needs 28 class photos, 56 party favors and 140 pieces of candy.

The Distributive Property works when we have numbers inside the parentheses. You can see this by looking at a simple problem and considering the **Order of Operations.**

Example 1

Determine the value of $11(2 + 6)$ *using both Order of Operations and the Distributive Property.*

First, we consider the problem with the Order of Operations – PEMDAS dictates that we evaluate the amount inside the parentheses first.

Solution

$$11(2 + 6) = 11(8) = 88$$

Next we will use the Distributive Property. We multiply the 11 by each term inside the parentheses.

Solution

$$11(2 + 6) = 11(2) + 11(6) = 22 + 66 = 88$$

Example 2

Determine the value of $11(2 - 6)$ *using both the Order of Operations and the Distributive Property.*

First, we consider the Order of Operations and evaluate the amount inside the parentheses first.

Solution

$$11(2-6) = 11(-4) = -44$$

Next, the Distributive Property.

Solution

$$11(2-6) = 11(2) + 11(-6) = 22 - 66 = -44$$

Note When applying the Distributive Property you **MUST** take note of any **negative signs**!

Example 3

Use the Distributive Property to determine the following.

a) $11(2x+6)$

b) $7(3x-5)$

c) $\frac{2}{7}\left(3y^2-11\right)$

d) $\frac{2x}{7}\left(3y^2-\frac{11}{xy}\right)$

a) Simply multiply each term by 11.

Solution

$$11(2x+6) = 22x + 66$$

b) Note the negative sign on the second term.

Solution

$$7(3x-5) = 21x - 35$$

c) $\frac{2}{7}\left(3y^2-11\right) = \frac{2}{7}\left(3y^2\right) + \frac{2}{7}(-11) = \frac{6y^2}{7} - \frac{22}{7}$

Solution

$$\frac{2}{7}\left(3y^2-11\right) = \frac{6y^2-22}{7}$$

d) $\frac{2x}{7}\left(3y^2-\frac{11}{xy}\right) = \frac{2x}{7}(3y^2) + \frac{2x}{7}\left(\frac{-11}{xy}\right) = \frac{6x^2y}{7} \frac{-22\not{x}}{7\not{x}y}$

Solution

$$\frac{2x}{7}\left(3y^2-\frac{11}{xy}\right) = \frac{6xy^3-22}{7y}$$

Identify Expressions That Involve the Distributive Property

The Distributive Property often appears in expressions, and many times it does not involve parentheses as grouping symbols. In Lesson 1.2, we saw how the fraction bar acts as a grouping symbol. The following example involves using the Distributive Property with fractions.

Example 4

Simplify the following expressions.

a) $\frac{2x+8}{4}$

b) $\frac{9y-2}{3}$

c) $\frac{z+6}{2}$

Even though these expressions are not written in a form we usually associate with the Distributive Property, the fact that the numerator of fractions should be treated as if it were in parentheses makes this a problem that the Distributive Property can help us solve.

a) $\frac{2x+8}{4}$ can be re-written as $\frac{1}{4}(2x+8)$.

We can then proceed to distribute the $\frac{1}{4}$.

$$\frac{1}{4}(2x+8) = \frac{2x}{4} + \frac{8}{4} = \frac{2x}{2 \cdot 2} + \frac{4 \cdot 2}{4}$$

Solution

$$\frac{2x+8}{4} = \frac{x}{2} + 2$$

b) $\frac{9y-2}{3}$ can be re-written as $\frac{1}{3}(9y-2)$.

We can then proceed to distribute the $\frac{1}{3}$.

$$\frac{1}{3}(9y-2) = \frac{9y}{3} - \frac{2}{3} = \frac{3 \cdot 3x}{3} - \frac{2}{3}$$

Solution

$$\frac{9y-2}{3} = 3y - \frac{2}{3}$$

c) $\frac{z+6}{2}$ can be re-written as $\frac{1}{6}(z+6)$.

We can then proceed to distribute the $\frac{1}{2}$.

$$\frac{1}{2}(z+6)) = \frac{z}{2} + \frac{6}{2}$$

Solution

$$\frac{z+6}{3} = \frac{z}{2} + 3$$

Solve Real-World Problems Using the Distributive Property

The Distributive Property is one of the most common mathematical properties to be seen in everyday life. It crops up in business and in geometry. Anytime we have two or more groups of objects, the Distributive Property can help us solve for an unknown.

Example 5

An octagonal gazebo is to be built as shown right. Building code requires five foot long steel supports to be added along the base and four foot long steel supports to be added to the roof-line of the gazebo. What length of steel will be required to complete the project?

Each side will require two lengths, one of five and four feet respectively. There are eight sides, so here is our equation.

$$\text{Steel required} = 8(4+5) \text{ feet.}$$

We can use the distributive property to find the total amount of steel:

$$\text{Steel required} = 8 \times 4 + 8 \times 5 = 32 + 40 \text{ feet.}$$

Solution

A total of 72 feet of steel is required for the project.

Example 6

Each student on a field trip into a forest is to be given an emergency survival kit. The kit is to contain a flashlight, a first aid kit, and emergency food rations. Flashlights cost \$12 *each, first aid kits are* \$7 *each and emergency food rations cost* \$2 *per day. There is* \$500 *available for the kits and* 17 *students to provide for. How many days worth of rations can be provided with each kit?*

The unknown quantity in this problem is the number of days' rations. This will be x in our expression. Each kit will contain the following items.

$$1 \cdot \$12 \text{ flashlight.}$$
$$1 \cdot \$7 \text{ first aid kit.}$$
$$x \cdot \$2 \text{ daily rations.}$$

The number of kits = 17, so the total cost is equal to the following equation.

$$\text{Total cost} = 17(12+7+2x)$$

We can use the Distributive Property on this expression.

$$17(12+7+2x) = 204 + 119 + 34x$$

We know that there is \$500 available to buy the kits. We can substitute the cost with the money available.

$$\begin{aligned} 204 + 119 + 34x &= 500 && \text{The sum of the numbers on the left equal to the money available} \\ 323 + 34x &= 500 && \text{Subtract 323 from both sides} \\ -323 - 323 & && \\ 34x &= 177 && \text{Divide both sides by 34} \\ x &= 5.20588\ldots && \end{aligned}$$

Since this represents the number of daily rations that can be bought, we must **round to the next lowest whole number.** We wouldn't have enough money to buy a sixth day of supplies.

Solution

Five days worth of emergency rations can be purchased for each survival kit.

Lesson Summary

- **Distributive Property** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Ex: $4 \times (6 + 3) = 4 \times 6 + 4 \times 3$

- When applying the Distributive Property you **MUST** take note of any **negative signs**!

Review Questions

1. Use the Distributive Property to simplify the following expressions.
   (a) $(x + 4) - 2(x + 5)$
   (b) $\frac{1}{2}(4z + 6)$
   (c) $(4 + 5) - (5 + 2)$
   (d) $(x + 2 + 7)$
   (e) $y(x + 7)$
   (f) $13x(3y + z)$
2. Use the Distributive Property to remove the parentheses from the following expressions.
   (a) $\frac{1}{2}(x - y) - 4$
   (b) $0.6(0.2x + 0.7)$
   (c) $6 + (x - 5) + 7$
   (d) $6 - (x - 5) + 7$
   (e) $4(m + 7) - 6(4 - m)$
   (f) $-5(y - 11) + 2y$
3. Use the Distributive Property to simplify the following fractions.
   (a) $\frac{8x+12}{4}$
   (b) $\frac{9x+12}{3}$
   (c) $\frac{11x+12}{2}$
   (d) $\frac{3y+2}{6}$
   (e) $-\frac{6z-2}{3}$
   (f) $\frac{7-6p}{3}$
4. A bookshelf has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
5. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates them with macadamia nuts. If Amar has 5 lbs of cookie dough (1 lb = 16 oz) and 60 macadamia nuts, calculate the following.
   (a) How many (**full**) cookies he can make?
   (b) How many macadamia nuts he can put on each cookie, if each is to be identical?

Review Answers

1.
2. (a) $-x-6$
   (b) $2z+3$
   (c) 2
   (d) $x+9$
   (e) $xy+7y$
   (f) $39xy+13xz$
3.
4. (a) $\frac{x}{2}-\frac{y}{2}-4$
   (b) $0.12x+0.42$
   (c) $x+8$
   (d) $18-x$
   (e) $10\text{ m}+4$
   (f) $55-3y$
5.
6. (a) $2x+3$
   (b) $3x+4$
   (c) $\frac{11x}{2}+6$
   (d) $\frac{y}{2}+\frac{1}{3}$
   (e) $\frac{2}{3}-2z$
   (f) $\frac{7}{3}-2p$
7. The bookshelf contains 35 poetry books and 55 novels.
8.
9. (a) Amar can make 13 cookies (2 oz leftover).
   (b) Each cookie has 4 macadamia nuts (8 left over).

2.6 Division of Rational Numbers

Learning Objectives

- Find multiplicative inverses.
- Divide rational numbers.
- Solve real-world problems using division.

Introduction – Identity elements

An **identity element** is a number which, when combined with a mathematical operation on a number, leaves that number unchanged. For addition and subtraction, the **identity element** is **zero**.

$$2+0=2$$
$$-5+0=-5$$
$$99-0=99$$

The inverse operation of addition is subtraction.

$x+5-5=x$ When we subtract what we have added, we get back to where we started!

When you add a number to its **opposite,** you get the identity element for addition.

$$5 + (-5) = 0$$

You can see that the **addition of an opposite is an equivalent operation to subtraction.**

For multiplication and division, the **identity element** is **one**.

$$2 \times 1 = 2$$
$$-5 \times 1 = -5$$
$$99 \div 1 = 99$$

In this lesson, we will learn about **multiplying by a multiplicative inverse** as an equivalent operation to division. Just as we can use **opposites** to turn a **subtraction** problem into an **addition** problem, we can use **reciprocals** to turn a **division** problem into a **multiplication** problem.

Find Multiplicative Inverses

The **multiplicative inverse** of a number, x, is the number when multiplied by x yields **one**. In other words, any number times the multiplicative inverse of that number equals one. The multiplicative inverse is commonly the reciprocal, and the multiplicative inverse of x is denoted by $\frac{1}{x}$.

Look at the following multiplication problem:

Simplify $\frac{2}{3} \times \frac{3}{2}$

We know that we can cancel terms that appear on both the numerator and the denominator. Remember we leave a one when we cancel all terms on either the numerator or denominator!

$\frac{2}{3} \times \frac{3}{2} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}} = 1$

It is clear that $\frac{2}{3}$ is the multiplicative inverse of $\frac{3}{2}$. Here is the rule.

To find the multiplicative inverse of a rational number, we simply ***invert the fraction***.

The multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$, as long as $a \neq 0$

Example 1

Find the multiplicative inverse of each of the following.

a) $\frac{3}{7}$

b) $\frac{4}{7}$

c) $3\frac{1}{2}$

d) $-\frac{x}{y}$

e) $\frac{1}{11}$

a) **Solution**

The multiplicative inverse of $\frac{3}{7}$ is $\frac{7}{3}$.

b) **Solution**

The multiplicative inverse of $\frac{4}{9}$ is $\frac{9}{r}$.

c) To find the multiplicative inverse of $3\frac{1}{2}$ we first need to convert $3\frac{1}{2}$ to an **improper fraction**:

$$3\frac{1}{2} = \frac{3}{1} + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$$

Solution

The multiplicative inverse of $3\frac{1}{2}$ is $\frac{2}{7}$.

d) Do not let the negative sign confuse you. The multiplicative inverse of a negative number is also negative!

Solution

The multiplicative inverse of $-\frac{x}{y}$ is $-\frac{y}{x}$.

e) The multiplicative inverse of $\frac{1}{11}$ is $\frac{11}{1}$. Remember that when we have a denominator of one, we omit the denominator.

Solution

The multiplicative inverse of $\frac{1}{11}$ is 11.

Look again at the last example. When we took the multiplicative inverse of $\frac{1}{11}$ we got a whole number, 11. This, of course, is expected. We said earlier that the multiplicative inverse of x is $\frac{1}{x}$.

The multiplicative inverse of a whole number is one divided that number.

Remember the idea of the **invisible denominator**. The idea that every integer is actually a rational number whose denominator is one. $5 = \frac{5}{1}$.

Divide Rational Numbers

Division can be thought of as the inverse process of multiplication. If we multiply a number by seven, we can divide the answer by seven to return to the original number. Another way to return to our original number is to multiply the answer by the **multiplicative inverse of seven.**

In this way, we can simplify the process of dividing rational numbers. We can turn a division problem into a multiplication process by replacing the divisor (the number we are dividing by) with its multiplicative inverse, or **reciprocal**.

To divide rational numbers, invert the divisor and multiply $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Also, $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$

Example 2

Divide the following rational numbers, giving your answer in the **simplest form.**

a) $\frac{1}{2} \div \frac{1}{4}$

b) $\frac{7}{3} \div \frac{2}{3}$

c) $\frac{x}{2} \div \frac{1}{4y}$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right)$

a) Replace $\frac{1}{4}$ with $\frac{4}{1}$ and multiply. $\frac{1}{2} \times \frac{4}{1} = \frac{1}{\not{2}} \times \frac{\not{2}.2}{2} = \frac{1}{2}$.

Solution

$$\frac{1}{2} \div \frac{1}{4} = 2$$

b) Replace $\frac{2}{3}$ with $\frac{3}{2}$ and multiply. $\frac{7}{\not{3}} \times \frac{\not{3}}{2} = \frac{7}{2}$.

Solution

$$\frac{7}{3} \div \frac{2}{3} = \frac{7}{2}$$

c) eplace $\frac{1}{4y}$ with $\frac{4y}{1}$ and multiply. $\frac{x}{2} \times \frac{4y}{1} = \frac{x}{\cancel{2}} \times \frac{\cancel{2}.2.y}{1} = \frac{x.2y}{1}$

Solution

$$\frac{x}{2} \div \frac{1}{4y} = 2xy$$

d) Replace $\left(-\frac{x}{y}\right)$ with $\left(-\frac{y}{x}\right)$ and multiply. $\frac{11}{2x} \times \left(-\frac{y}{x}\right) = -\frac{11.y}{2x.x}$.

Solution

$$\frac{11}{2x}\left(-\frac{x}{y}\right) = -\frac{11y}{2x^2}$$

Solve Real-World Problems Using Division

Speed, Distance and Time

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

Example 3

Andrew is driving down the freeway. He passes mile marker 27 *at exactly mid-day. At 12:35 he passes mile marker* 69. *At what speed, in miles per hour, is Andrew traveling?*

To determine speed, we need the distance traveled and the time taken. If we want our speed to come out in miles per hour, we will need distance in **miles** and time in **hours**.

$$\text{Distance} = 69 - 27 = 42 \text{ miles}$$
$$\text{Time taken} = 35 \text{ minutes} = \frac{35}{60} = \frac{\cancel{5}.7}{\cancel{5}.12} = \frac{7}{12} \text{ hour}$$

We now *plug in* the values for distance and time into our equation for speed.

$$\text{Speed} = \frac{42}{\left(\frac{7}{12}\right)} = \frac{42}{1} \div \frac{7}{12} \qquad \text{Replace} \frac{7}{12} \text{ with} \frac{12}{7} \text{ and multiply.}$$
$$\text{Speed} = \frac{42}{1} \times \frac{12}{7} = \frac{\cancel{7}.6}{1}\frac{12}{\cancel{7}} = \frac{6.12}{1}$$

Solution

Andrew is driving at 72 miles per hour .

Example 4

Anne runs a mile and a half in a quarter hour. What is her speed in miles per hour?

We already have the distance and time in the correct units (miles and hours). We simply write each as a rational number and plug them into the equation.

$$\text{Speed} = \frac{\left(\frac{3}{2}\right)}{\left(\frac{1}{4}\right)} = \frac{3}{2} \div \frac{1}{4} \qquad \text{Replace} \frac{1}{4} \text{ with } \frac{4}{1} \text{ and multiply.}$$
$$\text{Speed} = \frac{3}{2} \times \frac{4}{1} = \frac{12}{2} = 6$$

Solution

Anne runs at 6 miles per hour.

Example 5 – Newton's Second Law

Newton's second law ($F = ma$) *relates the force applied to a body* (F), *the mass of the body* (m) *and the acceleration* (a). *Calculate the resulting acceleration if a Force of* $7\frac{1}{3}$ *Newtons is applied to a mass of* $\frac{1}{5}$ kg.

First, we rearrange our equation to isolate the acceleraion, a

$$a = \frac{F}{m}$$ Substitute in the known values.

$$a = \frac{\left(7\frac{1}{3}\right)}{\left(\frac{1}{5}\right)} = \left(\frac{7.3}{3} + \frac{1}{3}\right) \div \left(\frac{1}{5}\right)$$ Determine improper fraction, then invert $\frac{1}{5}$ and multiply.

$$a = \frac{22}{3} \times \frac{5}{1} = \frac{110}{3}$$

Solution

The resultant acceleration is $36\frac{2}{3}$ m/s^2.

Lesson Summary

- The **multiplicative inverse** of a number is the number which produces one when multiplied by the original number. The multiplicative inverse of x is the reciprocal $\frac{1}{x}$.
- To find the multiplicative inverse of a rational number, we simply **invert the fraction**: $\frac{a}{b}$ inverts to $\frac{b}{a}$.
- To divide rational numbers, invert the divisor and multiply $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Review Questions

1. Find the multiplicative inverse of each of the following.
   (a) 100
   (b) $\frac{2}{8}$
   (c) $-\frac{19}{21}$
   (d) 7
   (e) $-\frac{z^3}{2xy^2}$
2. Divide the following rational numbers, be sure that your answer in the simplest form.
   (a) $\frac{5}{2} \div \frac{1}{4}$
   (b) $\frac{1}{2} \div \frac{7}{9}$
   (c) $\frac{5}{11} \div \frac{6}{7}$
   (d) $\frac{1}{2} \div \frac{1}{2}$
   (e) $-\frac{x}{2} \div \frac{5}{7}$
   (f) $\frac{1}{2} \div \frac{x}{4y}$
   (g) $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$
   (h) $\frac{7}{2} \div \frac{7}{4}$
   (i) $11 \div \left(-\frac{x}{4}\right)$
3. The label on a can of paint states that it will cover 50 square feet per pint. If I buy a $\frac{1}{8}$ pint sample, it will cover a square two feet long by three feet high. Is the coverage I get more, less or the same as that stated on the label?

4. The world's largest trench digger, "Bagger 288", moves at $\frac{3}{8}$ mph. How long will it take to dig a trench $\frac{2}{3}$ mile long?
5. A $\frac{2}{7}$ Newton force applied to a body of unknown mass produces an acceleration of $\frac{3}{10}$ m/s^2. Calculate the mass of the body. Note: Newton = kg m/s^2.

Review Answers

1.
2. (a) $\frac{1}{101}$
   (b) $\frac{2}{8}$
   (c) $-\frac{21}{19}$
   (d) $\frac{1}{7}$
   (e) $-\frac{2xy^2}{z^3}$
3.
4. (a) 10
   (b) $\frac{9}{14}$
   (c) $\frac{35}{66}$
   (d) 1
   (e) $-\frac{7x}{10}$
   (f) $\frac{2y}{x}$
   (g) $\frac{5}{9}$
   (h) 2
   (i) $-\frac{44}{x}$
5. At 48 square feet per pint *I* get less coverage.
6. Time = $\frac{16}{9}$ hour (1 hr46 min40 sec)
7. mass = $\frac{20}{21}$ kg

2.7 Square Roots and Real Numbers

Learning Objectives

- Find square roots.
- Approximate square roots.
- Identify irrational numbers.
- Classify real numbers.
- Graph and order real numbers.

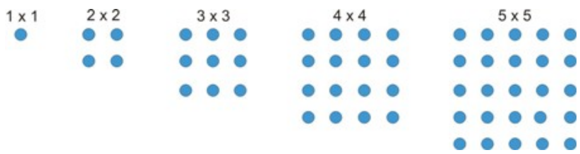
Find Square Roots

The square root of a number is a number which, when multiplied by itself gives the original number. In algebraic terms, the square root of x is a number, b, such that $b^2 = x$.

Note: There are two possibilities for a numerical value for b. The **positive** number that satisfies the equation $b^2 = x$ is called the **principal square root**. Since $(-b) \cdot (-b) = +b^2 = x, -b$ is also a valid solution.

The square root of a number, x, is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. For example, $2^2 = 4$, so the square root of 4, $\sqrt{4} = \pm 2$.

Some numbers, like 4, have integer square roots. Numbers with integer square roots are called **perfect squares**. The first five perfect squares $(1, 4, 9, 16, 25)$ are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. Further, to find the square root of that number, simply take one of each pair of factors and multiply them together.

Example 1

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

d) 576

a) $121 = 11 \times 11$

Solution

$$\sqrt{121} = 11$$

b) $225 = (5 \times 5) \times (3 \times 3)$

Solution

$$\sqrt{225} = 5 \times 3 = 15$$

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$

Solution

$$\sqrt{324} = 2 \times 3 \times 3 = 18$$

d) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$

Solution

$$\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$$

When we have an odd number of prime factors, we leave any unpaired factors under a radical sign. Any answer that contains both whole numbers and irreducible radicals should be written $A\sqrt{b}$.

Example 2

Find the principal square root of the following numbers.

a) 8

b) 48

c) 75

d) 216

a) $8 = (2 \times 2) \times 2$

Solution

$$\sqrt{8} = 2 \times \sqrt{2} = 2\sqrt{2}$$

b) $48 = (2 \times 2) \times (2 \times 2) \times 3$

Solution

$$\sqrt{48} = 2 \times 2 \times \sqrt{3} = 4\sqrt{3}$$

c) $75 = (5 \times 5) \times 3$

Solution

$$\sqrt{75} = 5 \times \sqrt{3} = 5\sqrt{3}$$

d) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$

Solution

$$\sqrt{216} = 2 \times \sqrt{2} \times 3 \times \sqrt{3} = 6\sqrt{2}\sqrt{3} = 6\sqrt{6}$$

Note that in the last example we collected the whole numbers and multiplied them first, **then** we collect unpaired primes under a single radical symbol. Here are the four rules that govern how we treat square roots.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab} \qquad \sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

$$A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab} \qquad A\sqrt{a} \div B\sqrt{b} = \frac{A}{B}\sqrt{\frac{a}{b}}$$

Example 3

Simplify the following square root problems

a) $\sqrt{8} \times \sqrt{2}$

b) $3\sqrt{4} \times 4\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

d) $12\sqrt{10} \div 6\sqrt{5}$

a) $\sqrt{8} \times \sqrt{2} = 16$

Solution

$$\sqrt{8} \times \sqrt{2} = 4$$

b) $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3}$

Solution

$$3\sqrt{4} \times 4\sqrt{3} = 24\sqrt{3}$$

c) $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4}$

Solution

$$\sqrt{12} \div \sqrt{3} = 2$$

d) $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}}$

Solution

$$12\sqrt{10} \div 6\sqrt{5} = 2\sqrt{2}$$

Approximate Square Roots

When we have perfect squares, we can write an exact numerical solution for the principal square root. When we have one or more unpaired primes in the factor tree of a number, however, we do not get integer values for the square root and we have seen that we leave a radical in the answer. Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{}$ or $\sqrt{x}$ button on a calculator. When the number we are finding the square root of is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the decimals will appear random and we will have an irrational number as our answer. We call this an **approximate answer.** Even though we may have an answer to eight or nine decimal places, it still represents an **approximation** of the real answer which has an **infinite number of non-repeating decimals**.

Example 4

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

a) The calculator returns 9.949874371.

Solution

$$\sqrt{8} \approx 9.950$$

b) The calculator returns 2.236067977.

Solution

$$\sqrt{5} \approx 2.236$$

c) The calculator returns 0.7071067812 .

Solution

$$\sqrt{0.5} \approx 0.707$$

d) The calculator returns 1.322875656.

Solution

$$\sqrt{1.75} \approx 1.323$$

Identify Irrational Numbers

Any square root that cannot be simplified to a form without a square root is **irrational,** but **not all** square roots are irrational. For example, $\sqrt{49}$ reduces to 7 and so $\sqrt{49}$ is **rational**, but $\sqrt{50}$ cannot be reduced further than $\sqrt{25} \times \sqrt{2} = 5\sqrt{2}$. The fact that we cannot remove the factor of square root of 2 makes $\sqrt{50}$ **irrational**.

Example 5

Identify which of the following are rational numbers and which are irrational numbers.

a) 23.7

b) 2.8956

c) π

d) $\sqrt{6}$

e) $3.\overline{27}$

a) $23.7 = 23\frac{7}{10}$. This is clearly a **rational number**.

b) $2.8956 = 2\frac{8956}{10000}$. Again, this is a **rational number**.

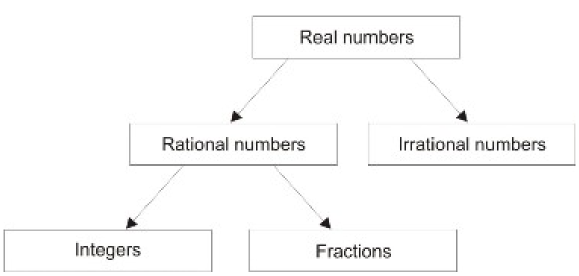
c) $\pi = 3.141592654\ldots$ The decimals appear random, and from the definition of π we know they do not repeat. This is an **irrational number**.

d) $\sqrt{6} = 2.44949489743\ldots$ Again the decimals appear to be random. We also know that $\sqrt{6} = \sqrt{2} \times \sqrt{3}$. Square roots of **primes** are irrational. $\sqrt{6}$ is an irrational number.

e) $3.\overline{27} = 3.2727272727\ldots$ Although these decimals are recurring they are certainly *not* unpredictable. This is a **rational number** (in actual fact, $3.\overline{27} = \frac{36}{11}$)

Classify Real Numbers

We can now see how real numbers fall into one of several categories.



If a real number can be expressed as a rational number, it falls into one of two categories. If the denominator of its **simplest form** is one, then it is an **integer**. If not, it is a fraction (this term is used here to also include decimals, as $3.27 = 3\frac{27}{100}$.

If the number cannot be expressed as the ratio of two integers (i.e. as a fraction), it is **irrational**.

Example 6

Classify the following real numbers.

a) 0

b) -1

c) $\frac{\pi}{3}$

d) $\frac{\sqrt{2}}{3}$

e) $\frac{\sqrt{36}}{9}$

a) **Solution**

Zero is an **integer.**

b) **Solution**

-1 is an **integer**.

c) Although $\frac{\pi}{3}$ is written as a fraction, the numerator (π) is irrational.

Solution

$\frac{\pi}{3}$ is an **irrational number**.

d) $\frac{\sqrt{2}}{3}$ cannot be simplified to remove the square root.

Solution

$\frac{\sqrt{2}}{3}$ is an **irrational number**.

e) $\frac{\sqrt{36}}{9}$ can be simplified to $\frac{\sqrt{36}}{3} = \frac{6}{9} = \frac{2}{3}$ **Solution**

$\frac{\sqrt{36}}{9}$ is a **rational number.**

Graph and Order Real Numbers

We have already talked about plotting integers on the number line. It gives a visual representation of which number is bigger, smaller, etc. It would therefore be helpful to plot non-integer rational numbers (fractions) on the number line also. There are two ways to graph rational numbers on the number line. You can convert them to a mixed number (graphing is one of the few instances in algebra when mixed numbers are preferred to improper fractions), or you can convert them to decimal form.

Example 7

Plot the following rational numbers on the number line.

a) $\frac{2}{3}$

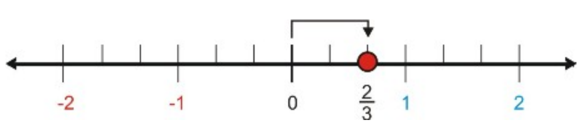
b) $-\frac{3}{7}$

c) $\frac{17}{3}$

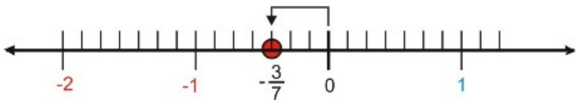
d) $\frac{57}{16}$

If we divide the intervals on the number line into the number on the denominator, we can look at the fraction's numerator to determine how many of these **sub-intervals** we need to include.

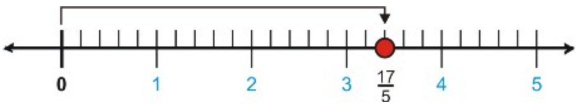
a) $\frac{2}{3}$ falls between 0 and 1. We divide the interval into three units, and include two sub-intervals.



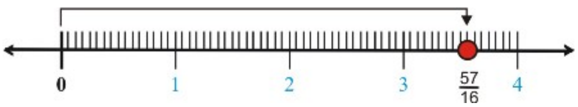
b) $-\frac{3}{7}$ falls between 0 and -1. We divide the interval into seven units, and move **left** from zero by three sub-intervals.



c) $\frac{17}{5}$ as a mixed number is $2\frac{2}{5}$ and falls between 3 and 4. We divide the interval into five units, and move over two sub-intervals.



d) $\frac{57}{16}$ as a mixed number is $3\frac{9}{16}$ and falls between 3 and 4. We need to make sixteen sub-divisions.



Example 8

Plot the following numbers, in the correct order, on a number line.

a) π

b) $\frac{22}{7}$

c) 3.14

d) $\sqrt{10}$

We will use a calculator to find decimal expansions for each of these, and use a number line divided into 1000 sub-divisions. When we have two extremely close numbers, we will ensure that we place them in the correct order by looking at the expansion to the 3^{rd} decimal place and writing as a fraction of 1000.

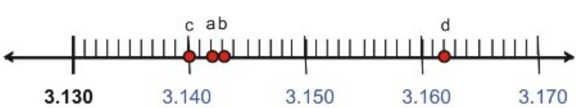
a) $\pi = 3.14159\ldots \approx 3\frac{142}{1000}$

b) $\frac{22}{7} = 3.14288\ldots \approx 3\frac{143}{1000}$

c) $3.14 \approx 3\frac{140}{1000}$

d) $\sqrt{10} = 3.16227\ldots \approx 3\frac{162}{1000}$

Solution



Lesson Summary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, the square root of x is a number, b, such that $b^2 = x$, or $b = \sqrt{x}$
- There are two possibilities for a numerical value for b. A positive value called the **principal square root** and a negative value (the opposite of the positive value).
- A **perfect square** is a number with an integer square root.

- Here are some mathematical properties of square roots.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab} \qquad \sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

$$A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab} \qquad A\sqrt{a} \div B\sqrt{b} = \frac{A}{B}\sqrt{\frac{a}{b}}$$

- Square roots of prime numbers are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

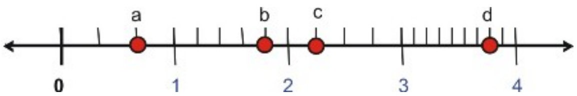
- Computing a square root on a calculator will produce an **approximate solution** since there are a finite number of digits after the decimal point.

Review Questions

1. Find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.
   (a) $\sqrt{25}$
   (b) $\sqrt{24}$
   (c) $\sqrt{20}$
   (d) $\sqrt{200}$
   (e) $\sqrt{2000}$
   (f) $\sqrt{\frac{1}{4}}$
   (g) $\sqrt{\frac{9}{4}}$
   (h) $\sqrt{0.16}$
   (i) $\sqrt{0.1}$
   (j) $\sqrt{0.01}$
2. Use a calculator to find the following square roots. Round to two decimal places.
   (a) $\sqrt{13}$
   (b) $\sqrt{99}$
   (c) $\sqrt{123}$
   (d) $\sqrt{2}$
   (e) $\sqrt{2000}$
   (f) $\sqrt{0.25}$
   (g) $\sqrt{1.35}$
   (h) $\sqrt{0.37}$
   (i) $\sqrt{0.7}$
   (j) $\sqrt{0.01}$
3. Classify the following numbers as an integer, a rational number or an irrational number.
   (a) $\sqrt{0.25}$
   (b) $\sqrt{1.35}$
   (c) $\sqrt{20}$
   (d) $\sqrt{25}$
   (e) $\sqrt{100}$
4. Place the following numbers in numerical order, from lowest to highest.

$$\frac{\sqrt{6}}{2} \qquad \frac{61}{50} \qquad \sqrt{1.5} \qquad \frac{16}{13}$$

5. Use the marked points on the number line and identify each proper fraction.



Review Answers

1.
2. (a) 5
   (b) $2\sqrt{6}$
   (c) $2\sqrt{5}$
   (d) $10\sqrt{2}$
   (e) $20\sqrt{5}$
   (f) $\frac{1}{2}$
   (g) $\frac{3}{2}$
   (h) 0.4
   (i) $\frac{1}{\sqrt{10}}$ *or* $\frac{\sqrt{10}}{10}$
   (j) 0.1
3.
4. (a) 3.61
   (b) 1.16
   (c) 9.95
   (d) 11.09
   (e) 44.72
   (f) 0.5
   (g) 1.16
   (h) 0.61
   (i) 0.84
   (j) 0.1
5.
6. (a) rational
   (b) irrational
   (c) irrational
   (d) integer
   (e) integer
7.
8.

$$\frac{61}{50} \qquad \frac{\sqrt{6}}{2} \qquad \frac{16}{13} \qquad \sqrt{1.6}$$

9.
10. (a) $\frac{2}{3}$
   (b) $\frac{9}{5}$
   (c) $\frac{9}{4}$
   (d) $\frac{34}{9}$

2.8 Problem-Solving Strategies: Guess and Check, Work Backward

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy: guess and check.
- Develop and use the strategy: work backward.
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we will continue using our problem solving plan to solve real-world problems. In this section, you will learn about the methods of **Guess and Check** and **Working Backwards.** These are very powerful strategies in problem solving and probably the most commonly used in everyday life. Let's review our problem-solving plan.

Step 1

Understand the problem.

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4

Look – Check and Interpret

Check to see if you used all your information and that the answer makes sense.

Let's now apply this plan to a few problems.

Read and Understand Given Problem Situations

The most difficult parts of problem-solving are most often the first two steps in our problem solving plan. First, you need to read the problem and make sure you understand what you are being asked. Then devise a strategy that uses the information you have been given to arrive at a solution.

Let's look at a problem without solving it. We will read through the problem and list the information we have been given and what we are trying to find. We will then try to devise a strategy for solving the problem.

Example 1

A book cost $18 *if bought online and* $22.50 *if bought at the store. The bookstore sold* 250 *books and took in* $4995. *How many books were bought online and how many were bought in the store?*

Problem set-up:

Step 1

Understand

A book bought online is $18

A book bought at the store is $22.50

The total takings equal $4995

The total number of books sold equals 250

How many books were bought online and how many books were bought in the store?

Step 2

Strategy

Total takings = Total for online sales + Total for in-store sales.

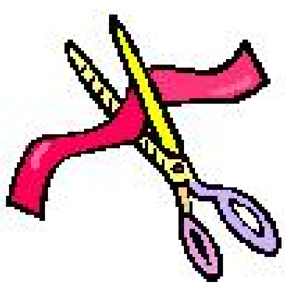
$4995 = $18 (number of books sold online) +$22.50 (number of books sold in-store)

Number of books sold online + Number of books sold in the store = 250 books.

We can guess values for each category and see which of them will give the correct answers.

Develop and Use the Strategy: Guess and Check

The strategy for the method "Guess and Check" is to guess a solution and use the guess in the problem to see if you get the correct answer. If the answer is too big or too small, then make another guess that will get you closer to the goal. You continue guessing until you arrive at the correct solution. The process might sound like a long one, however the guessing process will often lead you to patterns that you can use to make better guesses along the way.



Here is an example of how this strategy is used in practice.

Example 2

Nadia takes a ribbon that is 48 *inches long and cuts it in two pieces. One piece is three times as long as the other. How long is each piece?*

Solution

Step 1

Understand

We need to find two numbers that add to 48. One number is three times the other number.

Step 2

Strategy

We guess two random numbers, one three times bigger than the other and find the sum.

If the sum is too small we guess larger numbers and if the sum is too large we guess smaller numbers.

Then, we see if any patterns develop from our guesses.

Step 3

Apply Strategy/Solve

Guess	5 and 15	the sum is $5 + 15 = 20$	which is too small
Guess bigger numbers	6 and 18	the sum is $6 + 18 = 24$	which is too small

However, you can see that the answer is exactly half of 48.

Multiply 6 and 18 by two.

Our next guess is	12 and 36	the sum is $12 + 36 = 48$	This is correct.

Answer The pieces are 12 inches and 36 inches long.

Step 4

Check

$12 + 36 = 48$	The ribbon pieces add up to 48 inches.
$36 = 3(12)$	One piece is three times the length of the other piece.

The answer checks out.

Develop and Use the Strategy: Work Backward

The "Work Backward" method works well for problems in which a series of operations is applied to an unknown quantity and you are given the resulting number. The strategy in these problems is to start with the result and apply the operations in reverse order until you find the unknown. Let's see how this method works by solving the following problem.

Example 3

Anne has a certain amount of money in her bank account on Friday morning. During the day she writes a check for $24.50, makes an ATM withdrawal of \$80 *and deposits a check for* \$235*. At the end of the day* *she sees that her balance is* \$451.25. *How much money did she have in the bank at the beginning of the day?*

Solution:

Step 1

Understand

We need to find the money in Anne's bank account at the beginning of the day on Friday.

She took out \$24.50 and \$80 and put in \$235.

She ended up with \$451.25 at the end of the day.

Step 2

Strategy

From the unknown amount we subtract \$24.50 and \$80 and add \$235. We end up with \$451.25.

We need to start with the result and apply the operations in reverse.

Step 3

Apply Strategy/Solve

Start with \$451.25. Subtract \$235 and add \$80 and then add \$24.50.

$$451.25 - 235 + 80 + 24.50 = 320.75$$

Answer Anne had \$320.75 in her account at the beginning of the day on Friday.

Step 4

Check

Anne starts with	\$320.75
She writes a check for\$24.50	\$320.75 – \$24.50 = \$296.25
She withdraws\$80	\$296.25 – \$80 = \$216.25
She deposits\$235	\$216.25 + \$235 = \$451.25

The answer checks out.

Plan and Compare Alternative Approaches to Solving Problems

Most word problems can be solved in more than one way. Often one method is more straight forward than others. In this section, you will see how different approaches compare for solving different kinds of problems.

Example 4

Nadia's father is 36. *He is* 16 *years older than four times Nadia's age. How old is Nadia?*

Solution

This problem can be solved with either of the strategies you learned in this section. Let's solve the problem using both strategies.

Guess and Check Method

Step 1

Understand

We need to find Nadia's age.

We know that her father is 16 years older than four times her age. Or $4\times$ (Nadia's age) + 16

We know her father is 36 years old.

Step 2

Strategy

We guess a random number for Nadia's age.

We multiply the number by 4 and add 16 and check to see if the result equals to 36.

If the answer is too small, we guess a larger number and if the answer is too big then we guess a smaller number.

We keep guessing until we get the answer to be 36.

Step 3

Apply strategy/Solve

Guess Nadia's age	10	$4(10) + 16 = 56$	which is too big for her father's age
Guess a smaller number	9	$4(9) + 16 = 52$	which is too big

We notice that when we decreased Nadia's age by one, her father's age decreased by four.

We want the father's age to be 36, which is 16 years smaller than 52.

This means that we should guess Nadia's age to be 4 years younger than 9.

Guess	5	$4(5) + 16 = 36$	This is the right age.

Answer Nadia is 5 years old.

Step 4

Check

Nadia is 5 years old. Her father's age is $4(5) + 16 = 36$. This is correct.

The answer checks out.

Work Backward Method

Step 1

Understand

We need to find Nadia's age.

We know her father is 16 years older than four times her age. Or $4 \times (Nadia'sage) + 16$

We know her father is 36 years old.

Step 2

Strategy

Nadia's father is 36 years old.

To get from Nadia's age to her father's age, we multiply Nadia's age by four and add 16.

Working backwards means we start with the father's age, subtract 16 and divide by 4.

Step 3

Apply Strategy/Solve

Start with the father's age	36
Subtract16	$36 - 16 = 20$
Divide by4	$20 \div 4 = 5$

Answer Nadia is 5 years old.

Step 4

Check

Nadia is 5 years old. Her father's age is: $4(5) + 16 = 36$. This is correct.

The answer checks out.

You see that in this problem, the "Work Backward" strategy is more straight-forward than the Guess and Check method. The Work Backward method always works best when we perform a series of operations to get from an unknown number to a known result. In the next chapter, you will learn algebra methods for solving equations that are based on the Work Backward method.

Solve Real-World Problems Using Selected Strategies as Part of a Plan

Example 6

Nadia rents a car for a day. Her car rental company charges \$50 *per day and* \$0.40 *per mile. Peter rents a car from a different company that charges* \$70 *per day and* \$0.30 *per mile. How many miles do they have to drive before Nadia and Peter pay the same price for the rental for the same number of miles?*

Solution Let's use the Guess and Check method.

Step 1

Understand

Nadia's car rental costs \$50 plus \$0.40 per mile.

Peter's car rental costs \$70 plus \$0.30 per mile.

We want to know how many miles they have to drive to pay the same price of the rental for the same number of miles.

Step 2

Strategy

Nadia's total cost is \$50 plus \$0.40 times the number of miles.

Peter's total cost is \$70 plus \$0.30 times the number of miles.

Guess the number of miles and use this guess to calculate Nadia's and Peter's total cost.

Keep guessing until their total cost is the same.

Step 3

Apply Strategy/Solve

Guess	50 miles		
Check	$\$50 + \$0.40(50) = \$70$	$\$70 + \$0.30(50) = \$85$	too small
Guess	60 miles		
Check	$\$50 + \$0.40(60) = \$74$	$\$70 + \$0.30(60) = \$88$	too small

Notice that for an increase of 10 miles, the difference between total costs fell from \$15 to \$14.

To get the difference to zero, we should try increasing the mileage by 140 miles.

Guess	200 miles		
Check	$\$50 + \$0.40(200) = \$130$	$\$70 + \$0.30(200) = \$130$	correct

Answer: Nadia and Peter each have to drive 200 miles to pay the same total cost for the rental.

Step 4

Check

Nadia	$\$50 + \$0.40(200) = \$130$	Peter	$\$70 + \$0.30(200) = \$130$

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** are:

- **Understand the problem**
- **Devise a plan – Translate**
- **Carry out the plan – Solve**
- **Look – Check and Interpret**

Two common problem solving strategies are:

- **Guess and Check**

Guess a solution and use the guess in the problem to see if you get the correct answer. If the answer is too big or too small, then make another guess that will get you closer to the goal.

- **Work Backward**

This method works well for problems in which a series of operations is applied to an unknown quantity and you are given the resulting number. Start with the result and apply the operations in reverse order until you find the unknown.

Review Questions

1. Finish the problem we started in Example 1.
2. Nadia is at home and Peter is at school which is 6 miles away from home. They start traveling towards each other at the same time. Nadia is walking at 3.5 miles per hour and Peter is skateboarding at 6 miles per hour. When will they meet and how far from home is their meeting place?
3. Peter bought several notebooks at Staples for $2.25 each and he bought a few more notebooks at Rite-Aid for $2 each. He spent the same amount of money in both places and he bought 17 notebooks in total. How many notebooks did Peter buy in each store?
4. Andrew took a handful of change out of his pocket and noticed that he was only holding dimes and quarters in his hand. He counted that he had 22 coins that amounted to $4. How many quarters and how many dimes does Andrew have?
5. Anne wants to put a fence around her rose bed that is one and a half times as long as it is wide. She uses 50 feet of fencing. What are the dimensions of the garden?
6. Peter is outside looking at the pigs and chickens in the yard. Nadia is indoors and cannot see the animals. Peter gives her a puzzle He tells her that he counts 13 heads and 36 feet and asks her how many pigs and how many chickens are in the yard. Help Nadia find the answer.
7. Andrew invests $8000 in two types of accounts. A savings account that pays 5.25% interest per year and a more risky account that pays 9% interest per year. At the end of the year he has $450 in interest from the two accounts. Find the amount of money invested in each account.
8. There is a bowl of candy sitting on our kitchen table. This morning Nadia takes one-sixth of the candy. Later that morning Peter takes one-fourth of the candy that's left. This afternoon, Andrew takes one-fifth of what's left in the bowl and finally Anne takes one-third of what is left in the bowl. If there are 16 candies left in the bowl at the end of the day, how much candy was there at the beginning of the day?
9. Nadia can completely mow the lawn by herself in 30 minutes. Peter can completely mow the lawn by himself in 45 minutes. How long does it take both of them to mow the lawn together?

Review Answers

1. 140 online sales and 110 in-store sales.
2. 37.9 minutes 2.2 miles from home
3. 8 notebooks at Staples and 9 notebooks at Rite-Aid
4. 12 quarters and 10 dimes
5. 10 feet wide and 15 feet long
6. 5 pigs and 8 chickens
7. $7200 in the savings account and $800 in the high-risk account
8. 48 candies
9. 18 minutes

Chapter 3

Equations of Lines

3.1 One-Step Equations

Learning Objectives

- Solve an equation using addition.
- Solve an equation using subtraction.
- Solve an equation using multiplication.
- Solve an equation using division.

Introduction

Nadia is buying a new mp3 player. Peter watches her pay for the player with a $100 bill. She receives $22.00 in change, and from only this information, Peter works out how much the player cost. How much was the player?

In math, we can solve problems like this using an **equation.** An **equation** is an algebraic expression that involves an **equals** sign. If we use the letter x to represent the cost of the mp3 player we could write the following equation.

$$x + 22 = 100$$

This tells us that the value of the player **plus** the value of the change received is **equal** to the $100 that Nadia paid.

Peter saw the transaction from a different viewpoint. He saw Nadia receive the player, give the salesperson $100 then he saw Nadia receive $22 change. Another way we could write the equation would be:

$$x = 100 - 22$$

This tells us that the value of the player is **equal** to the amount of money Nadia paid $(100 - 22)$.

Mathematically, these two equations are equivalent. Though it is easier to determine the cost of the mp3 player from the second equation. In this chapter, we will learn how to solve for the variable in a one variable linear equation. Linear equations are equations in which each term is either a constant or the product of a constant and a single variable (to the first power). The term linear comes from the word line. You will see in later chapters that linear equations define lines when graphed.

We will start with simple problems such as the one in the last example.

Solve an Equation Using Addition

When we write an algebraic equation, equality means that whatever we do to one side of the equation, we have to do to the other side. For example, to get from the second equation in the introduction back to the first equation, we would add a quantity of 22 to both sides:

$$x = 100 - 22$$

$$x + 22 = 100 - 22 + 22 \qquad \text{or} \qquad x + 22 = 100$$

Similarly, we can add numbers to each side of an equation to help solve for our unknown.

Example 1

Solve $x - 3 = 9$

Solution

We need to **isolate** x. Change our equation so that x appears by itself on one side of the equals sign. Right now our x has a 3 subtracted from it. To reverse this, we could add 3, but we must do this to **both sides.**

$$\begin{aligned} x - 3 &= 9 \\ x - 3 + 3 &= 9 + 3 \text{ The } +3 \text{ and } -3 \text{ on the left cancel each other. We evaluate } 9 + 3 \\ x &= 12 \end{aligned}$$

Example 2

Solve $x - 3 = 11$

Solution

To isolate x we need to add 3 to both sides of the equation. This time we will add vertically.

$$\begin{aligned} x - \not{3} &= 11 \\ +\not{3} &= +3 \\ x &= 14 \end{aligned}$$

Notice how this format works. One term will always cancel (in this case the three), but we need to remember to carry the x down and evaluate the sum on the other side of the equals sign.

Example 3

Solve $z - 9.7 = -1.026$

Solution

This time our variable is z, but don't let that worry you. Treat this variable like any other variable.

$$\begin{aligned} z - 9.7 &= -1.026 \\ +9.7 &= +9.7 \\ zb &= 8.674 \end{aligned}$$

Make sure that you understand the addition of decimals in this example!

Solve an Equation Using Subtraction

When our variable appears with a number added to it, we follow the same process, only this time to isolate the variable we **subtract** a number from both sides of the equation.

Example 4

Solve $x+6=26$

Solution

To isolate x we need to subtract six from both sides.

$$\begin{aligned} \not{x}+\not{6} &= 26 \\ -\not{6} &= -6 \\ x &= 20 \end{aligned}$$

Example 5

Solve $x+20=-11$

Solution

To isolate x we need to subtract 20 from both sides of the equation.

$$\begin{aligned} x+20 &= -11 \\ -20 &= -20 \\ x &= -31 \end{aligned}$$

Example 6

Solve $x+\frac{4}{7}=\frac{9}{5}$

Solution

To isolate x we need to subtract $\frac{4}{7}$ from both sides.

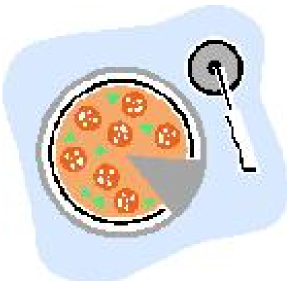
$$\begin{aligned} x+\frac{4}{7} &= \frac{9}{5} \\ -\frac{4}{7} &= -\frac{4}{7} \\ x &= \frac{9}{5}-\frac{4}{7} \end{aligned}$$

To solve for x, make sure you know how to subtract fractions. We need to find the lowest common denominator. 5 and 7 are both prime. So we can multiply to find the LCD, LCD $=5\cdot 7=35$.

$$\begin{aligned} x &= \frac{9}{5}-\frac{4}{7} \\ x &= \frac{7\cdot 9}{35}-\frac{4\cdot 5}{35} \\ x &= \frac{63-20}{35} \\ x &= \frac{43}{35} \end{aligned}$$

Make sure you are comfortable with decimals and fractions! To master algebra, you will need to work with them frequently.

Solve an Equation Using Multiplication



Suppose you are selling pizza for $1.50 a slice and you get eight slices out of a single pizza. How much do you get for a single pizza? It shouldn't take you long to figure out that you get $8 \times \$1.50 = \$12.00$. You solve this problem by multiplying. The following examples do the same algebraically, using the unknown variable x as the cost in dollars of the whole pizza.

Example 7

Solve $\frac{1}{8} \cdot x = 1.5$

Our x is being multiplied by one-eighth. We need to cancel this factor, so we multiply by the reciprocal 8. Do not forget to multiply **both sides** of the equation.

$$\not{8}\left(\frac{1}{\not{8}} \cdot x\right) = 8(1.5)$$
$$x = 12$$

In general, when x is multiplied by a fraction, we multiply by the reciprocal of that fraction.

Example 8

Solve $\frac{9x}{5} = 5$

$\frac{9x}{5}$ is equivalent to $\frac{9}{5} \cdot x$ so x is being multiplied by $\frac{9}{5}$. To cancel, multiply by the reciprocal, $\frac{5}{9}$.

$$\frac{5}{9}\left(\frac{9x}{5}\right) = \frac{5}{9} \cdot 5$$
$$x = \frac{25}{9}$$

Example 9

Solve $0.25x = 5.25$

0.25 is the decimal equivalent of one fourth, so to cancel the 0.25 factor we would multiply by 4.

$$4(0.25x) = 4(5.25)$$
$$x = 21$$

Solve an Equation Using Division

Solving by division is another way to cancel any terms that are being multiplied x. Suppose you buy five identical candy bars, and you are charged $3.25. How much did each candy bar cost? You might just divide $3.25 by 5. Or you may convert to cents and divide 325 by 5. Let's see how this problem looks in algebra.

Example 10

Solve $5x = 3.25$ *To cancel the* 5 *we divide both sides by* 5.

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{3.25}{5}$$
$$x = 0.65$$

Example 11

Solve $7x = \frac{5}{11}$ *Divide both sides by* 7.

$$x = \frac{5}{7 \cdot 11}$$
$$x = \frac{5}{77}$$

Example 12

Solve $1.375x = 1.2$ *Divide by* 1.375

$$x = \frac{1.2}{1.375}$$
$$x = 0.8\overline{72}$$

Notice the bar above the final two decimals. It means recurring or repeating: the full answer is 0.872727272 . . ..

Multimedia Link To see more examples of one- and two-step equation solving, watch the video series starting at Khan Academy Solving Equations . The narrator in these videos uses informal terms and

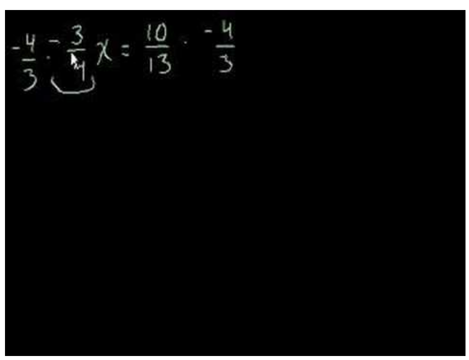


Figure 3.1: equations of the form AX=B (Watch on Youtube)

phrases to describe the process of solving equations, but the extra practice and seeing examples worked out may be helpful to reinforce procedural fluency.

Solve Real-World Problems Using Equations

Example 13

In the year 2017, *Anne will be* 45 *years old. In what year was Anne born?*

The unknown here is the year Anne was born. This is x. Here is our equation.

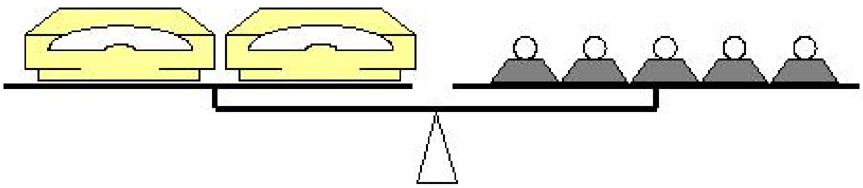
$$x + 45 = 2017$$
$$-45 = -45$$
$$x = 1972$$

Solution

Anne was born in 1972.

Example 14

A mail order electronics company stocks a new mini DVD player and is using a balance to determine the shipping weight. Using only one lb weights, the shipping department found that the following arrangement balances.



Knowing that each weight is one lb, calculate the weight of one DVD player.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity, the weight of the DVD player (in pounds), will be x. The combined weight on the right of the balance is $5 \times 1 \ lb = 5lb$.

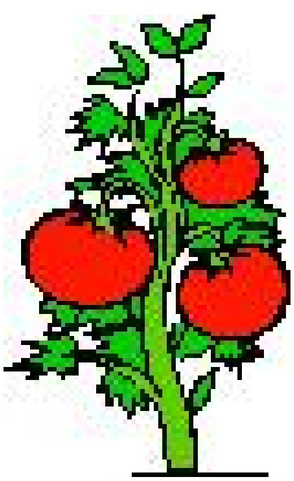
$2x = 5$ *Divide both sides by* 2.

$$\frac{\not{2}x}{\not{2}} = \frac{5}{2}$$

$x = 2.5$

Each DVD player weighs x 2.5 lbs.

Example 15



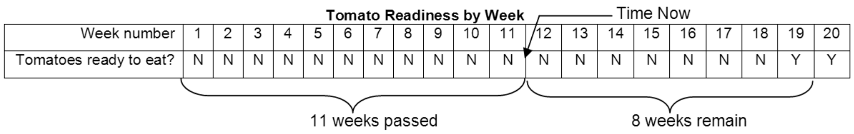
In good weather, tomato seeds can grow into plants and bear ripe fruit in as little as 19 weeks . *Lora planted her seeds* 11 *weeks ago. How long must she wait before her tomatoes are ready to eat?*

Solution

We know that the total time to bear fruit is 19 weeks , and that the time so far is 19 weeks . Our unknown is the time in weeks remaining, so we call that x. Here is our equation.

$$\begin{aligned} x + 11 &= 19 \\ -11 &= -11 \\ x &= 8 \end{aligned}$$

Lora will have to wait another 8 weeks before her tomatoes are ready. We can show this by designing a table.



Week number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Tomatoes ready to eat?	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	Y	Y

Example 16

In 2004, *Takeru Kobayashi, of Nagano, Japan, ate* $53\frac{1}{2}$ *hot dogs in* 12 minutes . *He broke his previous world record, set in* 2002, *by three hot dogs. Calculate:*

a) How many minutes it took him to eat one hot dog.

b) How many hot dogs he ate per minute.

c) What his old record was.

a) We know that the total time for 53.5 hot dogs is 12 minutes . If the time, in minutes, for each hot dog (the unknown) is x then we can write the following equation.

$$\begin{aligned} 53.5x &= 12 && \text{Divide both sides by 53.5} \\ x &= \frac{12}{53.5} = 0.224 && \text{minutes Convert to seconds, by multiplying by 60} \end{aligned}$$

.

Solution

The time taken to eat one hot dog is 0.224 minutes , or about 13.5 seconds .

Note: We round off our answer as there is no need to give our answer to an accuracy better than 0.1 (one tenth) of a second.

b) This time, we look at our data slightly differently. We know that he ate for 12 minutes . His **rate per-minute** is our new unknown (to avoid confusion with x, we will call this y). We know that the total number of hot dogs is 53.5 so we can write the following equation.

$$\begin{aligned} 12y &= 53.5 && \text{Divide both sides by 12} \\ y &= \frac{53.5}{12} = 4.458 \end{aligned}$$

Solution

Takeru Kobayashi ate approximately 4.5. hot dogs per minute.

c) We know that his new record is 53.5. and also that his new record is three more than his old record. We have a new unknown. We will call his old record z, and write the following equation.

$$
\begin{aligned}
x + 3 &= 53.5 \\
-3 &= -3 \\
x &= 50.5
\end{aligned}
$$

Solution

Takeru Kobayashi's old record was $50\frac{1}{2}$ hot dogs in 12 minutes .

Lesson Summary

- An equation in which each term is either a constant or a product of a constant and a single variable is a **linear equation.**
- Adding, subtracting, multiplying, or dividing both sides of an equation by the same value results in an equivalent equation.
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Review Questions

1. Solve the following equations for x.
   (a) $x + 11 = 7$
   (b) $x - 1.1 = 3.2$
   (c) $7x = 21$
   (d) $4x = 1$
   (e) $\frac{5x}{12} = \frac{2}{3}$
   (f) $x + \frac{5}{2} = \frac{2}{3}$
   (g) $x - \frac{5}{6} = \frac{3}{8}$
   (h) $0.01x = 11$
2. Solve the following equations for the unknown variable.
   (a) $q - 13 = -13$
   (b) $z + 1.1 = 3.0001$
   (c) $21\text{ s} = 3$
   (d) $t + \frac{1}{2} = \frac{1}{3}$
   (e) $\frac{7f}{11} = \frac{7}{11}$
   (f) $\frac{3}{4} = -\frac{1}{2} \cdot y$
   (g) $6r = \frac{3}{8}$
   (h) $\frac{9b}{16} = \frac{3}{8}$
3. Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.
   (a) How many more tokens he needs to collect, n.
   (b) How many tokens he collects per week, w.

(c) How many more weeks remain until he can send off for his boat, r.

4. Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him \$8.50, and he allowed \$1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements

(a) The amount of money that he sells the cake for (u).
(b) The amount of money he charges for each slice (c).
(c) The total profit he makes on the cake (w).

Review Answers

1.
2. (a) $x = -4$
(b) $x = 4.3$
(c) $x = 3$
(d) $x = 0.25$
(e) $x = 1.6$
(f) $x = -\frac{11}{6}$
(g) $x = \frac{29}{24}$
(h) $x = 1100$
3.
4. (a) $q = 0$
(b) $z = 1.9001$
(c) $s = 1/7$
(d) $t = -\frac{1}{6}$
(e) $f = 1$
(f) $y = -1.5$
(g) $r = \frac{1}{16}$
(h) $b = \frac{2}{3}$
5.
6. (a) $n + 10 = 25$, $n = 15$
(b) $8w = 10$, $w = 1.25$
(c) $r \cdot w = 15$ or $1.25r = 15$, $r = 12$
7.
8. (a) $u = 3(8.5 + 1.25)$
(b) $12v = u$
(c) $w = u - (8.5 + 1.25)$

3.2 Two-Step Equations

Learning Objectives

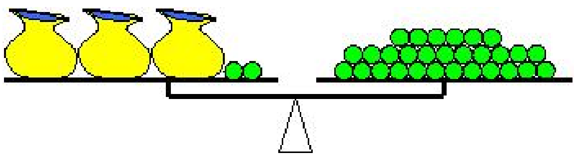
- Solve a two-step equation using addition, subtraction, multiplication, and division.
- Solve a two-step equation by combining like terms.
- Solve real-world problems using two-step equations.

Solve a Two-Step Equation

We have seen that in order to solve for an unknown variable we can isolate it on one side of the equal sign and evaluate the numbers on the other side. In this chapter we will expand our ability to do that, with problems that require us to combine more than one technique in order to solve for our unknown.

Example 1

Rebecca has three bags containing the same number of marbles, plus two marbles left over. She places them on one side of a balance. Chris, who has more marbles than Rebecca, added marbles to the other side of the balance. He found that with 29 *marbles, the scales balanced. How many marbles are in each bag? Assume the bags weigh nothing.*



Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity, the number of marbles in each bag, will be our x. We can see that on the left hand scale we have three bags (each containing x marbles) and two extra marbles. On the right scale we have 29 marbles. The balancing of the scales is similar to the balancing of the following equation.

$$3x + 2 = 29$$

"Three bags plus two marbles **equals** 29 marbles"

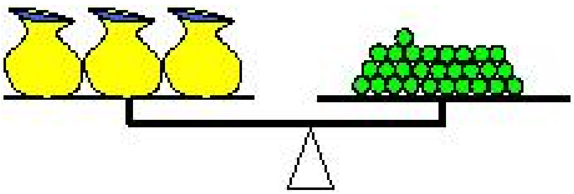
To solve for x we need to first get all the variables (terms containing an x) alone on one side. Look at the balance. There are no bags on the right. Similarly, there are no x terms on the right of our equation. We will aim to get all the constants on the right, leaving only the x on the left.

$$\begin{aligned} 3x + \cancel{2} &= 29 \\ \cancel{-2} &= -2 \qquad \text{Subtract 2 from both sides :} \\ \hline 3x &= 27 \\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{27}{3} \qquad \text{Divide both sides by 3} \\ x &= 9 \end{aligned}$$

Solution

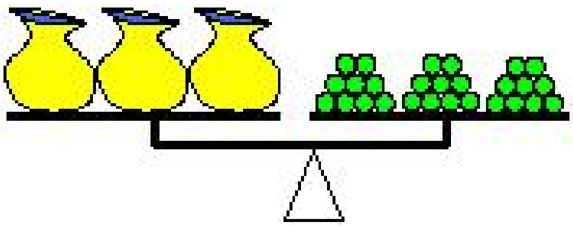
There are nine marbles in each bag.

We can do the same with the real objects as we have done with the equation. Our first action was to subtract two from both sides of the equals sign. On the balance, we could remove this number of marbles from each scale. Because we remove the same number of marbles from each side, we know the scales will still balance.



Next, we look at the left hand scale. There are three bags of marbles. To make our job easier, we divide the marbles on the right scale into three equal piles. You can see that there are nine marbles in each.

Three bags of marbles ***balances*** *three piles of nine marbles*



So each bag of marble balances nine marbles. Again you see we reach our solution:

Solution

Each bag contains nine marbles.

On the web: `http://www.mste.uiuc.edu/pavel/java/balance/` has interactive balance beam activities!

Example 2

Solve $6(x + 4) = 12$

Solution

This equation has the x *buried* in parentheses. In order to extract it we can proceed in one of two ways: we can either distribute the six on the left, or divide both sides by six to remove it from the left. Since the right hand side of the equation is a multiple of six, it makes sense to divide.

$$6(x + 4) = 12 \quad \text{Divide both sides by } 6.$$

$$\frac{\not{6}(x + 4)}{\not{6}} = \frac{12}{6}$$

$$x + \not{4} = 2 \quad \text{Subtract 4 from both sides.}$$

$$-\not{4} - 4$$

$$x = -2$$

Solution

$$x = -2$$

Example 3

Solve $\frac{x-3}{5} = 7$

This equation has a fraction in it. It is always a good idea to get rid of fractions first.

$$\left(x - \frac{3}{5}\right) = 7$$

Solution:

$$\cancel{5}\left(\frac{x-3}{\cancel{5}}\right) = 5 \cdot 7 \qquad \text{Multiply both sides by 5}$$

$$x - 3 = 35$$

$$+3 = +3 \qquad \text{Add 3 to both sides}$$

$$x = 38$$

Solution

$$x = 38$$

Example 4

Solve $\frac{5}{9}(x+1) = \frac{2}{7}$

First, we will cancel the fraction on the left (making the coefficient equal to one) by multiplying by the reciprocal (the multiplicative inverse).

$$\frac{\cancel{9}}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{9}}(x+1) = \frac{9}{5} \cdot \frac{2}{7}$$

$$x + 1 = \frac{18}{35} \qquad \text{Subtract1} \left(1 = \frac{35}{35}\right) \text{ from both sides.}$$

$$x = \frac{18}{35} - \frac{35}{35}$$

$$x = \frac{18 - 35}{35}$$

Solution

$$x = -\frac{17}{35}$$

These examples are called **two-step equations,** as we need to perform two separate operations to the equation to isolate the variable.

Solve a Two-Step Equation by Combining Like Terms

When we look at linear equations we predominantly see two terms, those that contain the unknown variable as a factor, and those that don't. When we look at an equation that has an x on both sides, we know that in order to solve, we need to get all the $x-$terms on one side of the equation. This is called **combining like terms.** They are **like terms** because they contain the same variable (or, as you will see in later chapters, the same combination of variables).

Like Terms

- $17x, 12x, -1.2x$, and $\frac{17x}{9}$
- $3y, 19y$, and $\frac{y}{99}$

- $xy, 6xy$, and $0.0001xy$

Unlike Terms

- $3x$ and $2y$
- $12xy$ and $2x$
- $0.001x$ and 0.001

To add or subtract like terms, we can use the Distributive Property of Multiplication instead of addition and subtraction.

$$3x + 4x = (3 + 4)x = 7x$$
$$0.03xy - 0.01xy = (0.03 - 0.01)xy = 0.02xy$$
$$-y + 16y - 5y = (-1 + 16 - 5)y = 10y$$
$$5z + 2z - 7z = (5 + 2 - 7)z = 0z = 0$$

To solve an equation with two or more like terms we need to combine them before we can solve for the variable.

Example 5

Solve $(x + 5) - (2x - 3) = 6$

There are two like terms. The x and the $-2x$ (do not forget that the negativesign multiplies everything in the parentheses).

Collecting like terms means we write all the terms with matching variables together. We will then add, or subtract them individually. We pull out the x from the first bracket and the $-2x$ from the second. We then rewrite the equation collecting the like terms.

$$(x - 2x) + (5 - (-3)) = 6 \qquad \text{Combine like terms and constants.}$$

$$-x + \cancel{8} = 6$$
$$\cancel{-8} = -8 \qquad \text{Subtract 8 from both sides}$$
$$-x = -2 \qquad \text{Mutliply both sides by } -1 \text{ to get the variable by itself}$$

Solution

$$x = 2$$

Example 6

Solve $\frac{x}{2} - \frac{x}{3} = 6$

Solution

This problem involves fractions. Combining the variable terms will require dealing with fractions. We need to write all the terms on the left over a common denominator of six.

$$\frac{3x}{6} - \frac{2x}{6} = 6 \qquad \text{Next we combine the fractions.}$$
$$\frac{x}{6} = 6 \qquad \text{Multiply both sides by 6.}$$
$$x = 36$$

Solve Real-World Problems Using Two-Step Equations

When we are faced with real world problems the thing that gives people the most difficulty is going from a problem in words to an equation. First, look to see what the equation is asking. What is the **unknown** for which you have to solve? That will determine the quantity we will use for our **variable.** The text explains what is happening. Break it down into small, manageable chunks. Then, follow what is going on with our variable all the way through the problem.

Example 7

An emergency plumber charges \$65 *as a call-out fee plus an additional* \$75 perhour per hour. He arrives at a house at 9 : 30 and works to repair a water tank. If the total repair bill is \$196.25*,* at what time was the repair completed?

In order to solve this problem, we collect the information from the text and convert it to an equation.

Unknown time taken in hours – this will be our x

The bill is made up of two parts: a call out fee and a per-hour fee. The call out is a flat fee, and independent of x. The per-hour part depends on x. Lets look at how this works algebraically.

\$65 as a call-out fee	65
Plus an additional \$75 per hour	$+75x$

So the bill, made up from the call out fee plus the per hour charge times the hours taken creates the following equation.

$$\text{Total Bill} = 65 + 75x$$

Lastly, we look at the final piece of information. The total on the bill was \$196.25. So our final equation is:

$$196.25 = 65 + 75x$$

We solve for x:

$$196.25 = \cancel{65} + 75x$$

$$-65 = \cancel{-65} \qquad \text{To isolate x first subtract 65 from both sides :}$$

$$131.25 = 75x \qquad \text{Divide both sides by 75}$$

$$\frac{131.25}{75} = x = 1.75 \qquad \text{The time taken was one and three quarter hours.}$$

Solution

The repair job was completed at 11:15AM.

Example 8

When Asia was young her Daddy marked her height on the door frame every month. Asia's Daddy noticed that between the ages of one and three, he could predict her height (in inches) by taking her age in months, adding 75 *inches and multiplying the result by one-third. Use this information to determine the following*

a) Write an equation linking her predicted height, h*, with her age in months,* m*.*

b) Determine her predicted height on her second birthday.

c) Determine at what age she is predicted to reach three feet tall.

a) To convert the text to an equation, first determine the type of equation we have. We are going to have an equation that links **two variables.** Our unknown will change, depending on the information we are given. For example, we could solve for height given age, or solve for age given height. However, the text gives us a way to determine **height.** Our equation will start with "$h =$".

Next we look at the text.

$$(m+75) \qquad \text{Take her age in months, and add75.}$$
$$\frac{1}{3}(m+75) \qquad \text{Multiply the result by one-third.}$$

Solution

Our full equation is $h = \frac{1}{3}(m+75)$.

b) To determine the prediction of Asia's height on her second birthday, we substitute $m = 24$ (2 years = 24 months) into our equation and solve for h.

$$h = \frac{1}{3}(24+75) \qquad \text{Combine terms in parentheses.}$$
$$h = \frac{1}{3}(99) \qquad \text{Multiply.}$$
$$h = 33$$

Solution

Asia's height on her second birthday was predicted to be 33 inches .

c) To determine the predicted age when she reached three feet, substitute $h = 36$ into the equation and solve for m.

$$36 = \frac{1}{3}(m+75) \qquad \text{Multiply both sides by 3.}$$
$$108 = m+75 \qquad \text{Subtract 75 from both sides.}$$
$$33 = m$$

Solution

Asia was predicted to be 33 months old when her height was three feet.

Example 9

To convert temperatures in Fahrenheit to temperatures in Celsius follow the following steps: Take the temperature in Fahrenheit and subtract 32. *Then divide the result by* 1.8 *and this gives temperature in degrees Celsius.*

a) Write an equation that shows the conversion process.

b) Convert 50 *degrees Fahrenheit to degrees Celsius.*

c) Convert 25 *degrees Celsius to degrees Fahrenheit.*

d) Convert −40 *degrees Celsius to degrees Fahrenheit.*

a) The text gives the process to convert Fahrenheit to Celsius. We can write an equation using two variables. We will use f for temperature in Fahrenheit, and c for temperature in Celsius. Follow the text to see it work.

$$(F-32) \qquad \text{Take the temperature in Fahrenheit and subtract 32.}$$
$$\frac{F-32}{1.8} \qquad \text{Then divide the result by 1.8.}$$
$$C=\frac{F-32}{1.8} \qquad \text{This gives temperature in degrees Celsius.}$$

In order to convert from one temperature scale to another, simply substitute in for the **known** temperature and solve for the **unknown.**

b) To convert 50degrees Fahrenheit to degrees Celsius substitute $F = 50$ into the equation.

$$C=\frac{50-32}{1.8} \qquad \text{Evaluate numerator.}$$
$$C=\frac{18}{1.8} \qquad \text{Perform division operation.}$$

Solution

$C = 10$, so 50 degrees Fahrenheit is equal to 10 degrees Celsius.

ci) To convert 25 degrees Celsius to degrees Fahrenheit substitute $C = 25$ into the equation:

$$25=\frac{F-32}{1.8} \qquad \text{Multiply both sides by 1.8}$$
$$45=F-32$$
$$+32=+32 \qquad \text{Add 32 to both sides.}$$
$$77=F$$

Solution

25 degrees Celsius is equal to 77 degrees Fahrenheit.

d) To convert −40 degrees Celsius to degrees Fahrenheit substitute $C = -40$ into the equation.

$$-40 = \frac{F-32}{1.8} \qquad \text{Multiply both sides by 1.8.}$$

$$-72 = F - 32$$

$$+32 = +32 \qquad \text{Add 32 to both sides.}$$

$$-40 = F$$

Solution

−40 degrees Celsius is equal to −40 degrees Fahrenheit.

Lesson Summary

- Some equations require more than one operation to solve. Generally it, is good to go from the outside in. If there are parentheses around an expression with a variable in it, cancel what is outside the parentheses first.
- Terms with the same variable in them (or no variable in them) are **like terms. Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Review Questions

1. Solve the following equations for the unknown variable.
   (a) $1.3x - 0.7x = 12$
   (b) $6x - 1.3 = 3.2$
   (c) $5x - (3x + 2) = 1$
   (d) $4(x + 3) = 1$
   (e) $5q - 7 = \frac{2}{3}$
   (f) $\frac{3}{5}x + \frac{5}{2} = \frac{2}{3}$
   (g) $s - \frac{3s}{8} = \frac{5}{6}$
   (h) $0.1y + 11 = 0$
   (i) $\frac{5q-7}{12} = \frac{2}{3}$
   (j) $\frac{5(q-7)}{12} = \frac{2}{3}$
   (k) $33t - 99 = 0$
   (l) $5p - 2 = 32$
2. Jade is stranded downtown with only \$10 to get home. Taxis cost \$0.75 per mile, but there is an additional \$2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money. Determine how many miles she can ride.
3. Jasmin's Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs \$150 dollars for the afternoon, and the food will cost \$3.00 per person. Andrew, Jasmin's Dad, has a budget of \$300 . Write an equation to help him determine the maximum number of guests he can invite.

Review Answers

1.
2. (a) $x = 20$

(b) $x = 0.75$
(c) $x = 1.5$
(d) $x = -2.75$
(e) $q = \frac{23}{15}$
(f) $= -\frac{55}{18}$
(g) $s = \frac{4}{3}$
(h) $y = -110$
(i) $q = 3$
(j) $q = \frac{43}{5}$
(k) $t = 3$
(l) $p = \frac{34}{5}$

3. $0.75x + 2.35 = 10$; $x = 10.2$ miles
4. $3x + 150 = 300$; $x = 50$ guests

3.3 Multi-Step Equations

Learning Objectives

- Solve a multi-step equation by combining like terms.
- Solve a multi-step equation using the distributive property.
- Solve real-world problems using multi-step equations.

Solving Multi-Step Equations by Combining Like Terms

We have seen that when we solve for an unknown variable, it can be a simple matter of moving terms around in one or two steps. We can now look at solving equations that take several steps to isolate the unknown variable. Such equations are referred to as **multi-step equations**.

In this section, we will simply be combining the steps we already know how to do. Our goal is to end up with all the constants on one side of the equation and all of the variables on the other side. We will do this by collecting "like terms". Don't forget, like terms have the same combination of variables in them.

Example 1

Solve $\frac{3x+4}{3} - 5x = 6$

This problem involves a fraction. Before we can combine the variable terms we need to deal with it. Let's put all the terms on the left over a common denominator of three.

$$\frac{3x+4}{3} - \frac{15x}{3} = 6 \qquad \text{Next we combine the fractions.}$$

$$\frac{3x+4-15x}{3} = 6 \qquad \text{Combine like terms.}$$

$$\frac{4-12x}{3} = 6 \qquad \text{Multiply both sides by 3.}$$

$$4 - 12x = 18 \qquad \text{Subtract 4 from both sides.}$$

$$-12x = 14 \qquad \text{Divide both sides by } -12$$

$$\frac{-12}{-12}x = -\frac{14}{12}$$

Solution

$x = -\frac{7}{6}$

Solving Multi-Step Equations Using the Distributive Property

You have seen in some of the examples that we can choose to divide out a constant or distribute it. The choice comes down to whether on not we would get a fraction as a result. We are trying to simplify the expression. If we can divide out large numbers without getting a fraction, then we avoid large coefficients. Most of the time, however, we will have to distribute and then collect like terms.

Example 2

Solve $17(3x+4) = 7$

This equation has the x buried in parentheses. In order to extract it we can proceed in one of two ways. We can either distribute the seventeen on the left, or divide both sides by seventeen to remove it from the left. If we divide by seventeen, however, we will end up with a fraction. We wish to avoid fractions if possible!

$$17(3x+4) = 7 \qquad \text{Distribute the 17.}$$
$$51x+68 = 7$$
$$-68 = -68 \qquad \text{Subtract 68 from both sides.}$$
$$51x = -61 \qquad \text{Divide by 51.}$$

Solution

$$x = -\frac{61}{51}$$

Example 3

Solve $4(3x-4) - 7(2x+3) = 3$

This time we will need to collect like terms, but they are hidden inside the brackets. We start by expanding the parentheses.

$$12x - 16 - 14x - 21 = 3 \qquad \text{Collect the like terms } (12x \text{ and } -14x).$$
$$(12x - 14x) + (-16 - 21) = 3 \qquad \text{Evaluate each set of like terms.}$$
$$-2x - 37 = 3$$
$$+37 + 37 \qquad \text{Add37 to both sides.}$$
$$-2x = 40$$
$$\frac{-2x}{-2} = \frac{40}{-2} \qquad \text{Divide both sides by } -2.$$

Solution

$$x = -20$$

Example 4

Solve the following equation for x.

$$0.1(3.2+2x) + \frac{1}{2}\left(3 - \frac{x}{5}\right) = 0$$

This function contains both fractions and decimals. We should convert all terms to one or the other. It is often easier to convert decimals to fractions, but the fractions in this equation are easily moved to decimal form. Decimals do not require a common denominator!

Rewrite in decimal form.

$$\begin{aligned}
0.1(3.2+2x)+0.5(3-0.2x)&=0 && \text{Multiply out decimals:}\\
0.32+0.2x+1.5-0.1x&=0 && \text{Collect like terms:}\\
(0.2x-0.1x)+(0.32+1.5)&=0 && \text{Evaluate each collection:}\\
0.1x+1.82&=0 && \text{Subtract 1.82 from both sides}\\
-1.82\ \ &-1.82\\
0.1x&=-1.82 && \text{Divide by } -0.1\\
\frac{0.1x}{0.1}&=\frac{-1.82}{0.1}
\end{aligned}$$

Solution

$$x=18.2$$

Solve Real-World Problems Using Multi-Step Equations

Real-world problems require you to translate from a problem in words to an equation. First, look to see what the equation is asking. What is the **unknown** you have to solve for? That will determine the quantity we will use for our **variable.** The text explains what is happening. Break it down into small, manageable chunks, and follow what is going on with our variable all the way through the problem.

Example 5

A grower's cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% *of all the money taken is removed for sales tax.* \$150 *is removed to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much money is taken in total if each grower receives a* \$175 *share?*

Let us translate the text above into an equation. The unknown is going to be the total money taken in dollars. We will call this x.

"8.5% *of all the money taken is removed for sales tax*". This means that 91.5% of the money remains. This is $0.915x$.

$$(0.915x-150)$$

"\$150 *is removed to pay the rent on the space they occupy"*

$$\frac{0.915x-150}{7}$$

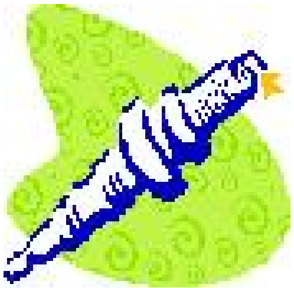
*"What remains is split evenly between the*7 *growers"*

If each grower's share is \$175, then we can write the following equation.

$$\begin{aligned}
\frac{0.915x-150}{7}&=175 && \text{Multiply by both sides 7.}\\
0.915x-150&=1225 && \text{Add 150 to both sides.}\\
0.915x&=1375 && \text{Divide by 0.915.}\\
\frac{0.915x}{0.915}&=\frac{1373}{0.915}\\
&=1502.7322\cdots && \text{Round to two decimal places.}
\end{aligned}$$

Solution

If the growers are each to receive a $175 share then they must take at least $1,502.73.



Example 6

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 *lb cargo. Each crate weighs twelve lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to* 1200 *lbs?*

The unknown quantity is the weight to put in each box. This is x. Each crate, when full will weigh:

$(x+12)$	16 crates must weigh.
$16(x+12)$	And this must equal 16 lbs.
$16(x+12) = 1200$	Isolate x first, divide both sides by 16.
$x+12 = 75$	Next subtract 12from both sides.
$x = 63$	

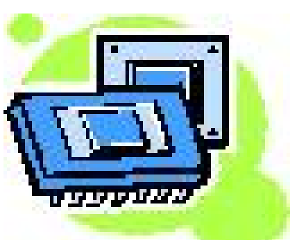
Solution

The manager should tell the workers to put 63 lbs of components in each crate.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$V = I \cdot R$ where R is the resistance (measured in Ohms - Ω)



Example 7

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component $x\Omega$. *The resistance of a circuit containing a number of these components is* $(5x+20)\Omega$. *If a* 120 *volt potential difference across the circuit produces a current of* 2.5 *amps, calculate the resistance of the unknown component.*

Substitute $V = 120$, $I = 2.5$ and $R = 5x+20$ into $V = I \cdot R$:

$$
\begin{aligned}
120 &= 2.5(5x+20) && \text{Distribute the 2.5.}\\
120 &= 12.5x+50 && \text{Subtract 50 from both sides.}\\
-50 &= -50 \\
70 &= 12.5x && \text{Divide both sides by 12.5.}\\
\frac{70}{12.5} &= \frac{12.5x}{12.5}\\
5.6\Omega &= x
\end{aligned}
$$

Solution

The unknown components have a resistance of 5.6Ω.

Distance,Speed and Time

The speed of a body is the distance it travels per unit of time. We can determine how far an object moves in a certain amount of time by multiplying the speed by the time. Here is our equation.

$$\text{distance} = \text{speed} \times \text{time}$$

Example 8

Shanice's car is traveling 10 miles *per hour slower than twice the speed of Brandon's car. She covers* 93 miles in 1 hour 30 minutes. How fast is Brandon driving?

Here we have two unknowns in this problem. Shanice's speed and Brandon's speed. We do know that Shanice's speed is ten less than twice Brandon's speed. Since the question is asking for Brandon's speed, it is his speed in miles per hour that will be x.

Substituting into the distance time equation yields:

$$
\begin{aligned}
93 &= 2x-10\times 1.5 && \text{Divide by 1.5.}\\
62 &= 2x-10 \\
+10 &= +10 && \text{Add10to both sides.}\\
72 &= 2x \\
\frac{72}{2} &= \frac{2x}{2} && \text{Divide both sides by 2.}\\
36 &= x
\end{aligned}
$$

Solution

Peter is driving at 36 miles per hour.

This example may be checked by considering the situation another way: We can use the fact that Shanice's covers 93 miles in 1 hour 30 minutes to determine her speed (we will call this y as x has already been defined as Brandon's speed):

$$
\begin{aligned}
93 &= y\cdot 1.5 \\
\frac{93}{1.5} &= \frac{1.5y}{1.5} && \text{Divide both sides by 1.5.}\\
y &= 62\text{mph}
\end{aligned}
$$

We can then use this information to determine Shanice's speed by converting the text to an equation.

"Shanice's car is traveling at 10 *miles per hour slower than twice the speed of Peter's car"*

Translates to

$$y = 2x - 10$$

It is then a simple matter to substitute in our value in for$y$ and then solve for x:

$$\begin{aligned}
62 &= (2x - \cancel{10}) \\
+10 &+ \cancel{10} && \text{Add 10 to both sides.} \\
\hline
72 &= 2x \\
72 &= 2x && \text{Divide both sides by 2.} \\
\frac{72}{2} &= \frac{2x}{2} \\
x &= 36 \text{ miles per hour.}
\end{aligned}$$

Solution

Brandon is driving at 36 milesperhour.

You can see that we arrive at exactly the same answer whichever way we solve the problem. In algebra, there is almost always more than one method of solving a problem. If time allows, it is an excellent idea to try to solve the problem using two different methods and thus confirm that you have calculated the answer correctly.

Speed of Sound

The speed of sound in dry air, v, is given by the following equation.

$v = 331 + 0.6T$ *where T is the temperature in Celsius and v is the speed of sound in meters per second.*

Example 9

Tashi hits a drainpipe with a hammer and 250 meters *away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time from her hitting the pipe and hearing Mihn's pipe at* 2.46 seconds . *What is the temperature of the air?*

This complex problem must be carefully translated into equations:

Distance traveled $= (331 + 0.6T) \times$ time

$$\begin{aligned}
\text{time } &= (2.46 - 1) && \text{Do not forget, for one second the sound is not traveling} \\
\text{Distance} &= 2 \times 250
\end{aligned}$$

Our equation is:

$$\begin{aligned}
2(250) &= (331 + 0.6T) \cdot (2.46 - 1) && \text{Simplify terms.} \\
\frac{500}{1.46} &= \frac{1.46(331 + 0.6T)}{1.46} && \text{Divide by 1.46.} \\
342.47 - 331 &= 331 + 0.6T - 331 && \text{Subtract 331 from both sides.} \\
\frac{11.47}{0.6} &= \frac{0.6T}{0.6} && \text{Divide by 0.6.} \\
19.1 &= T
\end{aligned}$$

Solution

The temperature is 19.1 degrees Celsius.

Lesson Summary

- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** (for example, $3(x+2) = 3x+6$) to expand the terms and then combine like terms to solve the equation.

Review Questions

1. Solve the following equations for the unknown variable.
   (a) $3(x-1) - 2(x+3) = 0$
   (b) $7(w+20) - w = 5$
   (c) $9(x-2) = 3x+3$
   (d) $2\left(5a - \frac{1}{3}\right) = \frac{2}{7}$
   (e) $\frac{2}{9}\left(i + \frac{2}{3}\right) = \frac{2}{5}$
   (f) $4\left(v + \frac{1}{4}\right) = \frac{35}{2}$
   (g) $\frac{s-4}{11} = \frac{2}{5}$
   (h) $\frac{p}{16} - \frac{2p}{3} = \frac{1}{9}$
2. An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
3. A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x+4)\Omega$ to a steady 12 Volt supply and finds that this produces a current of 1.2 Amps . What is the value of the unknown resistance?
4. Lydia inherited a sum of money. She split it into five equal chunks. She invested three parts of the money in a high interest bank account which added 10% to the value. She placed the rest of her inheritance plus $500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
5. Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes . How far away does his mother live?

Review Answers

1.
2. (a) $x = 9$
   (b) $w = -22.5$
   (c) $x = 3.5$
   (d) $a = \frac{2}{21}$
   (e) $i = \frac{17}{15}$
   (f) $v = \frac{33}{8}$
   (g) $s = \frac{42}{5}$
   (h) $p = -\frac{16}{87}$
3. $2(250) = 200 + 40x$; $x = 7.5 \rightarrow 7$ bags

4. 2Ω
5. \$1,176.50
6. 35 miles

3.4 Equations with Variables on Both Sides

Learning Objectives

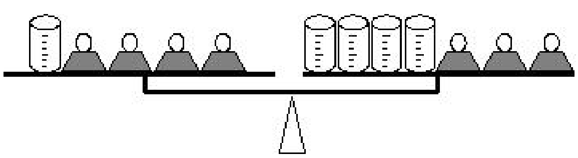
- Solve an equation with variables on both sides.
- Solve an equation with grouping symbols.
- Solve real-world problems using equations with variables on both sides.

Solve an Equation with Variables on Both Sides

When the variable appears on both sides of the equation, we need to manipulate our equation such that all variables appear on one side, and only constants remain on the other.

Example 1

Dwayne was told by his chemistry teacher to measure the weight of an empty beaker using a balance. Dwayne found only one lb weights, and so devised the following way of balancing the scales.



Knowing that each weight is one lb, calculate the weight of one beaker.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this fact. The unknown quantity, the weight of the beaker, will be our x. We can see that on the left hand scale we have one beaker and four weights. On the right scale, we have four beakers and three weights. The balancing of the scales is analogous to the balancing of the following equation.

$$x + 4 = 4x + 3$$

"One beaker plus 4 lbs **equals** 4 beakers plus 3 lbs"

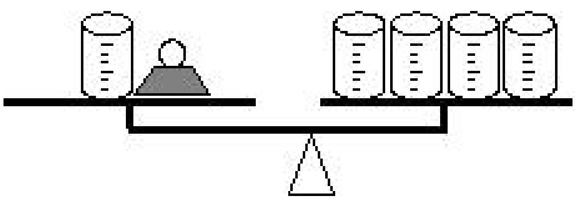
To solve for the weight of the beaker, we want all the constants (numbers) on one side and all the variables (terms with x in) on the other side. Look at the balance. There are more beakers on the right and more weights on the left. We will aim to end up with only x terms (beakers) on the right, and only constants (weights) on the left.

$$
\begin{aligned}
x + 4 &= 4x + 3 && \text{Subtract 3 from both sides.}\\
-3 &= -3 \\
\hline
x + 1 &= 4x \\
-x &= -x && \text{Subtract x from both sides.}\\
\hline
1 &= 3x \\
\frac{1}{3} &= \frac{\cancel{3}x}{\cancel{3}} && \text{Divide both sides by 3.}\\
x &= \frac{1}{3}
\end{aligned}
$$

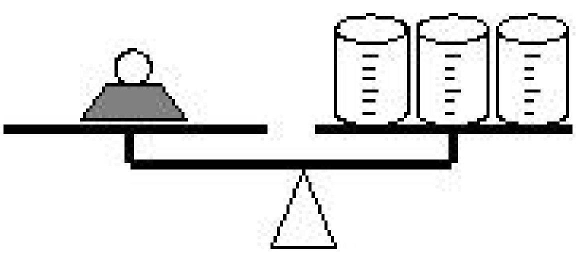
Answer The weight of the beaker is one-third of a pound.

We can do the same with the real objects as we have done with the equation. Our first action was to subtract three from both sides of the equals sign. On the balance, we could remove a certain number of weights or beakers from each scale. Because we remove the same number of weights from each side, we know the scales will still balance.

On the balance, we could remove three weights from each scale. This would leave one beaker and one weight on the left and four beakers on the right (in other words $x + 1 = 4x$).:



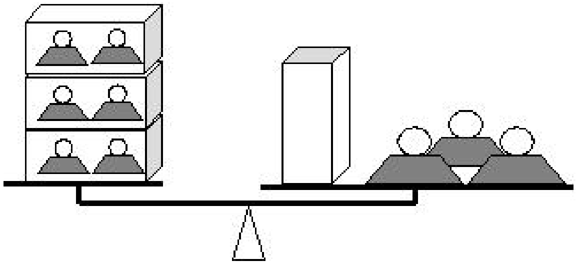
The next step we could do is remove one beaker from each scale leaving only one weight on the left and three beakers on the right and you will see our final equation: $1 = 3x$.



Looking at the balance, it is clear that the weight of the beaker is one-third of a pound.

Example 2

Sven was told to find the weight of an empty box with a balance. Sven found one lb weights and five lb weights. He placed two one lb weights in three of the boxes and with a fourth empty box found the following way of balancing the scales.



Knowing that small weights are one lb and big weights are five lbs, calculate the weight of one box.

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity, the weight of each empty box, in pounds, will be our x. A box with two 1 lb weights in it weighs $(x+2)$. Here is the equation.

$$\begin{aligned}
3(x+2) &= x+3(5) && \text{Distribute the 3.}\\
3x+6 &= \cancel{x}+5 \\
-x &= -\cancel{x} && \text{Subtract x from both sides.}\\
\hline
2x+\cancel{6} &= 15 \\
-\cancel{6} &= -6 && \text{Subtract 6 from both sides.}\\
\hline
2x &= 9 && \text{Divide both sides by2.}\\
x &= 4.5
\end{aligned}$$

Solution

Each box weighs 4.5 lbs.

Multimedia Link To see more examples of solving equations with variables on both sides of the equation, see Khan Academy Solving Linear Equations 3 . This video has several more examples of solving equations

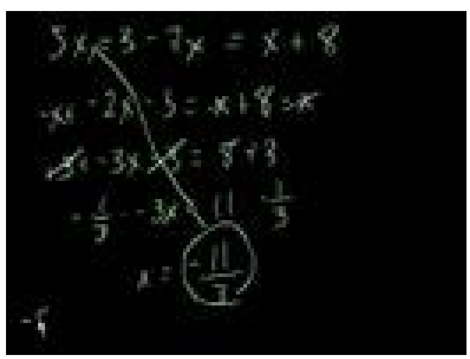


Figure 3.2: Linear equations with multiple variable and constant terms (Watch on Youtube)

and may help you practice the procedure of solving linear equations with variables on both sides of the equation.

Solve an Equation with Grouping Symbols

When we have a number of like terms on one side of the equal sign we **collect like terms** then add them in order to solve for our variable. When we move variables from one side of the equation to the other we sometimes call it grouping symbols. Essentially we are doing exactly what we would do with the constants. We can add and subtract variable terms just as we would with numbers. In fractions, occasionally we will have to multiply and divide by variables in order to get them all on the numerator.

Example 3

Solve $3x + 4 = 5x$

Solution

This equation has x on both sides. However, there is only a number term on the left. We will therefore move all the x terms to the right of the equal sign leaving the constant on the left.

$$\begin{aligned} \cancel{3x} + 4 &= 5x && \text{Subtract } 3x \text{ from both sides.} \\ -\cancel{3x} \quad &-3x \\ \hline 4 &= 2x \\ & && \text{Divide by 2} \\ \frac{4}{2} &= \frac{2x}{2} \end{aligned}$$

Solution

$$x = 2$$

Example 4

Solve $9x = 4 - 5x$

This time we will collect like terms (x terms) on the left of the equal sign.

$$\begin{aligned} 9x &= 4 - \cancel{5x} \\ +5x \quad &+\cancel{5x} && \text{Add } 5x \text{ to both sides.} \\ \hline 14x &= 4 \\ 14x &= 4 && \text{Divide by 14.} \\ \frac{14x}{14} &= \frac{4}{14} \\ x &= \frac{2}{7} \end{aligned}$$

Solution

$$x = \frac{2}{7}$$

Example 5

Solve $3x + 2 = \frac{5x}{3}$

This equation has x on both sides and a fraction. It is always easier to deal with equations that do not have fractions. The first thing we will do is get rid of the fraction.

$$\begin{aligned}
3x+2 &= \frac{5x}{3} \\
3(3x+2) &= 5x && \text{Multiply both sides by 3.} \\
& && \text{Distribute the 3.} \\
\cancel{9x}+6 &= 5x \\
-\cancel{9x} \quad & \quad -9x && \text{Subtract } 9x \text{ from both sides :} \\
\hline
& && \text{Divide by } -4. \\
\frac{6}{-4} &= \frac{-4x}{-4} \\
\frac{6}{-4} &= x \\
-\frac{3}{2} &= x
\end{aligned}$$

Solution

$$x=-1.5$$

Example 6

Solve $7x+2=\frac{5x-3}{6}$

Again we start by eliminating the fraction.

$$7x+2=\frac{5x-3}{6}$$

$$\begin{aligned}
6(7x+2) &= \frac{5x-3}{6}\cdot 6 && \text{Multiply both sides by 6.} \\
6(7x+2) &= 5x-3 && \text{Distribute the 6.} \\
42x+12 &= \cancel{5x}-3 && \text{Subtract } 5x \text{ from both sides :} \\
-5x \quad & \quad -\cancel{5x} \\
\hline
37x+\cancel{12} &= -3 && \text{Subtract 12 from both sides.} \\
-\cancel{12} \quad & \quad -12 \\
\hline
37x &= -15 \\
\frac{37x}{37} &= \frac{-15}{37} && \text{Divide by 37.}
\end{aligned}$$

Solution

$$x=-\frac{15}{37}$$

Example 7

Solve the following equation for x.

$$\frac{14x}{(x+3)} = 7$$

The form of the left hand side of this equation is known as a **rational function** because it is the ratio of two other functions $(14x)$ and $(x+3)$. However, we wish simply to solve for x so we start by eliminating the fraction. We do this as we have always done, by multiplying by the denominator.

$$\begin{aligned}
\frac{14x}{(x+3)}(x+3) &= 7(x+3) && \text{Multiply by } (x+3).\\
14x &= 7(x+3) && \text{Distribute the 7.}\\
14x &= \cancel{7x} + 21 &&\\
-7x &= -\cancel{7x} && \text{Subtract } 7x \text{ from both sides.}\\
\hline
7x &= 21 &&\\
\frac{7x}{7} &= \frac{21}{7} && \text{Divide both sides by 7}\\
x &= 3 &&
\end{aligned}$$

Solve Real-World Problems Using Equations with Variables on Both Sides

Build your skills in translating problems from words to equations. What is the equation asking? What is the **unknown** variable? What quantity will we use for our variable?

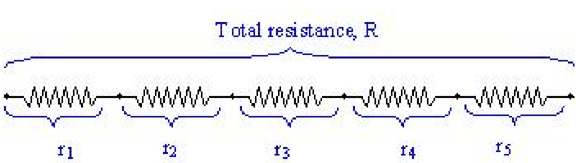
The text explains what is happening. Break it down into small, manageable chunks, and follow what is going on with our variable all the way through the problem.

More on Ohm's Law

The electrical current, I(amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$$V = I \cdot R \qquad \text{where R is the resistance (measured in Ohms)}$$

The resistance R of a number of components wired in a **series** (one after the other) is given by: $R = r_1 + r_2 + r_3 + r_4 + \ldots$



Example 8

In an attempt to find the resistance of a new component, a scientist tests it in series with standard resistors. A fixed voltage causes a 4.8 *amp current in a circuit made up from the new component plus a* 15Ω *resistor in series. When the component is placed in a series circuit with a* 50Ω *resistor the same voltage causes a* 2.0 *amp current to flow. Calculate the resistance of the new component.*

This is a complex problem to translate, but once we convert the information into equations it is relatively straight forward to solve. Firstly we are trying to find the resistance of the new component (in Ohms, Ω). This is our x. We do not know the voltage that is being used, but we can leave that as simple V. Our first situation has the unknown resistance plus 15Ω. The current is 4.8 amps. Substitute into the formula $V = I \cdot R$.

$$V = 4.8(x + 15)$$

Our second situation has the unknown resistance plus 50Ω. The current is 2.0 amps.

$$V = 2(x + 50)$$

We know the voltage is fixed, so the V in the first equation must equal the V in the second. This means that:

$$\begin{aligned}
4.8(x+15) &= 2(x+50) && \text{Distribute the constants.}\\
4.8x + 72 &= \cancel{2x} + 100 \\
-2x \qquad & \quad -\cancel{2x} && \text{Subtract } 2x \text{ from both sides.}\\
\hline
2.8x + \cancel{72} &= 100 \\
-72 \qquad & \quad -\cancel{72} && \text{Subtract 72 from both sides.}\\
\hline
2.8x &= 28 && \text{Divide both sides by 2.8.}\\
\frac{2.8x}{2.8} &= \frac{28}{2.8} \\
x &= 10
\end{aligned}$$

Solution

The resistance of the component is 10Ω.

Lesson Summary

- If an unknown variable appears on both sides of an equation, distribute as necessary. Then subtract (or add) one term to both sides to simplify the equation to have the unknown on only one side.

Review Questions

1. Solve the following equations for the unknown variable.
   (a) $3(x - 1) = 2(x + 3)$
   (b) $7(x + 20) = x + 5$
   (c) $9(x - 2) = 3x + 3$
   (d) $2\left(a - \frac{1}{3}\right) = \frac{2}{5}\left(a + \frac{2}{3}\right)$
   (e) $\frac{2}{7}\left(t + \frac{2}{3}\right) = \frac{1}{5}\left(t - \frac{2}{3}\right)$
   (f) $\frac{1}{7}\left(v + \frac{1}{4}\right) = 2\left(\frac{3v}{2} - \frac{5}{2}\right)$
   (g) $\frac{y-4}{11} = \frac{2}{5} \cdot \frac{2y+1}{3}$
   (h) $\frac{z}{16} = \frac{2(3z+1)}{9}$

(i) $\frac{q}{16} + \frac{q}{6} = \frac{(3q+1)}{9} + \frac{3}{2}$

2. Manoj and Tamar are arguing about how a number trick they heard goes. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer. What was Andrew's number?
3. I have enough money to buy five regular priced CDs and have \$6 left over. However all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them. How much are CDs on sale for today?
4. Five identical electronics components were connected in series. A fixed but unknown voltage placed across them caused a 2.3 amp current to flow. When two of the components were replaced with standard 10Ω resistors, the current dropped to 1.9 amps. What is the resistance of each component?
5. Solve the following resistance problems. Assume the same voltage is applied to all circuits.
   (a) Three unknown resistors plus 20Ω give the same current as one unknown resistor plus 70Ω.
   (b) One unknown resistor gives a current of 1.5 amps and a 15Ω resistor gives a current of 3.0 amps.
   (c) Seven unknown resistors plus 18Ω gives twice the current of two unknown resistors plus 150Ω.
   (d) Three unknown resistors plus 1.5Ω gives a current of 3.6 amps and seven unknown resistors plus 7 12Ω resistors gives a current of 0.2 amps.

Review Answers

1.
2. (a) $x = 9$
   (b) $x = -22.5$
   (c) $x = 3.5$
   (d) $a = \frac{7}{12}$
   (e) $t = -\frac{34}{9}$
   (f) $v = \frac{141}{80}$
   (g) $y = -\frac{82}{29}$
   (h) $z = -\frac{32}{87}$
   (i) $q = -\frac{232}{15}$
3. 9
4. \$7
5. 6.55Ω
6.
7. (a) unknown $= 25\Omega$
   (b) unknown $= 30\Omega$
   (c) unknown $= 94\Omega$
   (d) unknown $= 1.213\Omega$

3.5 Ratios and Proportions

Learning Objectives

- Write and understand a ratio.
- Write and solve a proportion.
- Solve proportions using cross products.

Introduction

Nadia is counting out money with her little brother. She gives her brother all the nickels and pennies. She keeps the quarters and dimes for herself. Nadia has four quarters (worth 25 cents each) and six dimes (worth 10 cents each). Her brother has fifteen nickels (worth 5 cents each) and five pennies (worth one cent each) and is happy because he has more coins than his big sister. How would you explain to him that he is actually getting a bad deal?

Write a ratio

A **ratio** is a way to compare two numbers, measurements or quantities. When we write a ratio, we divide one number by another and express the answer as a fraction. There are two distinct ratios in the problem above. For example, the ratio of the **number** of Nadia's coins to her brother's is:

$$\frac{4+6}{15+5} = \frac{10}{20}$$

When we write a ratio, the correct way is to simplify the fraction.

$$\frac{10}{20} = \frac{\cancel{2.5}}{2.\cancel{2.5}} = \frac{1}{2}$$

In other words, Nadia has half the number of coins as her brother.

Another ratio we could look at in the problem is the **value** of the coins. The value of Nadia's coins is $(4 \times 25) + (6 \times 10) = 160 \text{ cents}$. The value of her brother's coins is $(15 \times 5) + (5 \times 1) = 80 \text{ cents}$. The ratio of the **value** of Nadia's coins to her brother's is:

$$\frac{160}{80} = \frac{2}{1}$$

So the value of Nadia's money is twice the value of her brother's.

Notice that even though the denominator is one, it is still written. A ratio with a denominator of one is called a **unit rate**. In this case, it means Nadia is gaining money at twice the rate of her brother.

Example 1

The price of a Harry Potter Book on Amazon.com is $10.00. *The same book is also available used for* $6.50. *Find two ways to compare these prices.*

Clearly, the cost of a new book is greater than the used book price. We can compare the two numbers using a difference equation:

Difference in price = 10.00 − $6.50 = $3.50

We can also use a ratio to compare the prices:

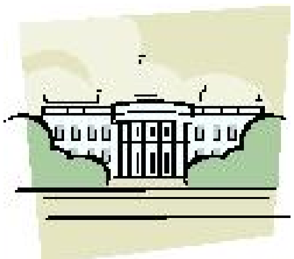
$$\frac{new\ price}{used\ price} = \frac{\$10.00}{\$6.50}$$ We can cancel the units of$ as they are the same.

$$\frac{10}{6.50} = \frac{1000}{650} = \frac{20}{13}$$ We remove the decimals and simplify the fraction.

Solution

The new book is \$3.50 more than the used book.

The new book costs $\frac{20}{13}$ times the cost of the used book.



Example 2

The State Dining Room in the White House measures approximately 48 *feet long by* 36 *feet wide. Compare the length of room to the width, and express your answer as a ratio.*

Solution

$$\frac{48\ feet}{36\ feet} = \frac{48}{36} = \frac{4}{3}$$

Example 3

A tournament size shuffleboard table measures 30 *inches wide by* 14 *feet long. Compare the length of the table to its width and express the answer as a ratio.*

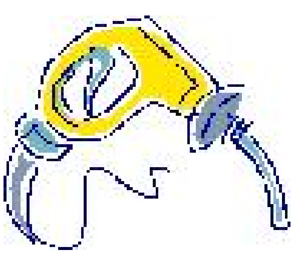
We could write the ratio immediately as:

$\frac{14\ feet}{30\ inches}$ *Notice that we cannot cancel the units.*

Sometimes it is OK to leave the units in, but as we are comparing two lengths, it makes sense to convert all the measurements to the same units.

Solution

$$\frac{14\ feet}{30\ inches} = \frac{14 \times 12\ inches}{30\ inches} = \frac{168}{30} = \frac{28}{5}$$



Example 4

A family car is being tested for fuel efficiency. It drives non-stop for 100 miles, *and uses* 3.2 *gallons of gasoline. Write the ratio of distance traveled to fuel used as a* **unit rate.**

$$Ratio = \frac{100\ miles}{3.2\ gallons}$$

A unit rate has a denominator of one, so we need to divide both numerator and denominator by 3.2.

$$Unit\ Rate \frac{(\frac{100}{3.2})\ miles}{(\frac{3.2}{3.2})\ gallons} = \frac{31.25\ miles}{1\ gallon}$$

Solution

The ratio of distance to fuel used is $\frac{31.25 \text{ miles}}{1 \text{ gallon}}$ or 31.25 miles per gallon.

Write and Solve a Proportion

When two ratios are equal to each other, we call it a proportion.

$$\frac{10}{15} = \frac{6}{9}$$

This statement is a proportion. We know the statement is true because we can reduce both fractions to $\frac{2}{3}$.

Check this yourself to make sure!

We often use proportions in science and business. For example, when scaling up the size of something. We use them to solve for an unknown, so we will use algebra and label our unknown variable x. We assume that a certain ratio holds true whatever the size of the thing we are enlarging (or reducing). The next few examples demonstrate this.

Example 5

A small fast food chain operates 60 *stores and makes* \$1.2 *million profit every year. How much profit would the chain make if it operated 250 stores?*

First, we need to write a **ratio**. This will be the ratio of profit to number of stores.

$$\text{Ratio} = \frac{\$1,200,000}{60 \text{ stores}}$$

We now need to determine our unknown, x which will be in dollars. It is the profit of 250 stores. Here is the ratio that compares unknown dollars to 250 stores.

$$\text{Ratio} = \frac{\$x}{250 \text{ stores}}$$

We now write equal ratios and solve the resulting **proportion**.

$\frac{\$1{,}200{,}000}{60 \ \text{stores}} = \frac{\$x}{250 \ \text{stores}}$ or $\frac{1{,}200{,}000}{60} = \frac{x}{250}$

Note that we can drop the units – not because they are the same on the numerator and denominator, but because they are the same on both sides of the equation.

$$\frac{1,200,000}{60} = \frac{x}{250} \qquad \text{Simplify fractions.}$$

$$20,000 = \frac{x}{250} \qquad \text{Multiply both sides by 250.}$$

$$5,000,000 = x$$

Solution

If the chain operated 250 stores the annual profit would be 5 milliondollars .

Example 6

A chemical company makes up batches of copper sulfate solution by adding 250 kg *of copper sulfate powder to* 1000 *liters of water. A laboratory chemist wants to make a solution of identical concentration, but only needs* 350 ml (0.35 liters) *of solution. How much copper sulfate powder should the chemist add to the water?*

First we write our ratio. The mass of powder divided by the volume of water used by the chemical company.

$$\text{Ratio} = \frac{250 \text{ kg}}{1000 \text{ liters}} \qquad \text{We can reduce this to : } \frac{1 \text{ kg}}{4 \text{ liters}}$$

Our unknown is the mass in kilograms of powder to add. This will be x. The volume of water will be 0.35 liters .

$$Ratio = \frac{x \ kg}{0.35 \ liters}$$

Our proportion comes from setting the two ratios equal to each other:

$\frac{1 \ kg}{4 \ liters} = \frac{x \ kg}{0.35 \ liters}$ which becomes $\frac{1}{4} = \frac{x}{0.35}$

We now solve for x.

$$\frac{1}{4} = \frac{x}{0.35} \qquad \text{Multiply both sides by 0.35.}$$

$$0.35 \cdot \frac{1}{4} = \frac{x}{0.35} \cdot 0.35$$

$$x = 0.0875$$

Solution

The mass of copper sulfate that the chemist should add is 0.0875 kg or 87.5 grams .

Solve Proportions Using Cross Products

One neat way to simplify proportions is to cross multiply. Consider the following proportion.

$$\frac{16}{4} = \frac{20}{5}$$

If we want to eliminate the fractions, we could multiply both sides by 4 and then multiply both sides by 5. In fact we ***could*** do both at once:

$$4 \cdot 5 \cdot \frac{16}{4} = 4 \cdot 5 \cdot \frac{20}{5}$$
$$5 \cdot 16 = 4 \cdot 20$$

Now comparing this to the proportion we started with, we see that the denominator from the left hand side ends up multiplying with the numerator on the right hand side.

You can also see that the denominator from the ***right*** hand side ends up multiplying the numerator on the ***left*** hand side.

In effect the two denominators have ***multiplied*** across the equal sign:

$\Rightarrow$

$$5 \cdot 16 = 4 \cdot 20$$

This movement of denominators is known as **cross multiplying**. It is extremely useful in solving proportions, especially when the unknown variable is on the denominator.

Example 7

Solve the proportion for x.

$$\frac{4}{3} = \frac{9}{x}$$

Cross multiply:

$$x \cdot 4 = 9 \cdot 3$$
$$\frac{4x}{4} = \frac{27}{4} \qquad \text{Divide both sides by4.}$$

Solution

$$x = 6.75$$

Example 8

Solve the following proportion for x.

$$\frac{0.5}{3} = \frac{56}{x}$$

Cross multiply:

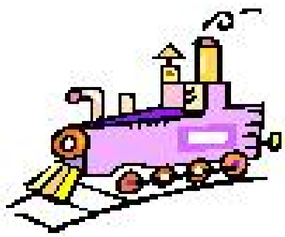
$$x \cdot 0.5 = 56 \cdot 3$$
$$\frac{0.5x}{0.5} = \frac{168}{0.5} \qquad \text{Divide both sides by 0.5.}$$

Solution:

$$x = 336$$

Solve Real-World Problems Using Proportions

When we are faced with a word problem that requires us to write a proportion, we need to identify both the unknown (which will be the quantity we represent as x) and the ratio which will stay fixed.



Example 9

A cross-country train travels at a steady speed. It covers 15 miles in 20 minutes . *How far will it travel in* 7 hours *assuming it continues at the same speed?*

This example is a Distance = speed × time problem. We came across a similar problem in Lesson 3.3. Recall that the speed of a body is the quantity distance/time. This will be our ratio. We simply plug in the known quantities. We will, however convert to hours from minutes.

$$\text{Ratio} = \frac{15 \text{ miles}}{20 \text{ minutes}} = \frac{15 \text{ miles}}{\frac{1}{3}\text{hour}}$$

This is a very awkward looking ratio, but since we will be cross multiplying we will leave it as it is. Next, we set up our proportion.

$$\frac{15 miles}{\frac{1}{3}\text{hour}} = \frac{x\ miles}{7\ hours}$$

Cancel the units and cross-multiply.

$$7 \cdot 15 = \frac{1}{3} \cdot x \qquad \text{Multiply both sides by 3.}$$
$$3 \cdot 7 \cdot 15 = 3 \cdot \frac{1}{3} \cdot x$$
$$315 = x$$

Solution

The train will travel 315 miles in 7 hours .

Example 10

Rain is falling at 1 *inch every* 1.5 hours. *How high will the water level be if it rains at the same rate for* 3 hours *?*

Although it may not look it, this again uses the Distance = speed × time relationship. The distance the water rises in inches will be our x. The ratio will again be $\frac{distance}{time}$.

$$\frac{1 \text{ inch}}{1.5 \text{ hours}} = \frac{x \text{ inch}}{3 \text{ hours}} \qquad \text{Cancel units and cross multiply.}$$

$$\frac{3(1)}{1.5} = \frac{1.5x}{1.5} \qquad \text{Divide by } 1.5$$

$$2 = x$$

Solution

The water will be 2 inches high if it rains for 3 hours .

Example 11

In the United Kingdom, Alzheimer's disease is said to affect one in fifty people over 65 *years of age. If approximately* 250000 *people over* 65 *are affected in the UK, how many people over* 65 *are there in total?*

The fixed ratio in this case will be the 1 person in 50. The unknown (x) is the number of persons over 65. Note that in this case, the ratio does not have units, as they will cancel between the numerator and denominator.

We can go straight to the proportion.

$$\frac{1}{50} = \frac{250000}{x} \qquad \text{Cross multiply :}$$

$$1 \cdot x = 250000 \cdot 50$$

$$x = 12,500,000$$

Solution

There are approximately 12.5 million people over the age of 65.

Multimedia Link For some advanced ratio problems and applications see Khan Academy Advanced Ratio Problems (9:57) . Can you think of an easier way to set up and solve these problems?

Lesson Summary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction. $\frac{2}{3}$, $\frac{32 \text{ miles}}{1.4 \text{ gallons}}$, and $\frac{x}{13}$ are all ratios.

- **A proportion** is formed when two ratios are set equal to each other.

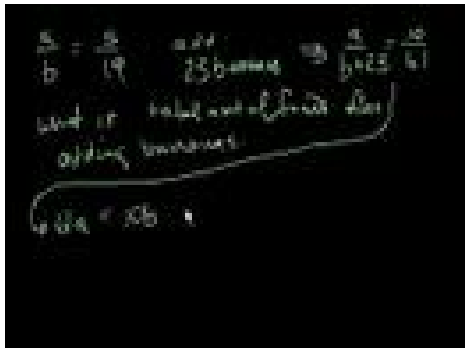
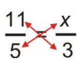


Figure 3.3: More advanced ratio problems (Watch on Youtube)

- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying
 results in $11 \cdot 3 = 5 \cdot x$.

Review Questions

1. Write the following comparisons as ratios. Simplify fractions where possible.
   (a) \$150 to \$3
   (b) 150 boys to 175 girls
   (c) 200 minutes to 1 hour
   (d) 10 days to 2 weeks
2. Write the following ratios as a unit rate.
   (a) 54 hotdogs to 12 minutes
   (b) 5000 lbs to 250 in^2
   (c) 20 computers to 80 students
   (d) 180 students to 6 teachers
   (e) 12 meters to 4 floors
   (f) 18 minutes to 15 appointments
3. Solve the following proportions.
   (a) $\frac{13}{6} = \frac{5}{x}$
   (b) $\frac{1.25}{7} = \frac{3.6}{x}$
   (c) $\frac{6}{19} = \frac{x}{11}$
   (d) $\frac{1}{x} = \frac{0.01}{5}$
   (e) $\frac{300}{4} = \frac{x}{99}$
   (f) $\frac{2.75}{9} = \frac{x}{(\frac{2}{9})}$
   (g) $\frac{1.3}{4} = \frac{x}{1.3}$
   (h) $\frac{0.1}{1.01} = \frac{1.9}{x}$
4. A restaurant serves 100 people per day and takes \$908. If the restaurant were to serve 250 people per day, what might the taking be?
5. The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert

is $\frac{220}{67}$ the height of Ben Nevis and $\frac{44}{48}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?

6. At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion owns a phone that is more than one year old?

Review Answers

1.
2. (a) $\frac{50}{1}$
   (b) $\frac{6}{7}$
   (c) $\frac{10}{3}$
   (d) $\frac{5}{7}$
3.
4. (a) 4.5 hot-dogs per minute
   (b) 20 lbs per in^2
   (c) 0.25 computers per student
   (d) 30 students per teacher
   (e) 3 meters per floor
   (f) 1.2 minutes per appointment
5.
6. (a) $x = \frac{30}{13}$
   (b) $x = 20.16$
   (c) $x = \frac{66}{19}$
   (d) $x = 500$
   (e) $x = 7425$
   (f) $x = \frac{11}{162}$
   (g) $x = 0.4225$
   (h) $x = \frac{100}{1919}$
7. $2270
8. 5960 meters .
9. $\frac{3}{10}$ or 30%

3.6 Scale and Indirect Measurement

Learning Objectives

- Use scale on a map.
- Solve problems using scale drawings.
- Use similar figures to measure indirectly.

Introduction

We are occasionally faced with having to make measurements of things that would be difficult to measure directly: the height of a tall tree, the width of a wide river, height of moon's craters, even the distance between two cities separated by mountainous terrain. In such circumstances, measurements can be made **indirectly**, using proportions and similar triangles. Such indirect methods link measurement with geometry and numbers. In this lesson, we will examine some of the methods for making indirect measurements.

Use Scale on a Map

A map is a two-dimensional, geometrically accurate representation of a section of the Earth's surface. Maps are used to show, pictorially, how various geographical features are arranged in a particular area. The **scale** of the map describes the relationship between distances on a map and the corresponding distances on the earth's surface. These measurements are expressed as a fraction or a ratio.

So far we have only written ratios as fractions, but outside of mathematics books, ratios are often written as two numbers separated by a colon (:). Here is a table that compares ratios written in two different ways.

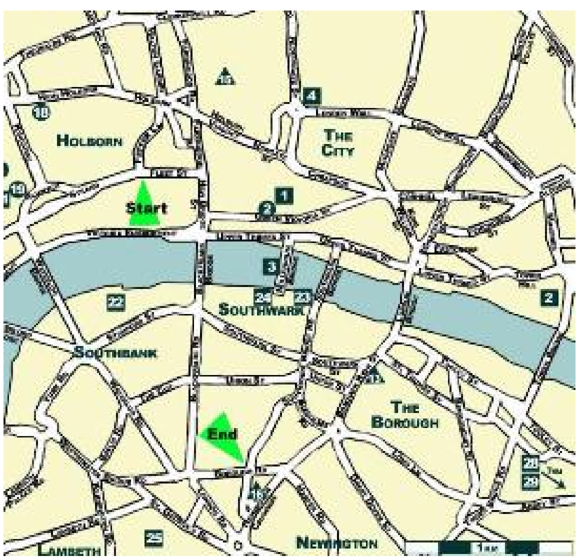
Table 3.1:

Ratio	Is Read As	Equivalent To
1 : 20	one to twenty	$\left(\frac{1}{20}\right)$
2 : 3	two to three	$\left(\frac{2}{3}\right)$
1 : 1000	one to one-thousand	$\left(\frac{1}{1000}\right)$

Look at the last row. In a map with a scale of 1 : 1000 ("one to one-thousand") one unit of measurement on the map (1 inch or 1 centimeter for example) would represent 1000 of the same units on the ground. A 1 : 1 (one to one) map would be a map as large as the area it shows!

Example 1

Anne is visiting a friend in London, and is using the map below to navigate from Fleet Street to Borough Road. She is using a $1 : 100,000$ *scale map, where* 1 cm on the map represents 1 km *in real life. Using a ruler, she measures the distance on the map as* 8.8 cm. *How far is the real distance from the start of her journey to the end?*



The scale is the ratio of distance on the map to the corresponding distance in real life.

$$\frac{\text{dist.on map}}{\text{real dist.}} = \frac{1}{100,000}$$

We can substitute the information we have to solve for the unknown.

$$\frac{8.8 \text{ cm}}{\text{real dist.}(x)} = \frac{1}{100,000} \qquad \text{Cross multiply.}$$

$$880000\text{cm} = x100 \qquad \text{cm} = 1 \text{ m.}$$

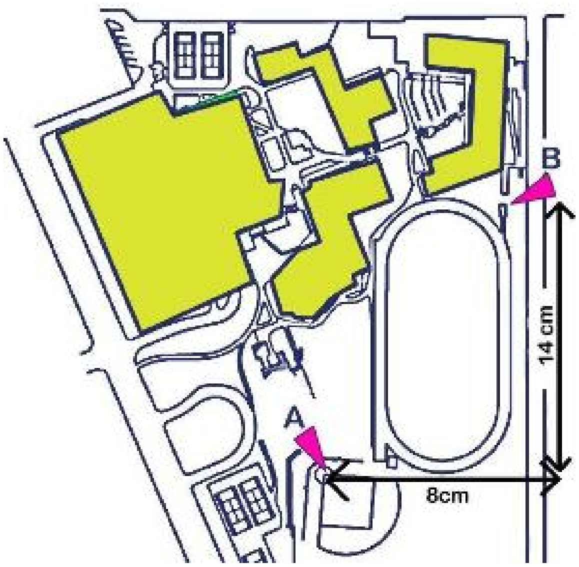
$$x = 8800 \text{ m} \qquad 1000 \text{ m} = 1 \text{ km.}$$

Solution

The distance from Fleet Street to Borough Road is 8.8 km.

We could, in this case, use our intuition: the 1 cm = 1 km scale indicates that we could simply use our reading in centimeters to give us our reading in km. Not all maps have a scale this simple. In general, you will need to refer to the map scale to convert between measurements on the map and distances in real life!

Example 2



Antonio is drawing a map of his school for a project in math. He has drawn out the following map of the school buildings and the surrounding area.

He is trying to determine the scale of his figure. He knows that the distance from the point marked **A** *on the baseball diamond to the point marked* **B** *on the athletics track is* 183 meters. *Use the dimensions marked on the drawing to determine the scale of his map.*

We know that the real-life distance is 183 m. To determine the scale we use the ratio:

$$\text{Scale} = \frac{\text{distance on map}}{\text{distance in real life}}$$

To find the distance on the map, we will use Pythagoras' Theorem $a^2 + b^2 = c^2$.

$$(\text{Distance})^2 = 8^2 + 14^2$$

$$(\text{Distance})^2 = 64 + 196$$

$$(\text{Distance})^2 = 260$$

$$\text{Distance} = \sqrt{260} = 16.12 \text{ cm}$$

$$\text{Scale} = \frac{16.12 \text{ cm}}{\text{real dist.}}$$

$$\text{Scale} = \frac{16.12 \text{ cm}}{183 \text{ m}} \qquad 1 \text{ m} = 100 \text{ cm.}$$

$$\text{Scale} = \frac{16.12 \text{ cm}}{18300 \text{ cm}} \qquad \text{Divide top and bottom by 16.12.}$$

$$\text{Scale} \approx \frac{1}{1135.23} \qquad \text{Round to two significant figures :}$$

Solution

The scale of Antonio's map is approximately $1 : 1100$.

Solve Problems Using Scale Drawings

Another visual use of ratio and proportion is in **scale drawings**. Scale drawings are used extensively by architects (and often called **plans**). They are used to represent real objects and are drawn to a specific ratio. The equations governing scale are the same as for maps. We will restate the equations in forms where we can solve for **scale, real distance**, or **scaled distance**.

$$\text{Scale} = \frac{\text{distance on diagram}}{\text{distance in real life}}$$

Rearrange to find the distance on the diagram and the distance in real life.

$$(\text{distance on diagram}) = (\text{distance in real life}) \times (\text{scale})$$

$$(\text{Distance in real life}) = \frac{\text{distance on diagram}}{\text{scale}} = (\text{distance on diagram}) \cdot \left(\frac{1}{\text{scale}}\right)$$

Example 3

Oscar is trying to make a scale drawing of the Titanic, which he knows was 883 feet *long. He would like his drawing to be* $1 : 500$ scale. How long, in inches, must the paper that he uses be?

We can reason intuitively that since the scale is $1 : 500$ that the paper must be $\frac{883}{500} = 1.766$ feet long.

Converting to inches gives the length at $12(1.766) \text{ in} = 21.192 \text{ in}$.

Solution

Oscar's paper should be at least 22 inches long.

Example 4

The Rose Bowl stadium in Pasadena California measures 880 feet *from north to south and* 695 feet *from east to west. A scale diagram of the stadium is to be made. If* 1 *inch represents* 100 feet, *what would be the dimensions of the stadium drawn on a sheet of paper? Will if fit on a standard (U.S.) sheet of paper* $(8.5 \text{ in} \times 11 \text{ in})$*?*

We will use the following relationship.

$$\text{(distance on diagram)} = \text{(distance in real life)} \times \text{(scale)}$$

$$\text{Scale} = 1 \text{ inch to } 100 \text{ feet} = \left(\frac{1 \text{ inch}}{100 \text{ feet}}\right)$$

$$\text{Width on paper} = 880 \text{ feet} \times \left(\frac{1 \text{ inch}}{100 \text{ feet}}\right) = 8.8 \text{ inches}$$

$$\text{Height on paper} = 695 \text{ feet} \times \left(\frac{1 \text{ inch}}{100 \text{ feet}}\right) = 6.95 \text{ inches}$$

Solution

The dimensions of the scale diagram would be 8.8 in $\times$ 6.95 in. Yes, this will fit on a 8.5 in $\times$ 11 in sheet of paper.

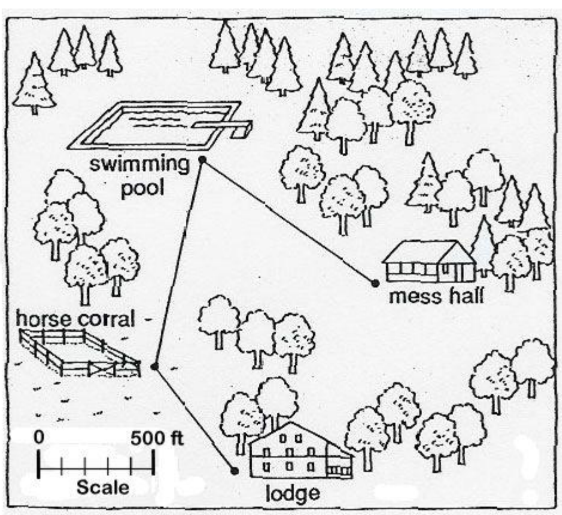
Example 5

The scale drawing below is sent to kids who attend Summer Camp. Use the scale to estimate the following:

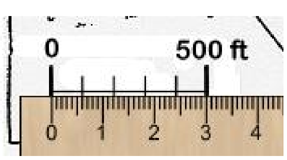
a) The distance from the mess hall to the swimming pool via the path shown.

b) The distance from the lodge to the swimming pool via the horse corral.

c) The ***direct*** *distance from the mess hall to the lodge*



To proceed with this problem, we need a **ruler**. It does not matter whether we use a ruler marked in inches or centimeters, but a centimeter scale is easier, as it is marked in **tenths**. For this example, the ruler used will be a centimeter ruler.



We first need to convert the scale on the diagram into something we can use. Often the scale will be stated on the diagram but it is always worth checking, as the diagram may have been enlarged or reduced from its original size. Here we see that 500 feet on the diagram is equivalent to 3.0 cm on the ruler. The scale we will use is therefore 3 cm = 500 ft. We can write this as a ratio.

$$\text{Scale} = \left(\frac{3 \text{ cm}}{500 \text{ ft}}\right) \qquad \text{Do not worry about canceling units this time!}$$

a) We are now ready to move to the next step. Measuring distances on the diagram. First, we need to know the distance from the mess hall to the swimming pool. We measure the distance with our ruler. We find that the distance is 5.6 cm. We divide this by the scale to find the real distance.

$$\frac{\text{disance on diagram}}{\text{scale}} = \frac{5.6 \text{ cm}}{\left(\frac{3 \text{ cm}}{500 ft}\right)} = 5.6 \text{ cm} \cdot \left(\frac{500 \text{ ft}}{3\text{cm}}\right)$$

Multiply this out. Note that the centimeter units will cancel leaving the answer in **feet**.

Solution

The distance from the mess hall to the swimming pool is approximately <u>930 feet</u> (rounded to the nearest 10 feet).

b)To find the distance from the lodge to the swimming pool, we have to measure two paths. The first is the distance from the lodge to the horse corral. This is found to be 3.4 cm.

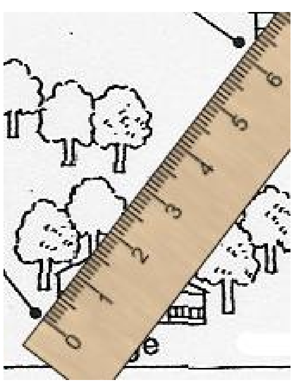
The distance from the corral to the swimming pool is 5.5 cm.

The total distance on the diagram is $(3.4 + 5.5) = 8.9$ cm.

$$\text{Distance in real life} = \frac{\text{disance on diagram}}{\text{scale}} \approx 8.9 \text{ cm} \left(\frac{500 \text{ ft}}{3 \text{ cm}}\right)$$

Solution

The distance from the lodge to the pool is approximately 1480 feet .



c) To find the **direct** distance from the lodge to the mess hall, we simply use the ruler to measure the distance from one point to the other. We do not have to go round the paths in this case.

$$\text{Distance on diagram} = 6.2 \text{ cm}$$
$$\text{Distance in real life} = \frac{\text{distance on diagram}}{\text{scale}} \approx 6.2 \text{ cm} \cdot \left(\frac{500 \text{ ft}}{3 \text{ cm}}\right)$$

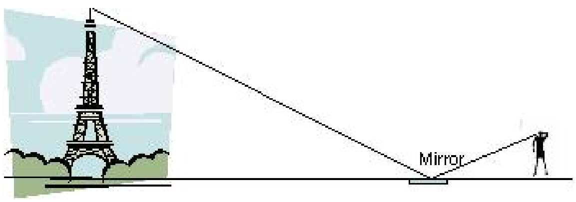
Solution

The distance from the lodge to the mess hall is approximately 1030 feet .

Use Similar Figures to Measure Indirectly

Similar figures are often used to make indirect measurements. Two shapes are said to be **similar** if they are the same shape but one is an enlarged (or reduced) version of the other. Similar triangles have the same angles, and are said to be "in proportion." The ratio of every measurable length in one figure to the corresponding length in the other is the same. **Similar triangles** crop up often in indirect measurement.

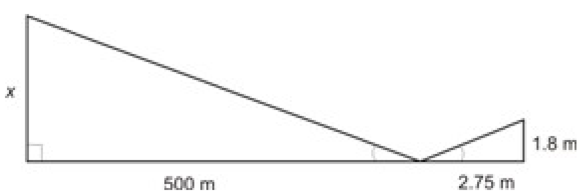
Example 6



Anatole is visiting Paris, and wants to know the height of the Eiffel Tower. Unable to speak French, he decides to measure it in three ways.

1. *He measures out a point* 500 meters *from the base of the tower, and places a small mirror flat on the ground.*
2. *He stands behind the mirror in such a spot that standing upright he sees the top of the tower reflected in the mirror.*
3. *He measures both the distance from the spot where he stands to the mirror* (2.75 meters) *and the height of his eyes from the ground* (1.8 meters) *.*

Explain how he is able to determine the height of the Eiffel Tower from these numbers and determine what that height is.



First, we will draw and label a scale diagram of the situation.

A fact about refection is that the angle that the light reflects off the mirror is the same as the angle that it hits the mirror.

Both triangles are right triangles, and both have one other angle in common. That means that all three angles in the large triangle match the angles in the smaller triangle. We say the triangles are **similar**: exactly the same shape, but enlarged or reduced.

- This means that the ratio of the long leg in the large triangle to the length of the long leg in the small triangle is the **same ratio** as the length of the short leg in the large triangle to the length of the short leg in the small triangle.

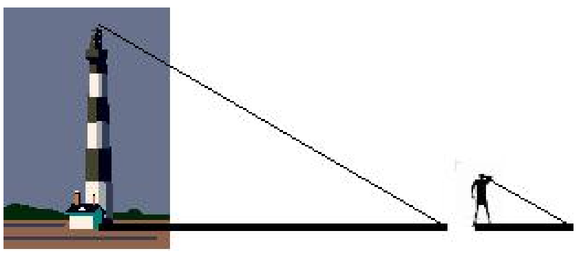
$$\frac{500\text{m}}{2.75\text{m}} = \frac{x}{1.8\text{m}}$$
$$1.8 \cdot \frac{500}{2.75} = \frac{x}{1.8} \cdot 1.8$$
$$327.3 = x$$

Solution

The Eiffel Tower, according to this calculation, is approximately 327.3 meters high.

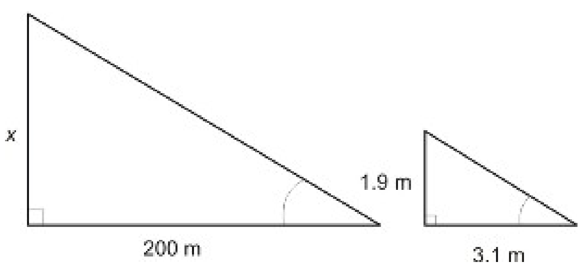
Example 7

Bernard is looking at a lighthouse and wondering how high it is. He notices that it casts a long shadow, which he measures at 200 meters *long. At the same time he measures his own shadow at* 3.1 meters *long. Bernard is* 1.9 meters *tall. How tall is the lighthouse?*



We will again draw a scale diagram:

Again, we see that we have two right triangles. The angle that the sun causes the shadow from the lighthouse to fall is the same angle that Bernard shadow falls. We have two similar triangles, so we can again say that the ratio of the corresponding sides is the same.



$$\frac{200\text{m}}{3.1\text{m}} = \frac{x}{1.9\text{m}}$$
$$1.9 \cdot \frac{200}{3.1} = \frac{x}{1.9} \cdot 1.9$$
$$122.6 = x$$

Solution

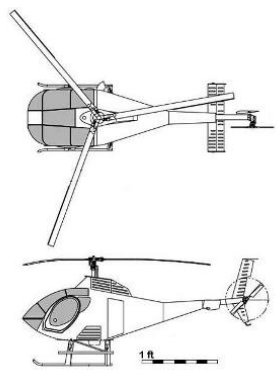
The lighthouse is 122.6 meters tall.

Lesson Summary

- **Scale** is a proportion that relates map distance to real life distance. scale $= \frac{\text{distance on map}}{\text{distance in real life}}$
- Two shapes, like triangles, are said to be **similar** if they have the same angles. The sides of similar triangles are **in proportion**. The ratio of every measurable length in one triangle to the corresponding length in the other is the same.

Review Questions

1. Use the map in Example One. Using the scale printed on the map, determine the distances (rounded to the nearest half km) between:
   (a) Points 1 and 4
   (b) Points 22 and 25
   (c) Points 18 and 13
   (d) Tower Bridge and London Bridge
2. The scale diagram in Example Five does **not** show the buildings themselves in correct proportion. Use the scale to estimate:
   (a) The real length the indicated pool would be if it *was* drawn in proportion.
   (b) The real height the lodge would be if it *was* drawn in proportion.
   (c) The length a 50 ft pool on the diagram.
   (d) The height a 20 ft high tree would be on the diagram.



3. Use the scale diagram to the right to determine:
   (a) The length of the helicopter (cabin to tail)
   (b) The height of the helicopter (floor to rotors)
   (c) The length of one main rotor
   (d) The width of the cabin
   (e) the diameter of the rear rotor system
4. On a sunny morning, the shadow of the Empire State Building is 600 feet long. At the same time, the shadow of a yardstick (3 feet long) is 1 foot, $5\frac{1}{4}$ inches . How high is the Empire State building?

Review Answers

1.
2. (a) 3 km
   (b) 7 km
   (c) $12\frac{1}{2}$ km
   (d) $4\frac{1}{2}$ km
3.
4. (a) Pool = 600 feet long
   (b) Lodge = 250 feet high
   (c) Pool should be 0.3 cm
   (d) Tree should be 0.12 cm
5.
6. (a) length = 21 ft

(b) height = 10 ft
(c) Main rotor = 12 ft
(d) cabin width = $5\frac{1}{2}$ ft
(e) rotor diameter = 4 ft

7. 1250 ft

3.7 Percent Problems

Learning Objectives

- Find a percent of a number.
- Use the percent equation.
- Find the percent of change.

Introduction

A **percent** is simply a ratio with a base unit of 100. When we write a ratio as a fraction, the base unit is the denominator. Whatever percentage we want to represent is the number on the numerator. For example, the following ratios and percents are equivalent.

Table 3.2:

Fraction	Percent	Fraction	Percent
$\left(\frac{50}{100}\right)$	50%	$\left(\frac{50}{1000}\right) = \left(\frac{0.5}{100}\right)$	0.5%
$\left(\frac{10}{100}\right)$	10%	$\left(\frac{1}{25}\right) = \left(\frac{4}{100}\right)$	4%
$\left(\frac{99}{100}\right)$	99%	$\left(\frac{3}{5}\right) = \left(\frac{60}{100}\right)$	60%
$\left(\frac{125}{100}\right)$	12.5%	$\left(\frac{1}{10{,}000}\right) = \left(\frac{0.01}{100}\right)$	0.01%

Fractions are easily converted to decimals, just as fractions with denominators of $10,\ 100,\ 1000,\ 10000$ are converted to decimals. When we wish to convert a percent to a decimal, we divide by 100, or simply move the decimal point two units to the left.

Table 3.3:

Percent	Decimal	Percent	Decimal	Percent	Decimal
10%	0.1	0.05%	.0005	0%	0
99%	0.99	0.25%	.0025	100%	1

Find a Percent of a Number

One thing we need to do before we work with percents is to practice converting between fractions, decimals and percentages. We will start by converting decimals to percents.

Example 1

Express 0.2 *as a percent.*

The word percent means "for every hundred". Therefore, to find the percent, we want to change the decimal to a fraction with a denominator of 100. For the decimal 0.2 we know the following is true:

$$0.2 = 0.2 \times 100 \times \left(\frac{1}{100}\right) \qquad \text{Since } 100 \times \left(\frac{1}{100}\right) = 1$$
$$0.2 = 20 \times \left(\frac{1}{100}\right)$$
$$0.2 = \left(\frac{20}{100}\right) = 20\%$$

Solution

$$0.2 = 20\%$$

We can take any number and multiply it by $100 \times \frac{1}{100}$ without changing that number. This is the key to converting numbers to percents.

Example 2

Express 0.07 as a percent.

$$0.07 = 0.07 \times 100 \times \left(\frac{1}{100}\right)$$
$$0.07 = 7 \times \left(\frac{1}{100}\right)$$
$$0.07 = \left(\frac{7}{100}\right) = 7\%$$

Solution

$$0.07 = 7\%$$

It is a simple process to convert percentages to decimals. Just remember that a percent is a ratio with a base (or denominator) of 100.

Example 3

Express 35% *as a decimal.*

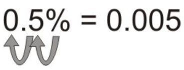
$$35\% = \left(\frac{35}{100}\right) = 0.35$$

Example 4

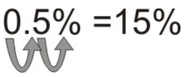
Express 0.5% *as a decimal.*

$$0.5\% = \left(\frac{0.5}{100}\right) = \left(\frac{5}{1000}\right) = 0.005$$

In practice, it is often easier to convert a percent to a decimal by moving the decimal point two spaces to the left.



The same trick works when converting a decimal to a percentage, just shift the decimal point two spaces to the right instead.



When converting fractions to percents, we can substitute $\frac{x}{100}$ for $x\%$, where x is the unknown percentage we can solve for.

Example 5

Express $\frac{3}{5}$ *as a percent.*

We start by representing the unknown as $x\%$ or $\frac{x}{100}$.

$$\begin{aligned}
\left(\frac{3}{5}\right) &= \frac{x}{100} && \text{Cross multiply.} \\
5x &= 100 \cdot 3 && \text{Divide both sides by 5 to solve for x.} \\
5x &= 300 \\
x &= \frac{300}{5} = 60
\end{aligned}$$

Solution

$$\left(\frac{3}{5}\right) = 60\%$$

Example 6

Express $\frac{13}{40}$ *as a percent.*

Again, represent the unknown percent as $\frac{x}{100}$, cross-multiply, and solve for x.

$$\begin{aligned}
\frac{13}{40} &= \frac{x}{100} \\
40x &= 1300 \\
x &= \frac{1300}{40} = 32.5
\end{aligned}$$

Solution

$$\left(\frac{13}{40}\right) = 32.5\%$$

Converting percentages to simplified fractions is a case of writing the percentage ratio with all numbers written as prime factors:

Example 7

Express 28% *as a simplified fraction.*

First write as a ratio, and convert numbers to prime factors.

$$28\%\left(\frac{28}{100}\right)=\left(\frac{2\cdot2\cdot7}{5\cdot5\cdot2\cdot2\cdot}\right)$$

Now cancel factors that appear on both numerator and denominator.

$$\left(\frac{\not{2}.\not{2}.7}{\not{2}.\not{2}.5.5}\right)=\frac{7}{25}$$

Solution

$$28\%=\left(\frac{7}{25}\right)$$

Multimedia Link The following video shows several more examples of finding percents and might be useful for reinforcing the procedure of finding the percent of a number. Khan Academy Taking Percentages (9:55)

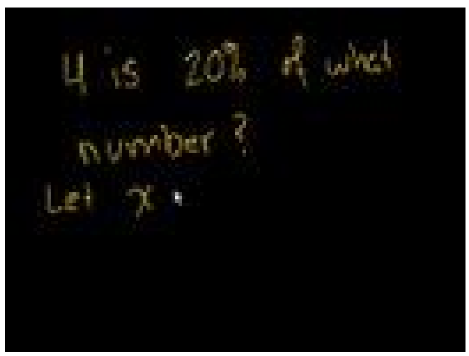


Figure 3.4: Taking a percentage of a number. (Watch on Youtube)

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Use the Percent Equation

The percent equation is often used to solve problems.

Percent Equation: Rate × Total = Part or "R% of Total is Part"

Rate is the ratio that the percent represents (***R***% in the second version).

Total is often called the ***base unit***.

Part is the amount we are comparing with the base unit.

Example 8

Find 25% *of* \$80

Use the percent equation. We are looking for the ***part.*** The ***total*** is \$80. 'of' means multiply. ***R***% is 25% so the ***rate*** is $\frac{25}{100}$ or 0.25.

$$0.25 \cdot \$80 = \$20$$

Solution

25% of \$80 is \$20.

Remember, to convert a percent to a decimal, you just need to move the decimal point two places to the left!

Example 9

Find 17% *of* \$93

Use the percent equation. We are looking for the ***part***. The ***total*** is \$93. ***R***% is 17% so the ***rate*** is 0.17.

$$0.17 \cdot 93 = 15.81$$

Solution

17% of \$93 is \$15.81.

Example 10

Express \$90 as a percentage of \$160.

Use the percent equation. This time we are looking for the ***rate.*** We are given the ***part*** (\$90) and the ***total*** (\$160). We will substitute in the given values.

$$\text{Rate} \times 160 = 90 \qquad \text{Divide both sides by 160}$$

$$\text{Rate} = \left(\frac{90}{160}\right) = 0.5625 = 0.5625\left(\frac{100}{100}\right) = \frac{56.25}{100}$$

Solution

\$90 is 56.25% of 160.

Example 11

\$50 *is* 15% *of what total sum?*

Use the percent equation. This time we are looking for the ***total.*** We are given the ***part*** (\$50) and the ***rate*** (15% or 0.15). The total is our unknown in dollars, or x. We will substitute in these given values.

$$0.15x = 50 \qquad \text{Solve for x by dividing both sides by 0.15.}$$

$$x = \frac{50}{0.15} \approx 333.33$$

Solution

\$50 is 15% of \$333.33.

Find Percent of Change

A useful way to express changes in quantities is through percents. You have probably seen signs such as "20% extra free", or "save 35% today." When we use percents to represent a change, we generally use the formula.

$$\text{Percent change} = \left(\frac{\text{final amount} - \text{original amount}}{\text{original amount}}\right) \times 100\%$$

Or

$$\frac{\text{percent change}}{100} = \left(\frac{\text{actual change}}{\text{original amount}}\right)$$

A **positive** percent change would thus be an **increase,** while a **negative** change would be a **decrease.**

Example 12

A school of 500 *students is expecting a* 20% *increase in students next year. How many students will the school have?*

The percent change is +20. It is positive because it is an **increase.** The original amount is 500. We will show the calculation using both versions of the above equation. First we will substitute into the first formula.

$$\text{Percent change} = \left(\frac{\text{final amount} - \text{original amount}}{\text{original amount}}\right) \times 100\%$$

$$20\% = \left(\frac{\text{final amount} - 500}{500}\right) \times 100\% \qquad \text{Divide both sides by 100\%.}$$

$$\text{Let x = final amount.}$$

$$0.2 = \frac{x - 500}{500} \qquad \text{Multiply both sides by 500.}$$

$$100 = x - 500 \qquad \text{Add 500 to both sides.}$$

$$600 = x$$

Solution

The school will have 600 students next year.

Example 13

A \$150 *mp3 player is on sale for* 30% *off. What is the price of the player?*

The percent change is given, as is the original amount. We will substitute in these values to find the final amount in dollars (our unknown x). Note that a decrease means the change is **negative.** We will use the first equation.

$$\text{Percent change} = \left(\frac{\text{final amount} - \text{original amount}}{\text{original amount}}\right) \times 100\%$$

$$\left(\frac{x - 150}{150}\right) \cdot 100\% = -30\% \qquad \text{Divide both sides by100\%.}$$

$$\left(\frac{x - 150}{150}\right) = \frac{30\%}{100\%} = -0.3\% \qquad \text{Multiply both sides by150.}$$

$$x - 150 = 150(-0.3) = -45 \qquad \text{Add 150 to both sides.}$$

$$x = -45 + 150$$

Solution

The mp3 player is on sale for \$105.

We can also substitute straight into the second equation and solve for the change y.

$$\frac{\text{percent change}}{100} = \left(\frac{\text{actual change}}{\text{original amount}}\right)$$

$$\frac{-30}{100} = \frac{y}{150} \qquad \text{Multiply both sides by 150.}$$

$$150(-0.3) = y$$

$$y = -45$$

Solution

Since the actual change is -45(\$), the final price is \$150 – \$45 = \$105.

A **mark-up** is an increase from the price a store pays for an item from its supplier to the retail price it charges to the public. For example, a 100% mark-up (commonly known in business as *keystone*) means that the price is doubled. Half of the retail price covers the cost of the item from the supplier, half is profit.

Example 14 – Mark-up

A furniture store places a 30% mark-up on everything it sells. It offers its employees a 20% discount from the sales price. The employees are demanding a 25% discount, saying that the store would still make a profit. The manager says that at a 25% discount from the sales price would cause the store to lose money. Who is right?

We will consider this problem two ways. First, let us consider an item that the store buys from its supplier for \$1000.

Item price \$1000	
Mark-up \$300	$(30\% \text{ of } 1000 = 0.30 \cdot 1000 = 300)$
Final retail price \$1300	

So a \$1000 item would retail for \$1300. \$300 is the profit available to the store. Now, let us consider discounts.

Retail Price	\$1300
20% discount	$0.20 \times \$1300 = \$260$
25% discount	$0.25 \times \$1300 = \$325$

So with a 20% discount, employees pay \$1300 – \$260 = \$1040

With a 25% discount, employees pay \$1300 – \$325 = \$975

With a 20% discount, employees pay \$40 more than the cost of the item.

At a 25% discount they pay \$975, which is \$25 less than the cost.

Finally, we will work algebraically. Consider an item whose wholesale price is x.

$$\begin{aligned}
\text{Mark-up} &= 0.3x \\
\text{Final retail price} &= 1.3x \\
\text{Price at 20\% discount} &= 0.80 \times 1.3x = 1.04x \\
\text{Price at 25\% discount} &= 0.75 \times 1.3x = 0.975x
\end{aligned}$$

Solution

The manager is right. A 20% discount from retail means the store makes around 4% profit. At a 25% discount, the store has a 2.5% loss.

Solve Real-World Problems Using Percents

Example 15

In 2004 *the* **US Department of Agriculture** *had* 112071 *employees, of which* 87846 *were Caucasian. Of the remaining minorities, African-American and Hispanic employees had the two largest demographic groups, with* 11754 *and* 6899 *employees respectively.* *

a) Calculate the total percentage of minority (non-Caucasian) employees at the USDA.
b) Calculate the percentage of African-American employees at the USDA.
c) Calculate the percentage of minority employees who were neither African-American nor Hispanic.

a) Use the percent equation $\text{Rate} \times \text{Total} = \text{Part}$.

The ***total*** number of employees is 112071. We know that the number of Caucasian employees is 87846, which means that there must be $(112071 - 87846) = 24225$ non-Caucasian employees. This is the ***part***.

$$\begin{aligned}
\text{Rate} \times 112071 &= 24225 && \text{Divide both sides by 112071.} \\
\text{Rate} &= 0.216 && \text{Multiply by 100 to obtain percent :} \\
\text{Rate} &= 21.6\%
\end{aligned}$$

Solution

21.6% of USDA employees in 2004 were from minority groups.

b) $\text{Total} = 112071$ $\text{Part} = 11754$

$$\begin{aligned}
\text{Rate} \times 112071 &= 11754 && \text{Divide both sides by 112071.} \\
\text{Rate} &= 0.105 && \text{Multiply by 100 to obtain percent :} \\
\text{Rate} &= 10.5\%
\end{aligned}$$

Solution

10.5% of USDA employees in 2004 were African-American.

c) We now know there are 24225 non-Caucasian employees. This is now our ***total***. That means there must be $(24225 - 11754 - 6899) = 5572$ minority employees who are neither African-American nor Hispanic. The part is 5572.

$$\text{Rate} \times 24225 = 5572 \qquad \text{Divide both sides by } 24225$$
$$\text{Rate} = 0.230 \qquad \text{Multiply by 100 to obtain percent.}$$
$$\text{Rate} = 23\%$$

Solution

23% of USDA minority employees in 2004 were neither African-American nor Hispanic.

Example 16

In 1995 *New York had* 18136000 *residents. There were* 827025 *reported crimes, of which* 152683 *were violent. By* 2005 *the population was* 19254630 *and there were* 85839 *violent crimes out of a total of* 491829 *reported crimes. Calculate the percentage change from* 1995 *to* 2005 *in:* †

a) Population of New York
b) Total reported crimes
c) violent crimes

This is a percentage change problem. Remember the formula for percentage change.

$$\text{Percent change} = \left(\frac{\text{final amount} - \text{original amount}}{\text{original amount}}\right) \times 100\%$$

In these cases, the final amount is the 2005 statistic. The initial amount is the 1995 statistic.

a) Population:

$$\text{Percent change} = \left(\frac{19,254,630 - 18,136,000}{18,136,000}\right) \times 100\%$$
$$\text{Percent change} = \left(\frac{1,118,630}{18,136,000}\right) \times 100\%$$
$$\text{Percent change} = 0.0617 \times 100$$
$$\text{Percent change} = 6.17\%$$

Solution

The population grew by 6.17%.

b) Total reported crimes

$$\text{Percent change} = \left(\frac{491,829 - 827,025}{827,025}\right) \times 100\%$$
$$\text{Percent change} = \left(\frac{-335,196}{827,025}\right) \times 100\%$$
$$\text{Percent change} = -0.4053 \times 100$$
$$\text{Percent change} = -40.53\%$$

Solution

The total number of reported crimes fell by 40.53%.

c) Violent crimes

$$\text{Percent change} = \left(\frac{85,839 - 152,683}{152,683}\right) \times 100\%$$
$$\text{Percent change} = \left(\frac{-66,844}{152,683}\right) \times 100\%$$
$$\text{Percent change} = -0.4377 \times 100$$
$$\text{Percent change} = -43.77\%$$

Solution

The total number of reported crimes fell by 43.77%. † Source: New York Law Enforcement Agency Uniform Crime Reports

Lesson Summary

- A **percent** is simply a ratio with a base unit of 100, i.e. $13\% = \frac{13}{100}$.
- The **percent equation** is: Rate × Total = Part or "R% of Total is Part".
- Percent change $= \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100$. A **positive** percent change means the value **increased,** while a **negative** percent change means the value **decreased.**

Review Questions

1. Express the following decimals as a percent.
   (a) 0.011
   (b) 0.001
   (c) 0.91
   (d) 1.75
   (e) 20
2. Express the following fractions as a percent (round to two decimal places when necessary).
   (a) $\frac{1}{6}$
   (b) $\frac{5}{24}$
   (c) $\frac{6}{7}$
   (d) $\frac{11}{7}$
   (e) $\frac{13}{97}$
3. Express the following percentages as a reduced fraction.
   (a) 11%
   (b) 65%
   (c) 16%
   (d) 12.5%
   (e) 87.5%
4. Find the following.
   (a) 30% of 90
   (b) 16.7% of 199
   (c) 11.5% of 10.01
   (d) y% of $3x$
5. A TV is advertised on sale. It is 35% off and has a new price of $195. What was the pre-sale price?

6. An employee at a store is currently paid $9.50 per hour. If she works a full year she gets a 12% pay rise. What will her new hourly rate be after the raise?
7. Store A and Store B both sell bikes, and both buy bikes from the same supplier at the same prices. Store A has a 40% mark-up for their prices, while store B has a 250% mark-up. Store B has a permanent sale and will always sell at 60% off those prices. Which store offers the better deal?

Review Answers

1.
2. (a) 1.1%
   (b) 0.1%
   (c) 91%
   (d) 175%
   (e) 2000%
3.
4. (a) 16.67%
   (b) 20.83%
   (c) 85.71%
   (d) 157.14%
   (e) −13.40%
5.
6. (a) $\frac{11}{100}$
   (b) $\frac{13}{20}$
   (c) $\frac{4}{25)}$
   (d) $\frac{1}{8}$
   (e) $\frac{7}{8}$
7.
8. (a) 27
   (b) 33.233
   (c) −1.15115
   (d) $\frac{3xy}{100}$
9. $300
10. $10.64
11. Both stores' final sale prices are identical.

3.8 Problem Solving Strategies: Use a Formula

Learning Objectives

- Read and understand given problem situations.
- Develop and apply the strategy: use a formula.
- Plan and compare alternative approaches to solving problems.

Introduction

In this chapter, we have been solving problems in which quantities vary directly with one another other. In this section, we will look at few examples of ratios and percents occurring in real-world problems. We will follow the **Problem Solving Plan.**

Step 1 Understand the problem

Read the problem carefully. Once you have read the problem, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2 Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation or formula.

Step 3 Carry out the plan – Solve

This is where you solve the formula you came up with in Step 2.

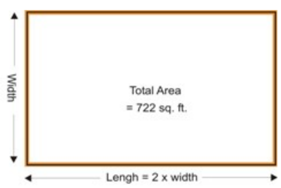
Step 4 Look – Check and Interpret

Check to see if you used all your information and that the answer makes sense.

It is important that you first know what you are looking for when solving problems in mathematics. Math problems often require that you extract information and use it in a definite **procedure**. You must collect the appropriate information and use it (using a strategy or strategies) to solve the problem Many times, you will be writing out an equation which will enable you to find the answer.

Example 1

An architect is designing a room that is going to be twice as long as it is wide. The total square footage of the room is going to be 722 *square feet. What are the dimensions in feet of the room?*



Step 1 Collect Relevant Information.

$$\begin{aligned}\text{Width of room} &= \text{unknown} = x\\ \text{Length of room} &= 2 \times \text{width}\\ \text{Area of room} &= 722 \text{ square feet}\end{aligned}$$

Step 2 Make an Equation

$$\begin{aligned}\text{Length of room} &= 2x\\ \text{Area of room} &= x \times 2x = 2x^2\\ 2x^2 &= 722\end{aligned}$$

Step 3 Solve

$$\begin{aligned}2x^2 &= 722 && \text{Divide both sides by 2}\\ x^2 &= 361 && \text{Take the 'square root' of both sides.}\\ x &= \sqrt{361} = 19\\ 2x &= 2 \times 19 = 38\end{aligned}$$

Solution

The dimensions of the room are 19 feet by 38 feet .

Step 4 Check Your Answer

Is 38 twice 19?

$$2 \times 19 = 38 \qquad \text{TRUE} \ - \ \text{This checks out.}$$

Is 38 times 19 equal to 722?

$$38 \times 19 = 722 \qquad \text{TRUE} - \text{This checks out}$$

The answer checks out.

Example 2

A passenger jet initially climbs at 2000 *feet per minute after take-off from an airport at sea level. At the four minute mark this rate slows to* 500 feet *per minute. How many minutes pass before the jet is at* 20000 feet *?*

Step 1

$$\begin{aligned}
\text{Initial climb rate} &= \frac{2000 \text{ feet}}{1 \text{ minute}} \\
\text{Initial climb time} &= 4 \text{ minutes} \\
\text{Final climb rate} &= \frac{500 \text{ feet}}{1 \text{ minute}} \\
\text{Final climb time} &= \text{unknown} = x \\
\text{Final altitude} &= 20000 \text{ feet}
\end{aligned}$$

The first two pieces on information can be combined. Here is the result.

Height at four minute mark $= 4 \ \text{ minutes } \cdot \frac{2000 \ feet}{1 \ minute} = 8000 \ feet.$

Step 2 Write an equation.

Since we know that the height at four minutes is 8000 feet , we need to find the time taken to climb the final $(20000 - 8000) = 12000$ feet .

We will use $\text{ distance } = \text{ speed } \times \text{ time}$ to give us an equation for time.

$$time = \frac{\text{distance}}{\text{speed}} = \text{distance } \cdot \left(\frac{1}{\text{speed}}\right)$$

Step 3 Solve.

$$\begin{aligned}
x &= 12,0\not{0}\not{0} \text{ feet} \cdot \left(\frac{1 \text{ minute}}{5\not{0}\not{0} \text{ feet}}\right) && \text{Note that the units of feet will also cancel} \\
x &= 24 \text{ minutes} \\
\text{total time} &= x + 4
\end{aligned}$$

Solution

The time taken to reach 20000 feet is 28 minutes .

Step 4 check your answer

What is 4 times 2000?

$$4 \times 2000 = 8000 \qquad \text{The initial climb is through 8000 feet.}$$

What is 24 times 500?

$$24 \times 500 = 12000 \qquad \text{The second part of the climb is through 8000 feet.}$$

The **total climb** = initial climb + secondary climb = $(8000 + 12000) = 20000$ feet .

The answer checks out.

Example 3

The time taken for a moving body to travel a given distance is given by **time** $= \frac{distance}{speed}$*. The speed of sound in air is approximately* 340 *meters per second. In water, sound travels much faster at around* 1500 *meters per second. A small meteor hits the ocean surface* 10 km *away. What would be the delay in seconds between the sound heard after traveling through the air and the same sound traveling through the ocean?*

Step 1 We will write out the most important information.

$$\begin{aligned}
\text{Distance} &= 10,000 \text{ meters}\\
\text{Speed in air} &= \frac{340 \text{ meters}}{1 \text{ second}}\\
\text{Speed in water} &= \frac{1500 \text{ meters}}{1 \text{ second}}\\
\text{Time through air} &= \text{unknown} x\\
\text{Time through water} &= \text{unknown} y\\
\text{Delay} &= x - y
\end{aligned}$$

Step 2 We will convert this information into equations.

Time in air $x = 10,000 \ \text{ meters} \cdot \frac{1 \ second}{340 \ meters}$

Time in water $y = 10,000 \ \text{ meters} \cdot \frac{1 \ second}{15000 \ meters}$

Step 3 Solve for x, y and the delay.

$$\begin{aligned}
x &= 29.41 \text{ seconds}\\
y &= 6.67 \text{ seconds}\\
\text{Delay} = x - y &= (29.41 - 6.67) \text{ seconds}
\end{aligned}$$

Solution

The delay between the two sound waves arriving is 22.7 seconds.

Step 4 Check that the answer works.

We need to think of a different way to explain the concept.

The **actual time** that the sound takes in air is 29.41 seconds . In that time, it crosses the following distance.

$$\text{Distance} = \text{speed} \times \text{time} = 340 \times 29.41 = 9999 \text{ meters}$$

The **actual time** that the sound takes in water is 6.67 seconds . In that time, it crosses the following distance of.

$$\text{Distance} = \text{speed}times\text{time} = 1500 \times 6.67 = 10005 \text{ meters}$$

Both results are close to the 10000 meters that we know the sound traveled. The slight error comes from rounding our answer.

The answer checks out.

Example 4:

Deandra is looking over her paycheck. Her boss took tax from her earnings at a rate of 15%*. A deduction to cover health insurance took one-twelfth of what was left. Deandra always saves one-third of what she gets paid after all the deductions. If Deandra worked* 16 *hours at* \$7.50 *per hour, how much will she save this week?*

Step 1 Collect relevant information.

Deductions:

$$\begin{aligned}
\text{Tax} &= 15\% = 0.15 \\
\text{Health} &= \frac{1}{12} \\
\text{Savings} &= \frac{1}{3} \\
\text{Hours} &= 16 \\
\text{Rate} &= \$7.50 \text{ per hour} \\
\text{Savings amount} &= \text{unknown} x
\end{aligned}$$

Step 2 Write an equation.

Deandra's earnings **before deductions** $= 16 \times \$7.50 = \$120$

Fraction remaining after tax $= 1 - 0.15 = 0.85$

Fraction remaining after health $= 0.85\left(1 - \frac{1}{12}\right) = 0.85\left(\frac{11}{12}\right) \approx 0.85 \cdot 0.91667 \approx 0.779167$

Fraction to be saved $= \frac{1}{3} \cdot 0.779167 \approx 0.25972$

Step 3 Solve

Amount to save $= 0.25972 \cdot \$120 = \$31.1664$ *Round to two decimal places.*

Solution

Deandra saves \$31.17.

Step 4 Check your answer by working backwards.

If Deandra saves \$31.17, then her take-home pay was $3 \times \$31.17 = \$93.51$

If Deandra was paid \$93.51, then before health deductions health she had $\$93.51 \cdot \frac{12}{11} = \$102.01$

If Deandra had \$102.01 after tax, then before tax she had $\$102.01 \cdot \frac{100}{85} = \$120.01$

If Deandra earned \$120.01 at \$7.50 per hour, then she worked for $\frac{\$102.01}{\$7.50} = 16.002$ hours

This is extremely close to the hours we know she worked (the difference comes from the fact we rounded to the nearest penny).

The answer checks out.

Lesson Summary

The four steps of the **Problem Solving Plan** are:

1. **Understand the problem**
2. **Devise a plan – Translate**
3. **Carry out the plan – Solve**
4. **Look – Check and Interpret**

Review Questions

Use the information in the problems to create and solve an equation.

1. Patricia is building a sandbox for her daughter. It is to be five feet wide and eight feet long. She wants the height of the sand box to be four inches above the height of the sand. She has 30 cubic feet of sand. How high should the sand box be?
2. A 500 sheet stack of copy paper is 1.75 inches high. The paper tray on a commercial copy machine holds a two foot high stack of paper. Approximately how many sheets is this?
3. It was sale day in Macy's and everything was 20% less than the regular price. Peter bought a pair of shoes, and using a coupon, got an additional 10% off the discounted price. The price he paid for the shoes was \$36. How much did the shoes cost originally?
4. Peter is planning to show a video file to the school at graduation, but is worried that the distance that the audience sits from the speakers will cause the sound and the picture to be out of sync. If the audience sits 20 meters from the speakers, what is the delay between the picture and the sound? (The speed of sound in air is 340 meters per second).
5. Rosa has saved all year and wishes to spend the money she has on new clothes and a vacation. She will spend 30% more on the vacation than on clothes. If she saved \$1000 in total, how much money (to the nearest whole dollar) can she spend on the vacation?
6. On a DVD, data is stored between a radius of 2.3 cm and 5.7 cm. Calculate the total area available for data storage in square *cm*.
7. If a Blu-ray ^{TM} DVD stores 25 gigabytes (GB), what is the ***storage density***, in GB per square cm?

Review Answers

1. 13 inches
2. Approximately 6860 sheets
3. \$50
4. 0.06 seconds

5. Approximately $565
6. 85.45 cm^2
7. 0.293 GB/cm^2

Chapter 4

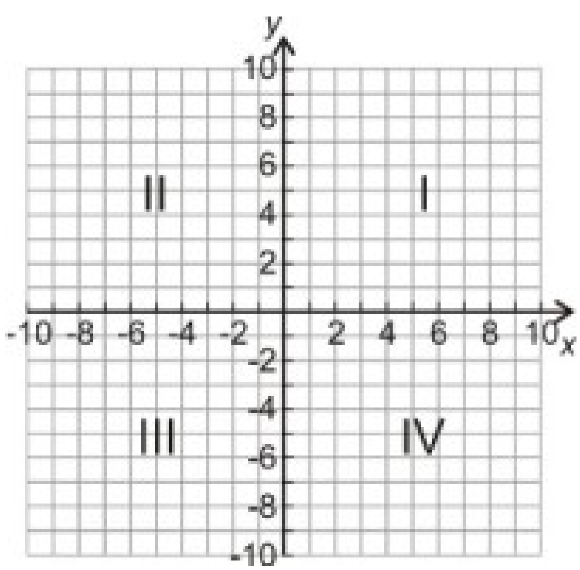
Graphs of Equations and Functions

4.1 The Coordinate Plane

Learning Objectives

- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction



We now make our transition from a number line that stretches in only one dimension (left to right) to something that exists in two dimensions. The **coordinate plane** can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**. The first quadrant (I) contains all the positive numbers from both axes. It is the top right quadrant. The other quadrants are numbered sequentially (II, III, IV) moving counter–clockwise from the first.

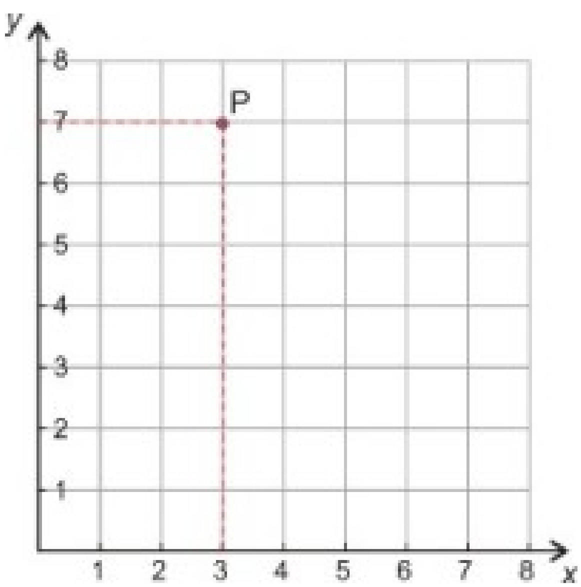
Identify Coordinates of Points

When given a point on a coordinate plane, it is a relatively easy task to determine its **coordinates**. The coordinates of a point are two numbers – written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parenthesis, with the $x-$**coordinate** (also called the **ordinate**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1,7)$	An ordered pair with an x-value of one and a y-value of seven
$(0,5)$	– an ordered pair with an x-value of zero and a y-value of five
$(-2.5,4)$	An ordered pair with an x-value of -2.5 and a y-value of four
$(-107.2,-.005)$	An ordered pair with an x-value of -107.2 and a y-value of -0.005.

The first thing to do is realize that identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram to the right.

Imagine you are standing at the origin (the points where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** x direction).

The $x-$coordinate of P is $+3$.

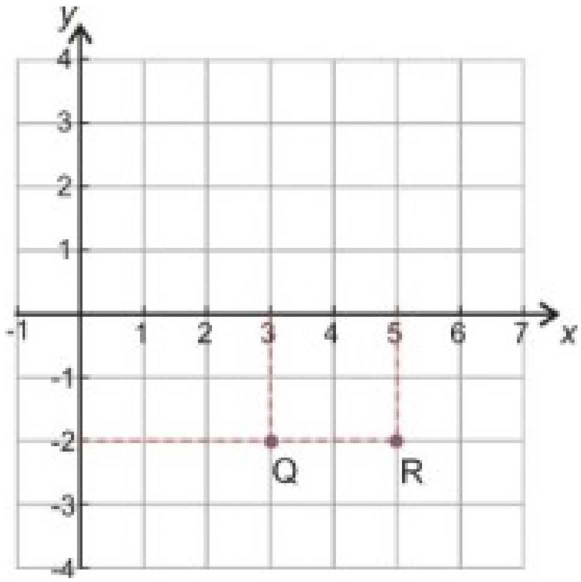
Now if you were standing at the three marker on the $x-$axis, point P would be 7 units above you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is $+7$.

Solution

The coordinates of point P are $(3,7)$.

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

In order to get to Q we move three units to the right, in the positive–x direction, then two units **down**. This time we are moving in the **negative** y direction. The x coordinate of Q is $+3$, the y coordinate of Q is -2.

The coordinates of R are found in a similar way. The x coordinate is $+5$ (five units in positive x) and the $y-$coordinate is again -2.

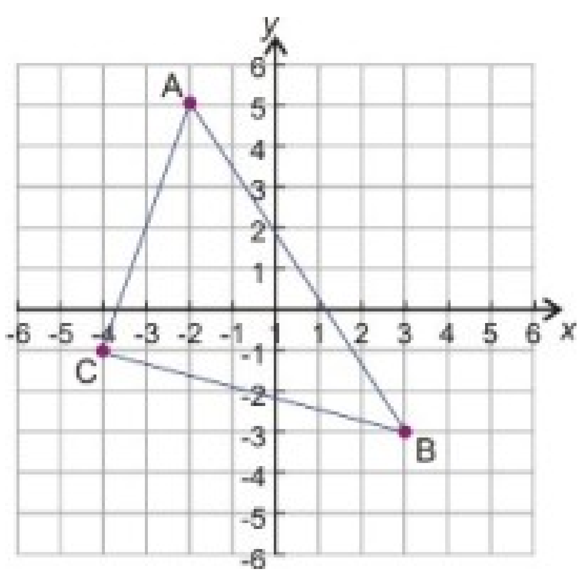
Solution

$Q\ (3,-2)$

$R\ (5,-2)$

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B and C.



Point A:

$x-$coordinate $= -2$

$y-$coordinate $= +5$

Point B:

$x-$coordinate $= +3$

$y-$coordinate $= -3$

Point C:

$x-$coordinate $= -4$

$y-$coordinate $= -1$

Solution

$$A(-2,5)$$
$$B(3,-3)$$
$$C(-4,-1)$$

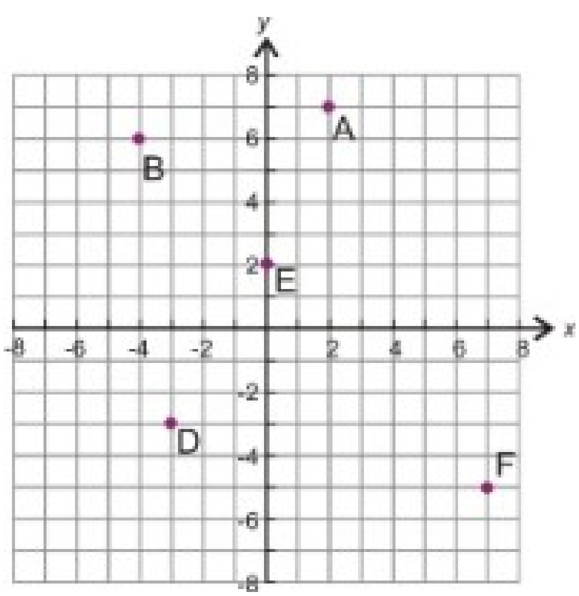
Plot Points in a Coordinate Plane

Plotting points is a simple matter once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$A(2,7)$ $B(-5,6)$ $C(-6,0)$ $D(-3,-3)$ $E(0,2)$ $F(7,-5)$



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-5,6)$ is 5 units left, 6 units up. It is in Quadrant II.

Point $C(-6,0)$ is 6 units left, 0 units up. It is **on the x axis**.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

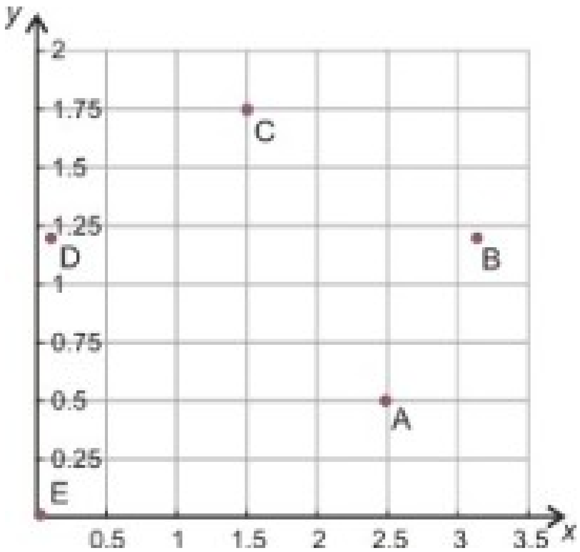
Point $E(0,2)$ is 2 units up from the origin. It is **on the y axis**.

Point $F(7, -5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$A(2.5, 0.5)$ $\quad$ $B(\pi, 1.2)$ $\quad$ $C(2, 1.75)$ $\quad$ $D(0.1, 1.2)$ $\quad$ $E(0, 0)$



Choice of axes is always important. In Example Four, it was important to have all four quadrants visible. In this case, all the coordinates are positive. There is no need to show the negative values of x or y. Also, there are no x values bigger than about 3.14, and 1.75 is the largest value of y. We will therefore show these points on the following scale $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$. The points are plotted to the right.

Here are some important points to note about this graph.

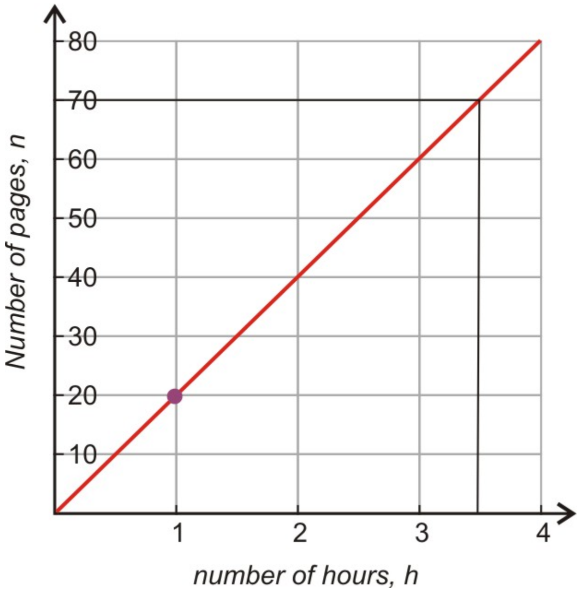
- The tick marks on the axes do not correspond to unit increments (i.e. the numbers do not go up by one).
- The scale on the $x-$axis is different than the scale on the $y-$axis.
- The scale is **chosen** to maximize the clarity of the plotted points.

Graph a Function Given a Table

Once we know how to plot points on a coordinate plane, we can think about how we would go about plotting a relationship between x and y values. So far we have been plotting sets of ordered pairs. This is called a **relation,** and there isn't necessarily a relationship between the x values and y values. In a relation, the set of x values is called the **domain** and the set of y values is called the **range.** If there is a relationship between the x and y values, and each x value corresponds to exactly one y value, then the relation is called a *function.* Remember that a function is a particular way to relate one quantity to another. If you read a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either:

$$m = 20 \cdot h \qquad n = \text{number of pages; } h = \text{time measured in hours. OR...}$$

$$h = \frac{n}{20}$$



So you could use the **function** that related n and h to determine how many pages you could read in $3\frac{1}{2}$ hours, or even to find out how long it took you to read forty-six pages. The graph of this function is shown right, and you can see that if we plot number of pages against number of hours, then we can simply read off the number of pages that you could read in 3.5 hours as seventy pages. You can see that in a similar way it would be possible to estimate how long it would take to read forty-six pages, though the time that was obtained might only be an approximation.

Generally, the graph of a **function** appears as a line or curve that goes through all points that satisfy the relationship that the function describes. If the domain of the function is all real numbers, then we call this a **continuous function.** However, if the domain of the function is a particular set of values (such as whole numbers), then it is called a **discrete function.** The graph will be a series of dots that fall along a line or curve.

In graphing equations, we assume the domain is all real numbers, unless otherwise stated. Often times though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. Sometimes the graph is still shown as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

Example 6

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

Table 4.1:

Number of Friends	Number of Presents
0	4
1	5
2	6

Table 4.1: (continued)

Number of Friends	Number of Presents
3	7
4	8
5	9

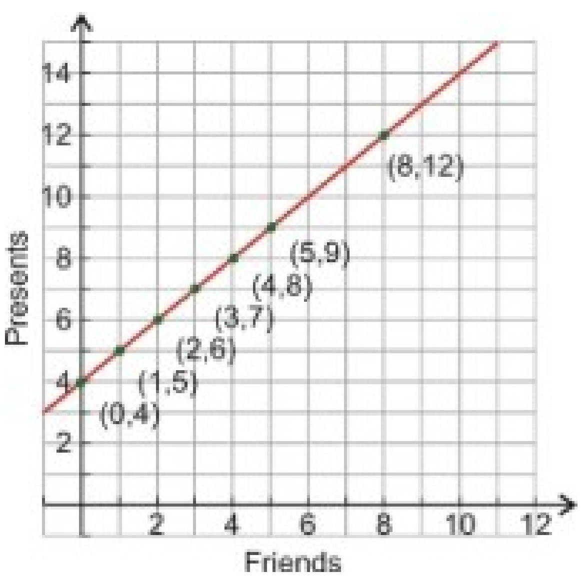
The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependant variable is. Clearly in this case, the number of friends can vary **independently** (the domain). The number of presents must **depend** on the number of friends who show up (the range).

We will therefore plot friends on the $x-$axis and presents on the $y-$axis. Let's add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

Table 4.2:

No. of friends (x)	no. of presents (y)	coordinates (x, y)
0	4	$(0,4)$
1	5	$(1,5)$
2	6	$(2,6)$
3	7	$(3,7)$
4	8	$(4,8)$
5	9	$(5,9)$

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we do not need to worry about anything other than Quadrant I. We need to choose a suitable scale for the x and y axes. We need to consider no more than eight friends (look again at the question to confirm this), but it always pays to allow a **little** extra room on your graph. We also need the y scale to accommodate the presents for eight people. We can see that this is still going to be under 20!



The scale of the graph on the right shows room for up to 12 friends and 15 presents. This will be fine, but there are many other scales that would be equally as good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lay on a straight line, so the function that describes the relationship between x and y will be **linear**. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a **discrete** problem since Sarah can only invite a whole numbers of friends. For instance, it would be impossible for 2.4 friends to show up. Keep in mind that the only permissible points for the function are those points on the line which have integer x and y values.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah would get if eight friends come to her party. Don't forget that x represents the number of friends and y represents the number of presents. If we look at $x = 8$ we can see that the function has a y value of 12.

Solution

If 8 friends show up, Sarah will receive a total of 12 presents.

Graph a Function Given a Rule

If we are given a rule instead of a table, we can proceed to graph the function in one of two ways. We will use the following example to show each way.

Example 7

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number. Then double it. Then add five to what he got. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the result of applying the rule. He wrote his rule in the form of an equation.

$$y = 2x + 5$$

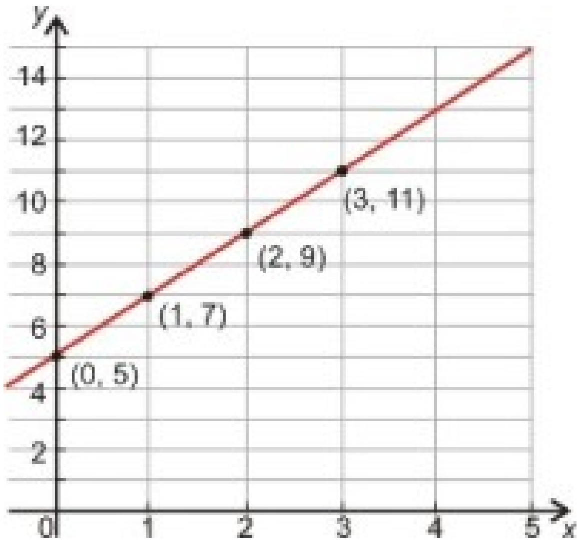
Help him visualize what is going on by graphing the function that this rule describes.

Method One – Construct a Table of Values

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few x, y pairs. We will use $0, 1, 2$ and 3 for x values.

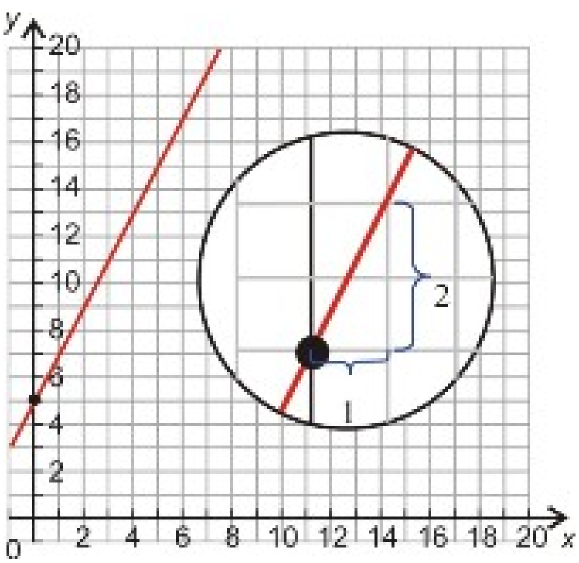
x	y
0	$2 \cdot 0 + 5 = 0 + 5 = 5$
1	$2 \cdot 1 + 5 = 2 + 5 = 7$
2	$2 \cdot 2 + 5 = 4 + 5 = 9$
3	$2 \cdot 3 + 5 = 6 + 5 = 11$

Next, we plot the points and join them with our line.



This method is nice and simple. Plus, with linear relationships there is no need to plot more than two or three points. In this case, the function is continuous because the domain (the number Ali is asked to think of) is all real numbers, even though he may only be thinking of positive whole numbers.

Method Two – Intercept and Slope



One other way to graph this function (and one that we will learn in more detail in the next lesson) is the **slope–intercept method.** To do this, follow the following steps:

1. Find the y value when $x = 0$.

$y(0) = 2 \cdot 0 + 5 = 5$ *So our y–intercept is* $(0, 5)$

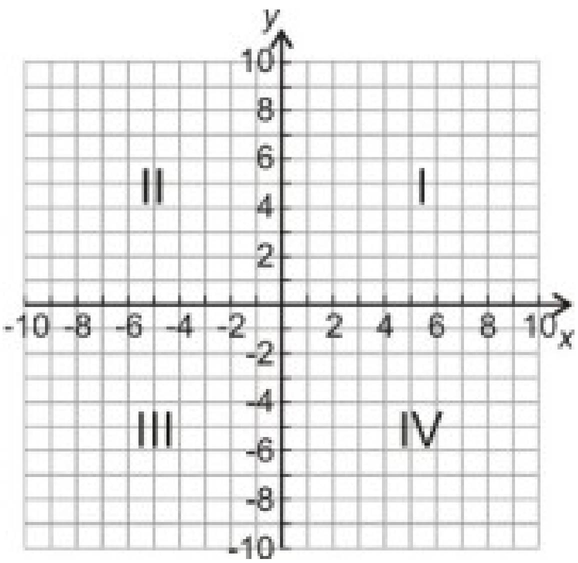
2. Look at the coefficient multiplying the x.

Every time we increase x by one, y increases by two so our slope is $+2$.

3. Plot the line with the given **slope** that goes through the **intercept.** We start at the point $(0, 5)$ and move over one in the x direction, then up two in the y direction. This gives the slope for our line, which we extend in both directions.

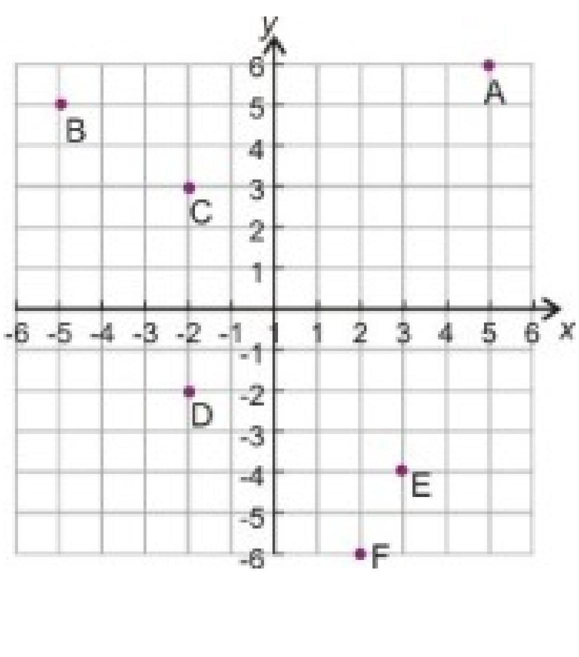
We will properly examine this last method in the next lesson!

Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram at right.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate.** Here is an exaxmple (x, y).
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Review Questions



1. Identify the coordinates of each point, $A - F$, on the graph to the right.
2. Plot the following points on a graph and identify which quadrant each point lies in:

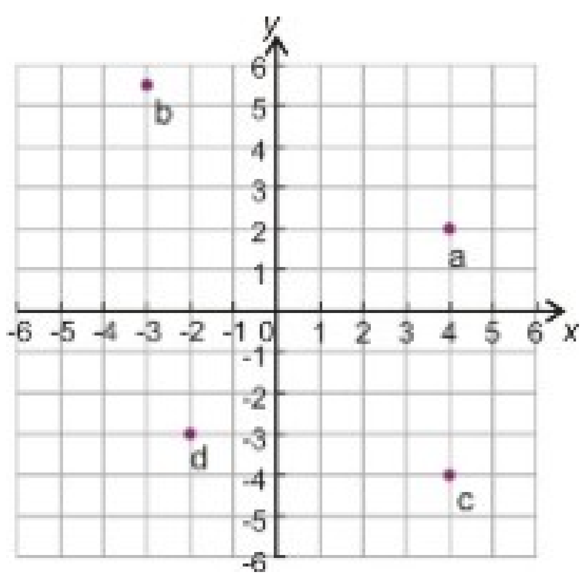
   (a) $(4, 2)$
   (b) $(-3, 5.5)$
   (c) $(4, -4)$
   (d) $(-2, -3)$

3. The following three points are three vertices of square $ABCD$. Plot them on a graph then determine what the coordinates of the fourth point, D, would be. Plot that point and label it. A $(-4, -4)$
   B $(3, -4)$
   C $(3, 3)$
4. Becky has a large bag of M&Ms that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three M&Ms in return. If x is the number of Starburst that Jaeyun gives Becky, and y is the number of M&Ms he gets in return then complete each of the following.

   (a) Write an algebraic rule for y in terms of x
   (b) Make a table of values for y with x values of $0, 1, 2, 3, 4, 5$.
   (c) Plot the function linking x *and* y on the following scale $0 \leq x \leq 10, 0 \leq y \leq 10$.

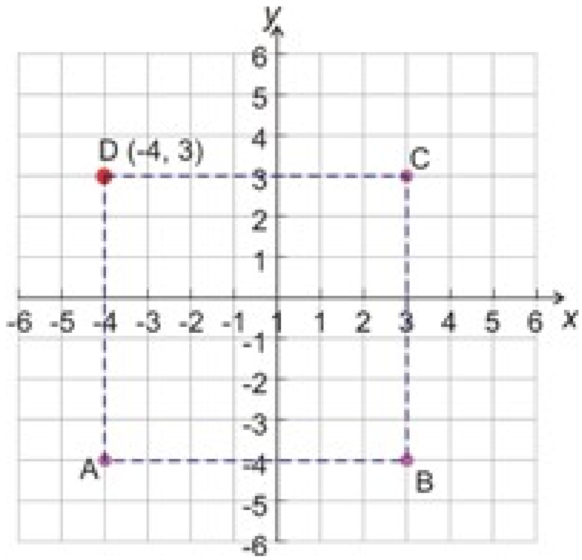
Review Answers

1. $A(5, 6)$ $B(-5, 5)$ $C(-2, 3)$ $D(-2, -2)$ $E(3, -4)$ $F(2, -6)$
2.



   (a) Quadrant I
   (b) Quadrant II
   (c) Quadrant IV
   (d) Quadrant III

3.



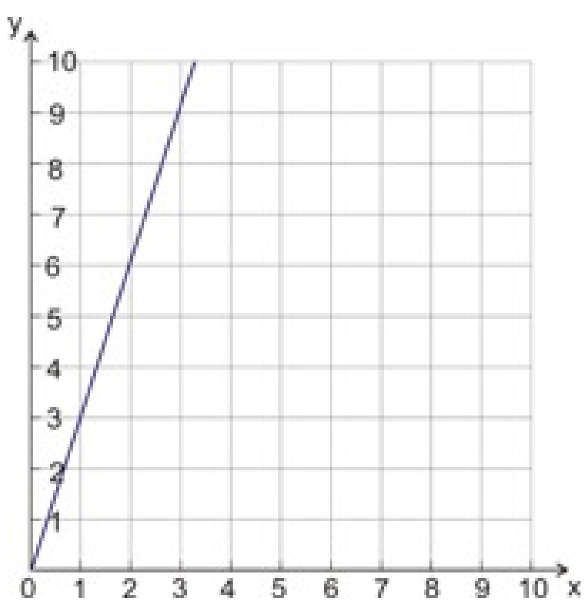
4.

(b)

5. (a) $y = 3x$

x	y
0	0
1	3
2	6
3	9
4	12
5	15

(c)



4.2 Graphs of Linear Equations

Learning Objectives

- Graph a linear function using an equation.
- Write equations and graph horizontal and vertical lines.
- Analyze graphs of linear functions and read conversion graphs.

Graph a Linear Equation

At the end of Lesson 4.1 we looked at graphing a function from a rule. A rule is a way of writing the relationship between the two quantities we are graphing. In mathematics, we tend to use the words **formula** and **equation** to describe what we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that you will use frequently in math.

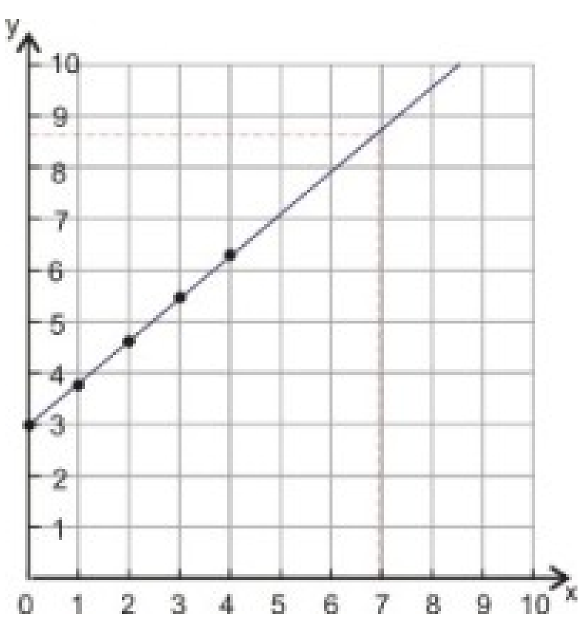
Example 1

A taxi fare costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges \$3 *as a set fee and* \$0.80 *per mile traveled. Here is the equation linking the cost in dollars (y) to hire a taxi and the distance traveled in miles (x).*

$$y = 0.8x + 3$$

Graph the equation and use your graph to estimate the cost of a seven mile taxi ride.

We will start by making a table of values. We will take a few values for x $0, 1, 2, 3, 4$, find the corresponding y values and then plot them. Since the question asks us to find the cost for a seven mile journey, we will choose a scale that will accommodate this.



x	y
0	3
1	3.8
2	4.6
3	5.4
4	6.2

The graph is shown to the right. To find the cost of a seven mile journey we first locate $x = 7$ on the horizontal axis and draw a line up to our graph. Next we draw a horizontal line across to the y axis and read where it hits. It appears to hit around half way between $y = 8$ and $y = 9$. Let's say it is 8.5.

Solution

A seven mile taxi ride would cost approximately \$8.50 (\$8.60 exactly).

There are a few interesting points that you should notice about this graph and the formula that generated it.

- The graph is a straight line (this means that the equation is **linear**), although the function is **discrete** and will graph as a series of points.
- The graph crosses the $y-$axis at $y = 3$ (look at the equation – you will see a +3 in there!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by 0.8 squares (look at the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by 3×0.8 squares.

Example 2

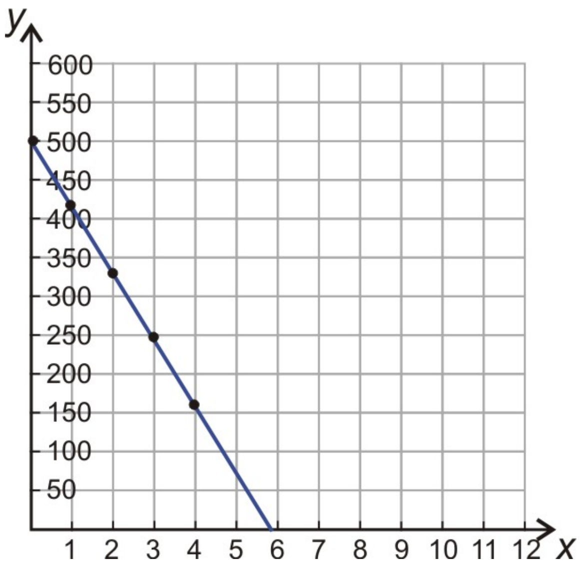
A small business has a debt of \$500000 *incurred from start-up costs. It predicts that it can pay off the debt at a rate of* \$85000 *per year according to the following equation governing years in business (x) and debt measured in thousands of dollars(y).*

$$y = -85x + 500$$

Graph the above equation and use your graph to predict when the debt will be fully paid.

First, we start with our table of values. We plug in $x-$values and calculate our corresponding $y-$values.

x	y
0	500
1	415
2	330
3	245
4	160



Then we plot our points and draw the line that goes through them.

Take note of the scale that has been chosen. There is no need to have any points above $y = 500$, but it is still wise to allow a little extra.

We need to determine how many years (the x value) that it takes the debt (y value) to reach zero. We know that it is greater than four (since at $x = 4$ the y value is still positive), so we need an x scale that goes well past $x = 4$. In this case the x value runs from 0 to 12, though there are plenty of other choices that would work well.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 0$ (the x axis). It looks as if the line hits pretty close to $x = 6$. So the debt will definitely be paid off in six years.

Solution

The debt will be paid off in six years.

Multimedia Link To see more simple examples of graphing linear equations by hand see the video Khan Academy Graphing Lines 1 (9:49) . The narrator models graphing several linear equations using a table of

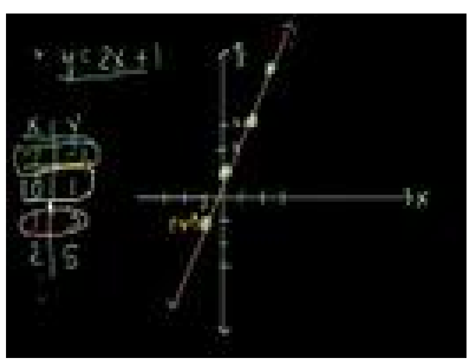


Figure 4.1: Graphing linear equations (Watch on Youtube)

values to plot points and then connecting the points with a line. This reinforces the procedure of graphing lines by hand.

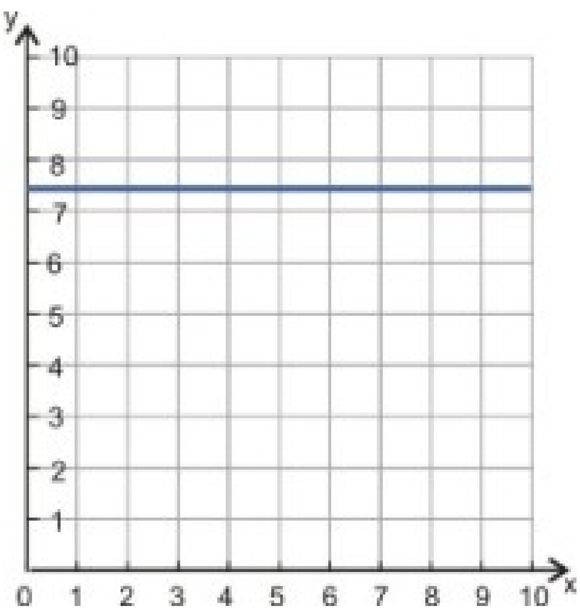
Example 3

"Mad-cabs" have an unusual offer going on. They are charging \$7.50 *for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) to the length of the journey in miles (x).*

To proceed, the first thing we need is an **equation.** You can see from the problem that the cost of a journey does not depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. In fact, any value of x results in the same value of y (7.5). Here is the equation.

$$y = 7.5$$

The graph of this function is shown to the right. You can see that the graph $y = 7.5$ is simply a horizontal line.



Any time you see an equation of the form $y =$ constant then the graph is a horizontal line that intercepts the $y-$axis at the value of the constant.

Similarly, when you see an equation of the form $x =$ constant then the graph is a vertical line that intercepts the $x-$axis at the value of the constant. Notice that this a relation, and not a function because each x value (there's only one in this case) corresponds to many (actually an infinite number) y values.

Example 4

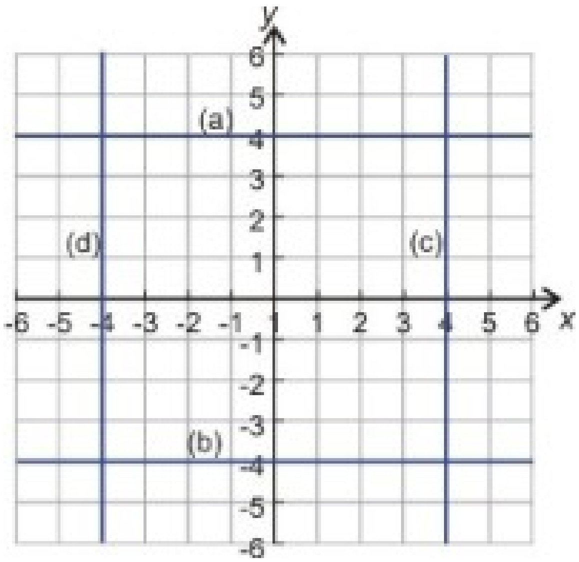
Plot the following graphs.

(a) $y = 4$

(b) $y = -4$

(c) $x = 4$

(d) $x = -4$



(a) $y = 4$ is a horizontal line that crosses the $y-$axis at 4

(b) $y = -4$ is a horizontal line that crosses the $y-$axis at -4

(c) $x = 4$ is a vertical line that crosses the $x-$axis at 4

(d) $x = -4$ is a vertical line that crosses the $x-$axis at -4

Example 5

Find an equation for the $x-$axis and the $y-$axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 0$ and $y = 0$). The following definition could easily work for each axis.

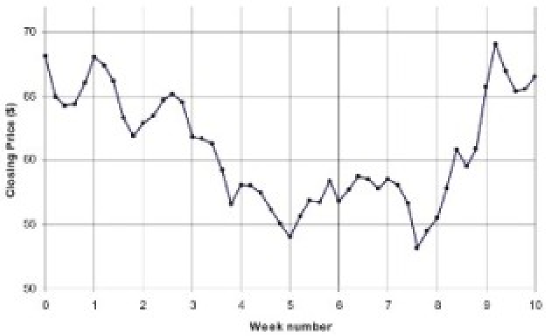
$x-$axis: *A horizontal line crossing the $y-$axis at zero.*

$y-$axis: *A vertical line crossing the $x-$axis at zero.*

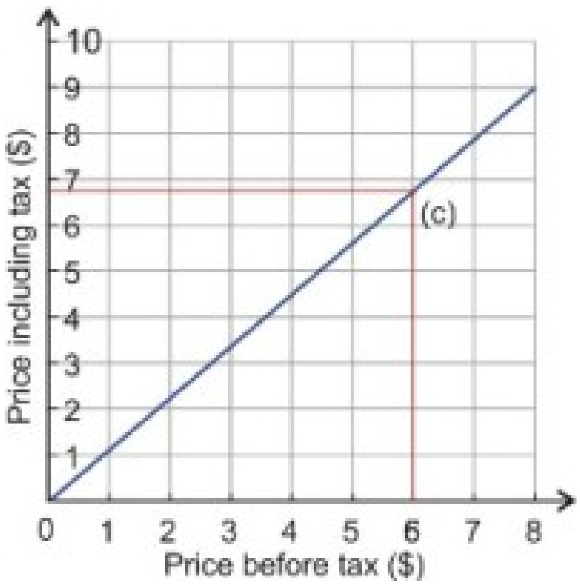
So using example 3 as our guide, we could define the $x-$axis as the line $y = 0$ and the $y-$axis as the line $x = 0$.

Analyze Graphs of Linear Functions

We often use line graphs to represent relationships between two linked quantities. It is a useful skill to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about $68, and at the end of the third week it was at about $62. You may also see that in the first five weeks it lost about 20% of its value and that it made about 20% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected for ease of interpretation.



Analyzing line graphs is a part of life – whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many of these graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to the complicated tasks, like reading the graph above. In this section, we will look at reading information from simple linear conversion graphs.



Example 6

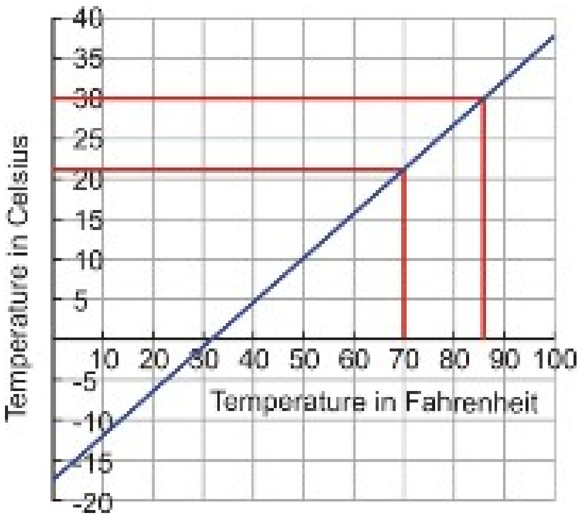
The graph shown at the right shows a chart for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost inclusive of sales tax for a $6.00 *pen in the store.*

To find the relevant price with tax we find the correct pre-tax price on the $x-$axis. This is the point $x = 6$.

Draw the line $x = 6$ up until it meets the function, then draw a horizontal line to the $y-$axis. This line hits at $y \approx 6.75$ (about three fourths of the way from $y = 6$ to $y = 7$).

Solution

The approximate cost including tax is $6.75



Example 7

The chart for converting temperature from Fahrenheit to Celsius is shown to the right. Use the graph to convert the following:

1. 70° *Fahrenheit to Celsius*
2. 0° *Fahrenheit to Celsius*
3. 30° *Celsius to Fahrenheit*
4. 0° *Celsius to Fahrenheit*

1. To find 70° Fahrenheit we look along the Fahrenheit-axis (in other words the $x-$axis) and draw the line $x = 70$ up to the function. We then draw a horizontal line to the Celsius-axis ($y-$axis). The horizontal line hits the axis at a little over 20 (21 or 22).

Solution

70° Fahrenheit is approximately equivalent to 21° Celsius

2. To find 0° Fahrenheit, we are actually looking at the $y-$axis. Don't forget that this axis is simply the line $x = 0$. We just look to see where the line hits the $y-$axis. It hits just below the half way point between -15 and -20.

Solution: 0° Fahrenheit is approximately equivalent to $-18°$ Celsius .

3. To find 30° Celsius, we look up the Celsius-axis and draw the line $y = 30$ along to the function. When this horizontal line hits the function, draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 85.

Solution

30° Celsius is approximately equivalent to 85° Fahrenheit.

4. To find 0° Celsius we are looking at the Fahrenheit-axis (the line $y = 0$). We just look to see where the function hits the $x-$axis. It hits just right of 30.

Solution

0° Celsius is equivalent to 32° Fahrenheit.

Lesson Summary

- Equations with the variables y and x can be graphed by making a chart of values that fit the equation and then plotting the values on a coordinate plane. This graph is simply another representation of the equation and can be analyzed to solve problems.

- Horizontal lines are defined by the equation $y =$ constant and vertical lines are defined by the equation $x =$ constant.

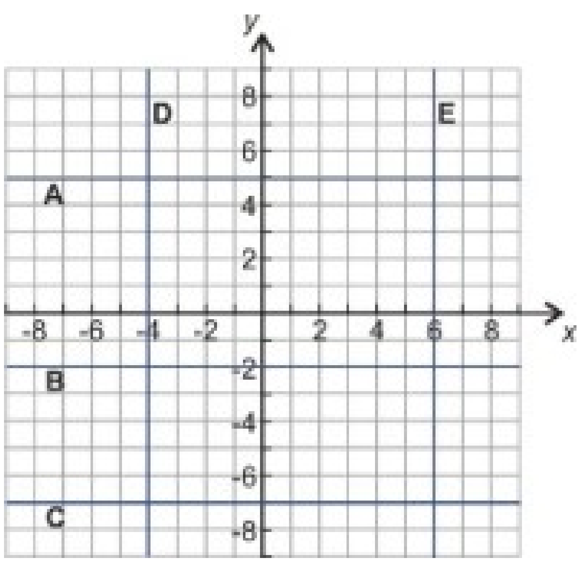
- Be aware that although we graph the function as a line to make it easier to interpret, the function may actually be discrete.

Review Questions

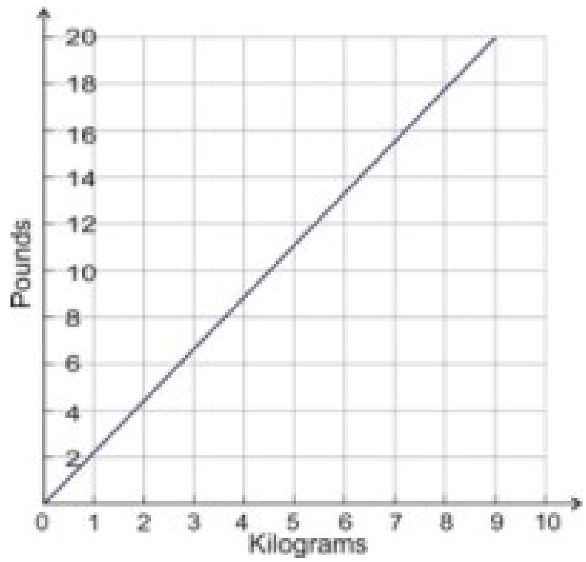
1. Make a table of values for the following equations and then graph them.

   (a) $y = 2x + 7$
   (b) $y = 0.7x - 4$
   (c) $y = 6 - 1.25x$

2. ***"Think of a number. Triple it, and then subtract seven from your answer"***. Make a table of values and plot the function that represents this sentence.

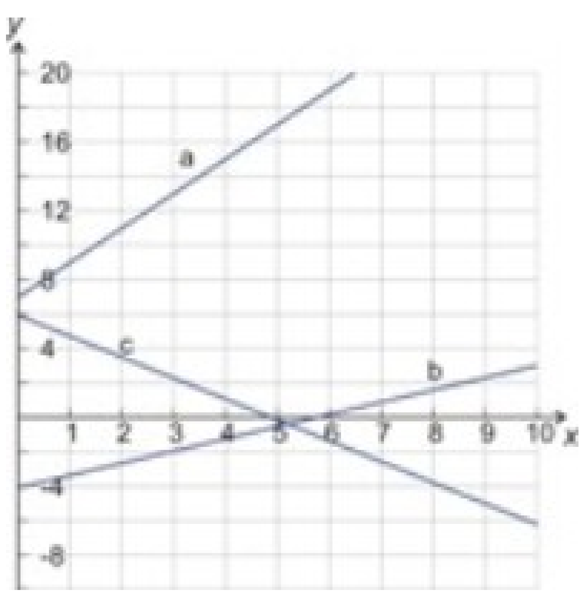


3. Write the equations for the five (A through E) lines plotted in the graph to the right.
4. At the Airport, you can change your money from dollars into Euros. The service costs \$5, and for every additional dollar you get 0.7 Euros. Make a table for this and plot the function on a graph. Use your graph to determine how many Euros you would get if you give the office \$50.
5. The graph to below shows a conversion chart for converting between weight in kilograms to weight in pounds. Use it to convert the following measurements.

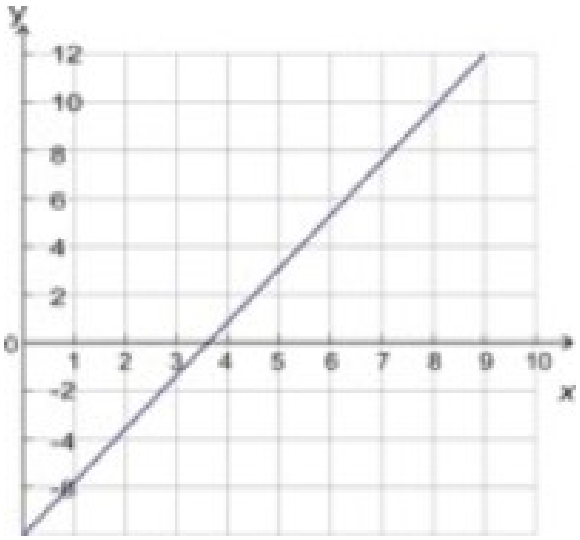


(a) 4 kilograms into weight in pounds
(b) 9 kilograms into weight in pounds
(c) 12 pounds into weight in kilograms
(d) 17 pounds into weight in kilograms

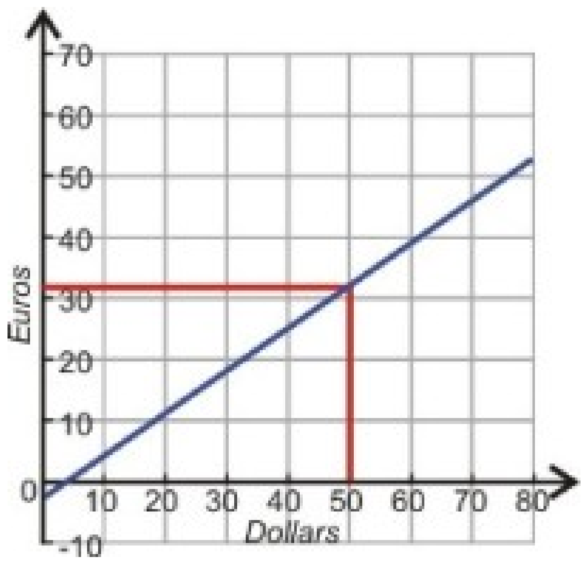
Review Answers

1. 

2. $y = 3x - 7$



3. $Ay = 5$ $By = -2$ $Cy = -7$ $Dx = -4$ $Ex = 6$
4. $y = 0.7(x - 5)$



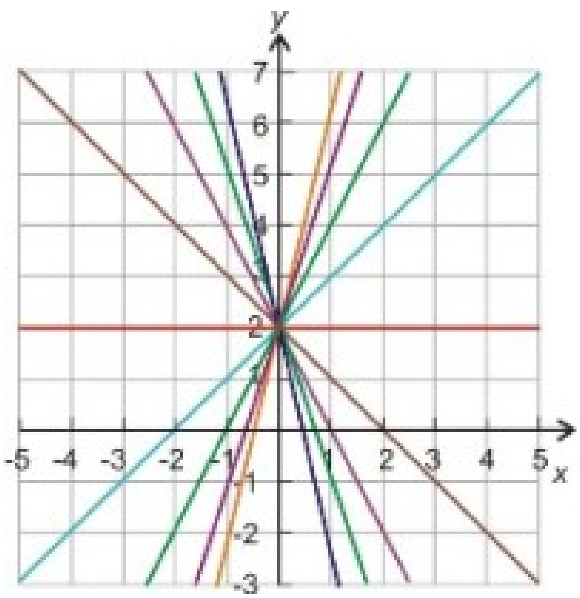
5.
6. (a) 9 lb
   (b) 20 lb
   (c) 5.5 kg
   (d) 7.75 kg

4.3 Graphing Using Intercepts

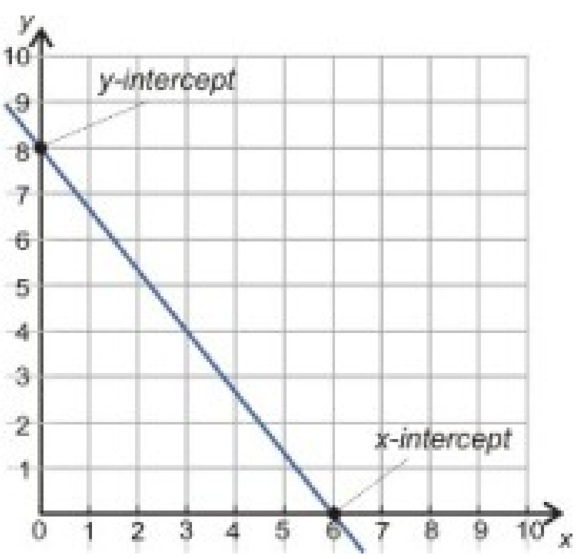
Learning Objectives

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph.

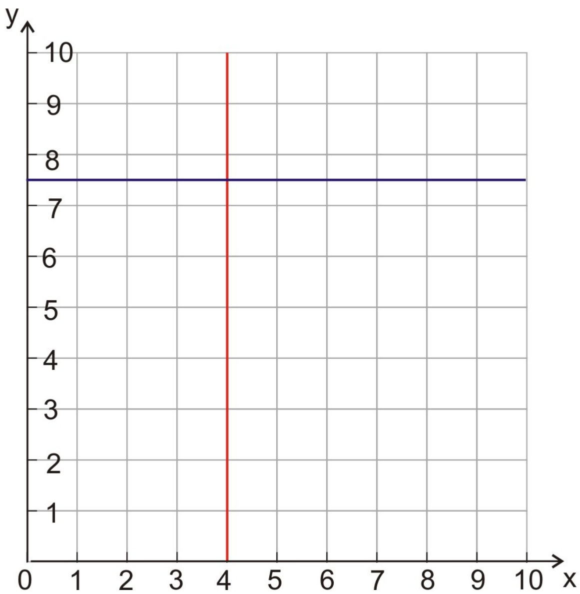
Introduction



Only two distinct points are needed to uniquely define a graph of a line. After all, there are an infinite number of lines that pass through a single point (a few are shown in the graph at right). But if you supplied just one more point, there can only be one line that passes through both points. To plot the line, just plot the two points and use a ruler, edge placed on both points, to trace the graph of the line.



There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we will focus on two points that are rather convenient for graphing: the points where our line crosses the x and y axes, or **intercepts.** We will be finding intercepts algebraically and using them to quickly plot graphs. Similarly, the $x-$intercept occurs at the point where the graph crosses the $x-$axis. The $x-$value in the graph at the right is 6.



Look at the graph to the right. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is 8.

Similarly the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is 6.

Now we know that the x value of all the points on the $y-$axis is zero, and the y value of all the points on the $x-$axis is also zero. So if we were given the coordinates of the two intercepts $(0, 8)$ and $(6, 0)$ we could quickly plot these points and join them with a line to recreate our graph.

Note: Not all lines will have both intercepts but most do. Specifically, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis. For examples of these special case lines, see the graph at right.

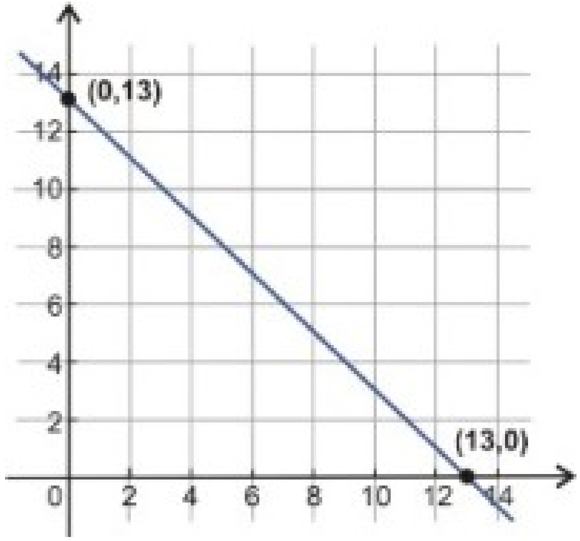
Finding Intercepts by Substitution

Example 1

Find the intercepts of the line $y = 13 - x$ *and use them to graph the function.*

The first intercept is easy to find. The $y-$intercept occurs when $x = 0$ Substituting gives:

$$y = 13 - 0 = 13 \qquad (0, 13) \text{ is the } x - \text{intercept.}$$



We know that the $x-$intercept has, by definition, a $y-$value of zero. Finding the corresponding $x-$value is a simple case of substitution:

$$\begin{aligned} 0 &= 13 - x && \text{To isolate } x \text{ subtract 13 from both sides.} \\ -13 &= -x && \text{Divide by } -1. \end{aligned}$$

Solution

$(13, 0)$ is the $x-$intercept.

To draw the graph simply plot these points and join them with a line.

Example 2

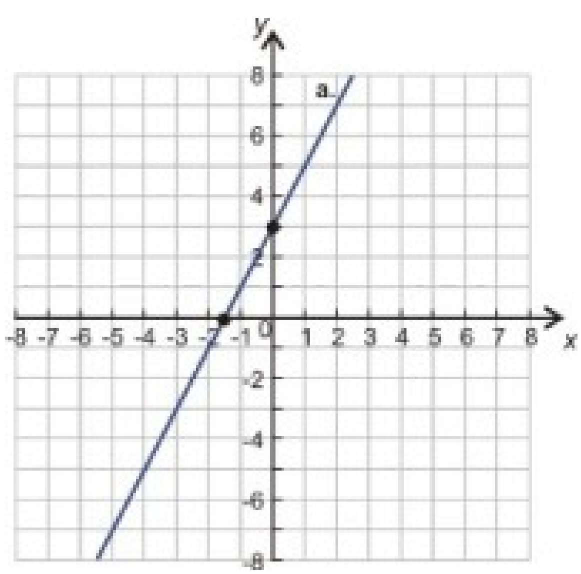
Graph the following functions by finding intercepts.

a. $y = 2x + 3$

b. $y = 7 - 2x$

c. $4x - 2y = 8$

d. $2x + 3y = -6$

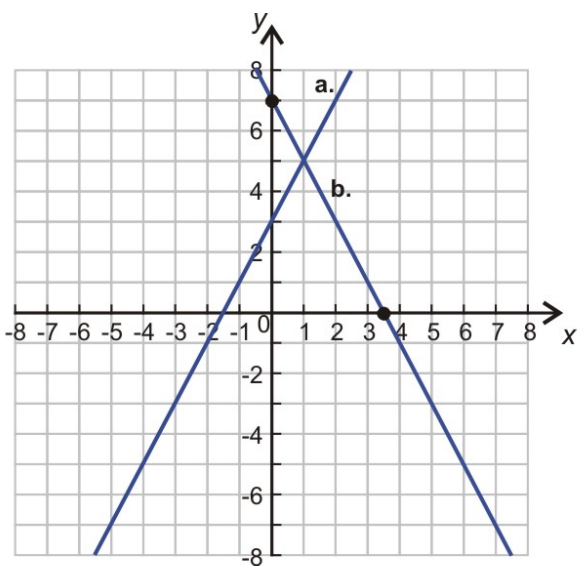


a. Find the $y-$intercept by plugging in $x = 0$.

$$y = 2 \cdot 0 + 3 = 3 \qquad \text{The } y-\text{intercept is } (0, 3)$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 0 &= 2x + 3 && \text{Subtract 3 from both sides.} \\ -3 &= 2x && \text{Divide by 2.} \\ -\frac{3}{2} &= x && \text{The } x-\text{intercept is } (-1.5, 0). \end{aligned}$$

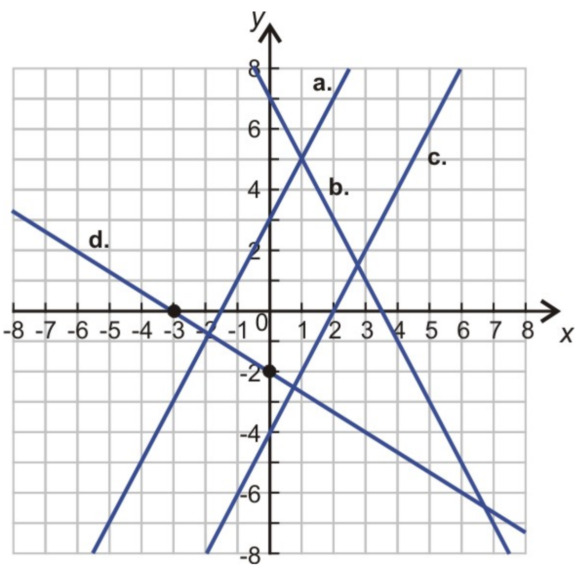


b. Find the $y-$intercept by plugging in $x = 0$.

$$y = 7 - 2 \cdot 0 = 7 \qquad \text{The } y-\text{intercept is } (0, 7).$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 0 &= 7 - 2x && \text{Subtract 7 from both sides.} \\ -7 &= -2x && \text{Divide by } -2. \\ \frac{7}{2} &= x && \text{The } x-\text{intercept is } (3.5, 0). \end{aligned}$$



c. Find the $y-$intercept by plugging in $x = 0$.

$$4 \cdot 0 - 2y = 8$$

$$-2y = 8 \qquad \text{Divide by } -2.$$

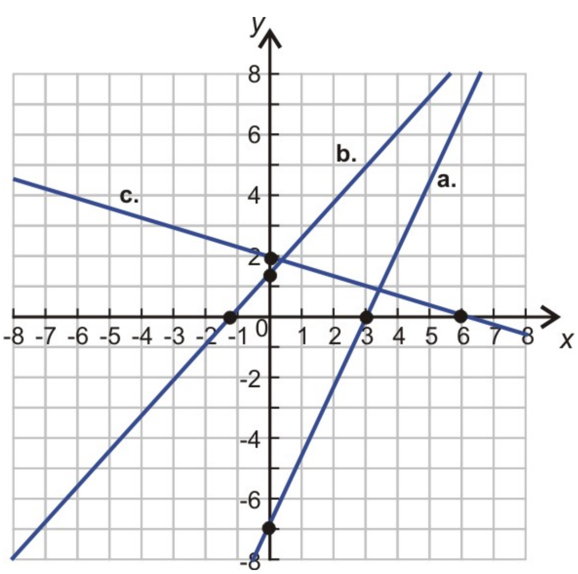
$$y = -4 \qquad \text{The } y-\text{intercept is } (0, -4).$$

Find the $x-$intercept by plugging in $y = 0$.

$$4x - 2 \cdot 0 = 8$$

$$4x = 8 \qquad \text{Divide by } 4.$$

$$x = 2 \qquad \text{The } x-\text{intercept is } (2, 0).$$



d. Find the $y-$intercept by plugging in $x = 0$.

$$2 \cdot 0 + 3y = -6$$

$$3y = -6 \qquad \text{Divide by } 3.$$

$$y = -2 \qquad \text{The } y-\text{intercept is } (0, -2).$$

Find the $x-$intercept by plugging in $y = 0$.

$$\begin{aligned} 2x + 3 \cdot 0 &= -6 \\ 2x &= -6 && \text{Divide by 2.} \\ x &= -3 && \text{The } x-\text{intercept is } (-3, 0) \end{aligned}$$

Finding Intercepts for Standard Form Equations Using the Cover-Up Method

Look at the last two equations in Example 2. These equations are written in **standard form**. Standard form equations are always written "**positive coefficient** times x plus (or minus) **positive coefficient** times y equals **value".** Note that the x term *always* has a positive value in front of it while the y value may have a negative term. The equation looks like this:

$$ax + by + c \text{ or } ax - by = c \qquad (a \text{ and } b \text{ are positive numbers})$$

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example 3

Find the intercepts of the following equations.

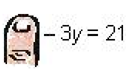
a. $7x - 3y = 21$

b. $12x - 10y = -15$

c. $x + 3y = 6$

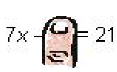
To solve for each intercept, we realize that on the intercepts the value of **either** x **or** y is zero, and so any terms that contain the zero variable effectively disappear. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a. To solve for the $y-$intercept we set $x = 0$ and cover up the x term:



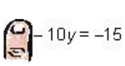
$$\begin{aligned} -3y &= 21 \\ y &= -7 && (0, -7) \text{ is the } y-\text{intercept} \end{aligned}$$

Now we solve for the $x-$intercept:



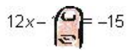
$$\begin{aligned} 7x &= 21 \\ x &= 3 && (3, 0) \text{ is the } x-\text{intercept.} \end{aligned}$$

b. Solve for the $y-$intercept ($x = 0$) by covering up the x term.



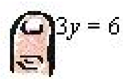
$$\begin{aligned} -10y &= -15 \\ y &= -1.5 && (0, -1.5) \text{ is the } y-\text{intercept.} \end{aligned}$$

Solve for the $x-$intercept $(y = 0)$:

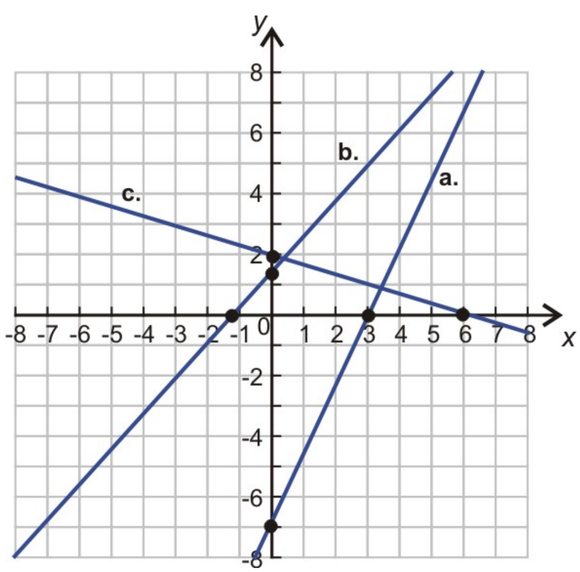


$$\begin{aligned} 12x &= -15 \\ x &= -\frac{5}{4} \end{aligned} \qquad (-1.25, 0) \text{ is the } x-\text{intercept.}$$

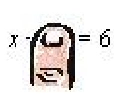
c. Solve for the $y-$intercept $(x = 0)$ by covering up the x term:



$$\begin{aligned} 3y &= 6 \\ y &= 2 \end{aligned} \qquad (0, 2) \text{ is the } y-\text{intercept.}$$



Solve for the $y-$intercept:



$$x = 6 \qquad (6, 0) \text{ is the } x-\text{intercept.}$$

The graph of these functions and the intercepts is shown in the graph on the right.

Solving Real-World Problems Using Intercepts of a Graph

Example 4

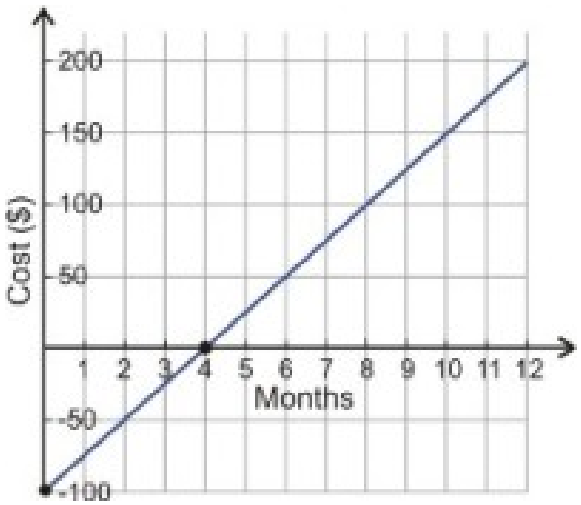
The monthly membership cost of a gym is \$25 *per month. To attract members, the gym is offering a* \$100 *cash rebate if members sign up for a full year. Plot the cost of gym membership over a* 12 *month period. Use the graph to determine the final cost for a* 12 *month membership.*

Let us examine the problem. Clearly the cost is a function of the number of months (not the other way around). Our independent variable is the number of months (the domain will be whole numbers) and this will be our x value. The cost in dollars is the dependent variable and will be our y value. Every month that passes the money paid to the gym goes up by \$25. However, we start with a \$100 cash gift, so our **initial cost** ($y-$intercept) is \$100. This pays for four months ($4 \times \$25 = 100$) so after four months the cost of membership ($y-$value) is zero.

The $y-$intercept is $(0, -100)$. The $x-$intercept is $(4, 0)$.

We plot our points, join them with a straight line and extend that line out all the way to the $x = 12$ line. The graph is shown below.

Cost of Gym Membership by Number of Months



To find the cost of a 12 month membership we simply read off the value of the function at the 12 month point. A line drawn up from $x = 12$ on the x axis meets the function at a y value of \$200.

Solution

The cost of joining the gym for one year is \$200.

Example 5

Jesus has \$30 *to spend on food for a class barbeque. Hot dogs cost* \$0.75 *each (including the bun) and burgers cost* \$1.25 *(including bun and salad). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than* \$30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that John buys is x, then the money spent on burgers is $1.25x$.

If the number of hot dogs he buys is y then the money spent on hot dogs is $0.75y$.

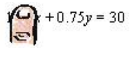
$$1.25x + 0.75y \qquad \text{The total cost of the food.}$$

The total amount of money he has to spend is \$30. If he is to spend it ALL, then we can use the following equation.

$$1.25x + 0.75y = 30$$

We solve for the intercepts using the cover-up method.

First the $y-$intercept ($x = 0$.



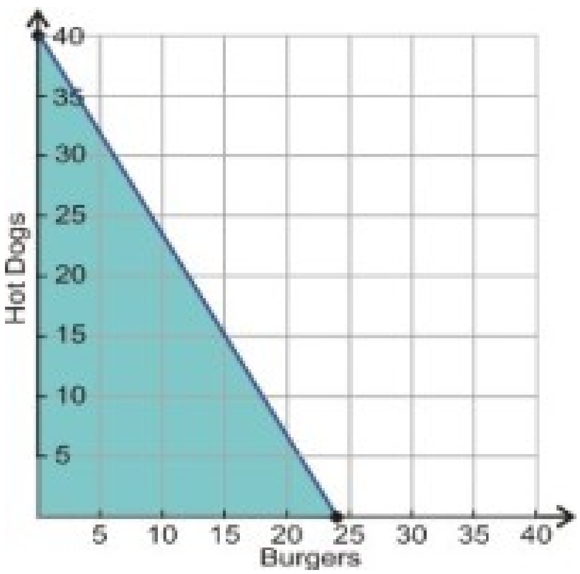
$$0.75y = 30$$
$$y = 40 \qquad\qquad y-\text{intercept}(0,40)$$

Then the $x-$intercept $(y = 0)$

$1.25x + 0(\ ) = 30$

$$1.25x = 30$$
$$x = 24 \qquad\qquad x-\text{intercept}(24,0)$$

Possible Numbers of Hot Dogs and Hamburgers Purchased for $30



We can now plot the points and join them to create our graph, shown right.

Here is an alternative to the equation method.

If Jesus were to spend ALL the money on hot dogs, he could buy $\frac{30}{0.75} = 40$ hot dogs. If on the other hand, he were to buy only burgers, he could buy $\frac{30}{1.25} = 24$ burgers. So you can see that we get two intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We would plot these in an identical manner and design our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an **inequality.** He can, in fact, spend any amount up to \$30. The only thing he cannot do is spend more than \$30. So our graph reflects this. The shaded region shows where Jesus' solutions all lie. We will see inequalities again in Chapter 6.

Lesson Summary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis $(x = 0)$ and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis $(y = 0)$.
- The $y-$intercept can be found by **substituting** $x = 0$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 0$ into the equation and solving for x.

- A linear equation is in **standard form** if it is written as "positive coefficient times x plus (or minus) positive coefficient times y equals value". Equations in standard form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

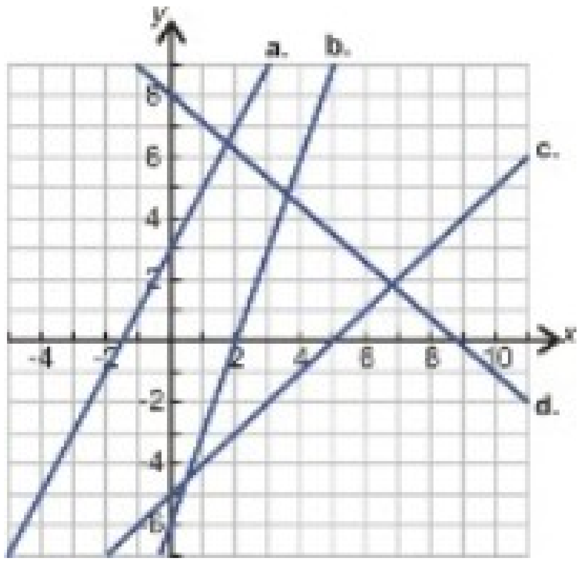
Review Questions

1. Find the intercepts for the following equations using substitution.
   (a) $y = 3x - 6$
   (b) $y = -2x + 4$
   (c) $y = 14x - 21$
   (d) $y = 7 - 3x$
2. Find the intercepts of the following equations using the cover-up method.
   (a) $5x - 6y = 15$
   (b) $3x - 4y = -5$
   (c) $2x + 7y = -11$
   (d) $5x + 10y = 25$
3. Use any method to find the intercepts and then graph the following equations.
   (a) $y = 2x + 3$
   (b) $6(x - 1) = 2(y + 3)$
   (c) $x - y = 5$
   (d) $x + y = 8$
4. At the local grocery store strawberries cost $3.00 per pound and bananas cost $1.00 per pound. If I have $10 to spend between strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly $10.
5. A movie theater charges $7.50 for adult tickets and $4.50 for children. If the theater takes $900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
6. Why can't we use the intercept method to graph the following equation? $3(x + 2) = 2(y + 3)$

Review Answers

1.
2. (a) $(0, -6)$, $(2, 0)$
   (b) $(0, 4)$, $(2, 0)$
   (c) $(0, -21)$, $(1.5, 0)$
   (d) $(0, 7)$, $(\frac{7}{3}, 0)$
3.
4. (a) $(0, -2.5)$, $(3, 0)$
   (b) $(0, 1.25)$, $(-\frac{5}{3}, 0)$
   (c) $(0, -\frac{11}{7})$, $(-\frac{11}{2}, 0)$
   (d) $(0, 2.5)$, $(5, 0)$

5.



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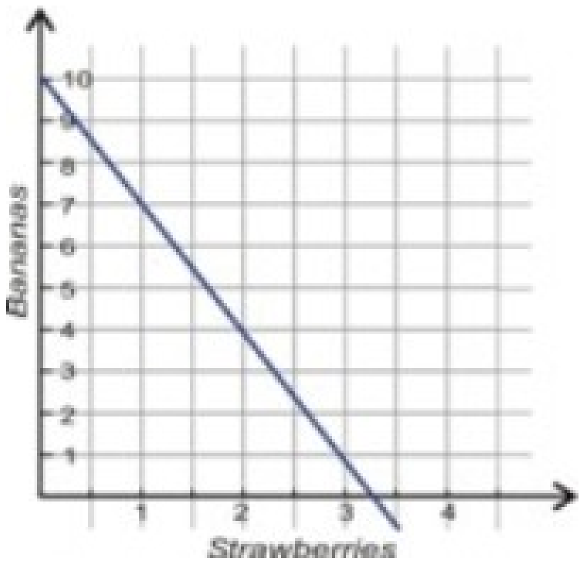
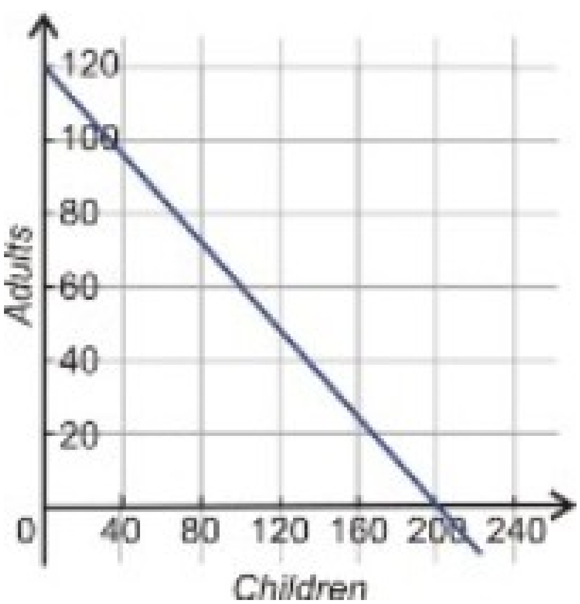


7.



8. This equation reduces to $3x = 2y$, which passes through $(0, 0)$ and therefore only has **one intercept**.

Two intercepts are needed for this method to work.

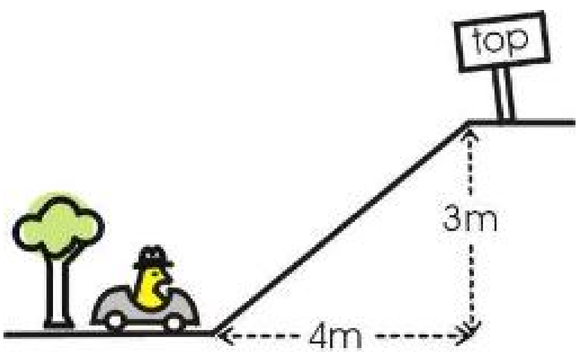
4.4 Slope and Rate of Change

Learning Objectives

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Introduction

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, and the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.



$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

This is often reworded to be easier to remember:

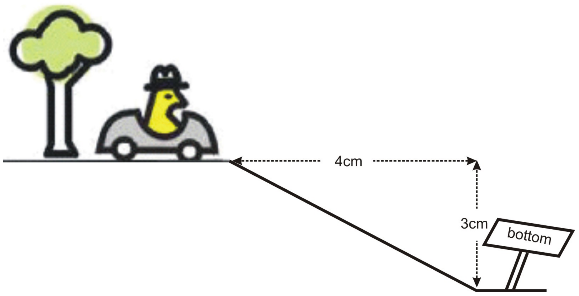
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

Essentially, slope is the change in y if x increases by 1.

In the picture to the right, the slope would be the ratio of the **height** of the hill (the **rise**) to the horizontal **length** of the hill (the **run**).

$$\text{Slope} = \frac{3}{4} = 0.75$$

If the car were driving to the **right** it would **climb** the hill. We say this is a positive slope. Anytime you see the graph of a line that goes up as you move to the right, the slope is **positive**.



If the car were to keep driving after it reached the top of the hill, it may come down again. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**. The picture at right has a **negative slope** of -0.75.

Do not get confused! If the car turns around and drives back down the hill shown, we would still classify the slope as positive. This is because the rise would be -3, but the run would be -4 (think of the x−axis – if you move from right to left you are moving in the negative x−direction). Our ratio for moving **left** is:

$$\text{Slope} = \frac{-3}{-4} = 0.75 \qquad \text{A negative divided by a negative is a positive.}$$

So as we move from left to right, positive slopes increase while negative slopes decrease.

Find a Positive Slope

We have seen that a function with a positive slope increases in y as we increase x. A simple way to find a value for the slope is to draw a right angled triangle whose hypotenuse runs along the line. It is then a simple matter of measuring the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown right.

There are already right-triangles drawn for each of the lines. In practice, you would have to do this yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. the coordinates are all integers).

a. The rise shown in this triangle is 4 units, the run is 2 units.

$$\text{Slope} = \frac{4}{2} = 2$$

b. The rise shown in this triangle is 4 units, the run is also 4 units.

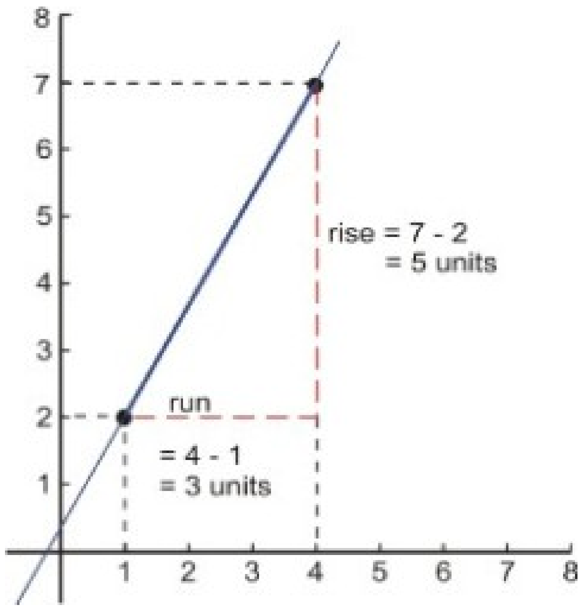
$$\text{Slope} = \frac{4}{4} = 1$$

c. The rise shown in this triangle is 2 units, the run is 4 units.

$$\text{Slope} = \frac{2}{4} = \frac{1}{2}$$

Example 2

Find the slope of the line that passes through the points $(1,2)$ *and* $(4,7)$.



We already know how to graph a line if we are given two points. We simply plot the points and connect them with a line. Look at the graph shown at right.

Since we already have coordinates for our right triangle, we can quickly work out that the rise would be 5 and the run would be 3 (see diagram). Here is our slope.

$$\text{Slope} = \frac{7-2}{4-1} = \frac{5}{3}$$

If you look closely at the calculations for the slope you will notice that the 7 and 2 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1, y_1) and (x_2, y_2).

Slope between (x_1, y_1) and $(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{\Delta y}{\Delta x}$

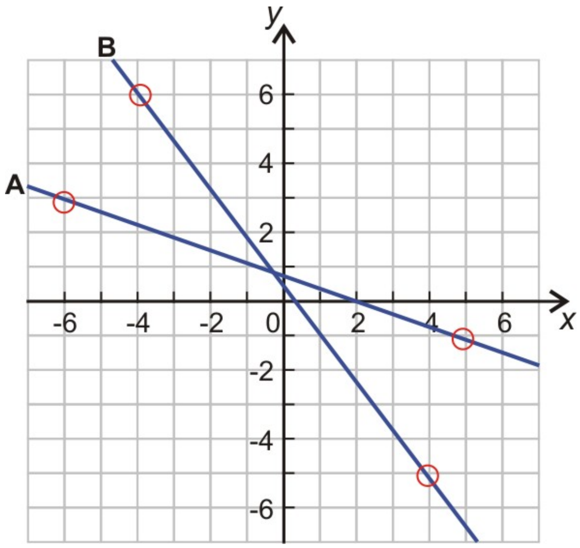
In the second equation, the letter m denotes the slope (you will see this a lot in this chapter) and the Greek letter delta (Δ) means **change**. So another way to express slope is *change in* y divided by *change in* x. In the next section, you will see that it does not matter which point you choose as point 1 and which you choose as point 2.

Find a Negative Slope

Any function with a negative slope is simply a function that decreases as we increase x. If you think of the function as the incline of a road a negative slope is a road that goes **downhill** as you drive to the **right**.

Example 3

Find the slopes of the lines on the graph to the right.



Look at the lines. Both functions fall (or decrease) as we move from left to right. Both of these lines have a **negative slope**.

Neither line passes through a great number of lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been identified (with rings) and we will use these to determine the slope. We will also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For line *A*:

$$(x_1, y_1) = (-6, 3) \ (x_2, y_2) = (5, -1)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364$$

$$(x_1, y_1) = (5, -1) \ (x_2, y_2) = (-6, 3)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (-5)} = \frac{-4}{11} \approx -0.364$$

For line *B*:

$$(x_1, y_1) = (-4, 6) \ (x_2, y_2) = (4, -5)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375$$

$$(x_1, y_1) = (4, -5) \ (x_2, y_2) = (-4, 6)$$
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

You can see that whichever way you select the points, the answers are the same!

Solution

Line *A* has slope −0.364. Line *B* has slope −1.375.

Multimedia Link The series of videos starting at Khan Academy Slope (8:28) models several more examples of finding the slope of a line given two points.

Find the Slopes of Horizontal and Vertical lines

Example 4

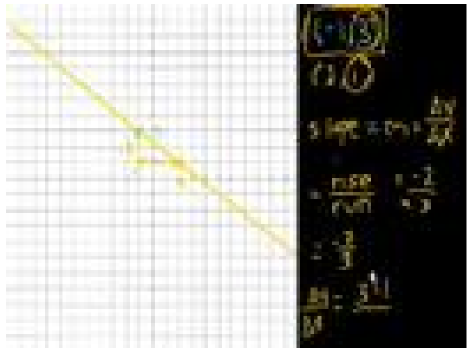
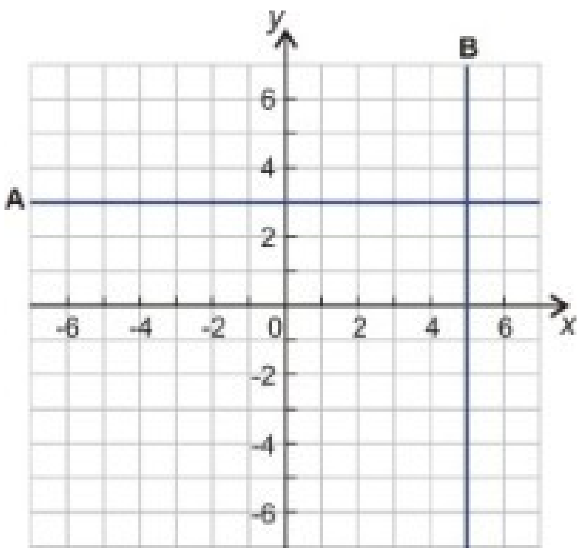


Figure 4.2: Figuring out the slope of a line (Watch on Youtube)



Determine the slopes of the two lines on the graph at the right.

There are two lines on the graph. A $(y = 3)$ and B $(x = 5)$.

Let's pick two points on line A. say, $(x_1, y_1) = (-4, 3)$ and $(x_2, y_2) = (5, 3)$ and use our equation for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0$$

If you think about it, this makes sense. If there is no change in y as we increase x then there is no slope, or to be correct, a slope of zero. You can see that this must be true for all horizontal lines.

Horizontal lines ($y =$ constant) all have a slope of 0.

Now consider line B. Pick two distinct points on this line and plug them in to the slope equation.

$(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-3)}{(5) - (5)} = \frac{7}{0} \qquad \text{A division by zero!}$$

Divisions by zero lead to infinities. In math we often use the term **undefined** for any division by zero.

Vertical lines ($x =$ constant) all have an infinite (or undefined) slope.

Find a Rate of Change

The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case, the slope refers to a change in one quantity (y) **per** unit change in another quantity (x).

Example 5

Andrea has a part time job at the local grocery store. She saves for her vacation at a rate of \$15 *every week. Express this rate as money saved* ***per day*** *and money saved* ***per year****.*

Converting rates of change is fairly straight forward so long as you remember the equations for rate (i.e. the equations for slope) and know the conversions. In this case 1 week = 7 days and 52 weeks = 1 year.

$$\text{rate} = \frac{\$15}{1 \text{ week}} \cdot \frac{1 \text{ week}}{7 \text{ days}} = \frac{\$15}{7 \text{ days}} = \frac{15}{7} \text{ dollars per day} \approx \$2.14 \text{ per day}$$
$$\text{rate} = \frac{\$15}{1 \text{ week}} \cdot \frac{52 \text{ week}}{1 \text{ year}} = \$15 \cdot \frac{52}{\text{year}} = \$780 \text{ per year}$$

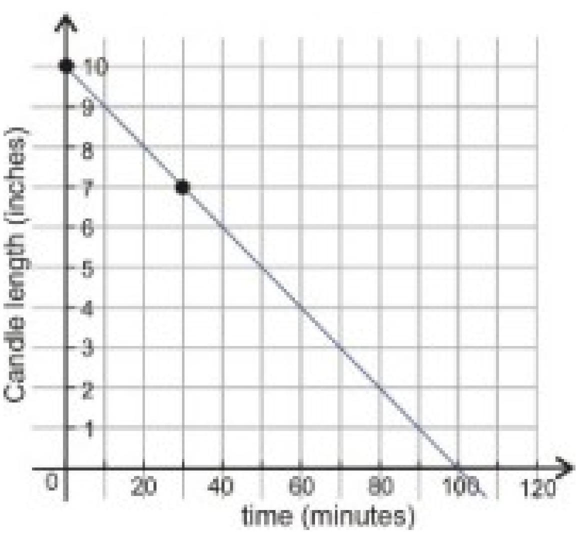
Example 6

A candle has a starting length of 10 *inches. Thirty minutes after lighting it, the length is* 7 inches *. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.*

In this case, we will graph the function to visualize what is happening.

We have two points to start with. We know that at the moment the candle is lit (time = 0) the length of the candle is 10 inches. After thirty minutes (time = 30) the length is 7 inches . Since the candle length is a function of time we will plot time on the horizontal axis, and candle length on the vertical axis. Here is a graph showing this information.

Candle Length by Burning Time



The rate of change of the candle is simply the slope. Since we have our two points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$ we can move straight to the formula.

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})} = \frac{-3 \text{ inches}}{30 \text{ minutes}} = -0.1 \text{ inches per minute}$$

The slope is negative. A negative rate of change means that the quantity is decreasing with time.

We can also convert our rate to inches per hour.

$$\text{rate} = \frac{-0.1 \text{ inches}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{-6 \text{ inches}}{1 \text{ hour}} = -6 \text{ inches per hour}$$

To find the point when the candle reaches zero length we can simply read off the graph (100 minutes). We can use the rate equation to verify this algebraically.

$$\text{Length burned } = \text{rate} \times \text{time}$$

$$0.1 \times 100 = 10$$

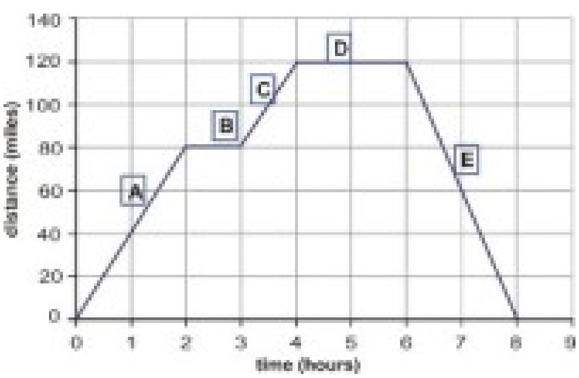
Since the candle length was originally 10 inches this confirms that 100 minutes is the correct amount of time.

Interpret a Graph to Compare Rates of Change

Example 7

Examine the graph below. It represents a journey made by a large delivery truck on a particular day. During the day, the truck made two deliveries, each one taking one hour. The driver also took a one hour break for lunch. Identify what is happening at each stage of the journey (stages A through E)

Truck's Distance from Home by Time



Here is the driver's journey.

A. The truck sets off and travels 80 miles in 2 hours .

B. The truck covers no distance for 1 hours .

C. The truck covers $(120 - 80) = 40$ miles in 1 hours

D. the truck covers no distance for 2 hours .

E. The truck covers 120 miles in 2 hours .

Lets look at the rates of change for each section.

A. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40$ miles per hour

- The rate of change is a **velocity!** This is a very important concept and one that deserves a special note!

The slope (or rate of change) of a distance-time graph is a velocity.

You may be more familiar with calling **miles per hour** a **speed. Speed** is the **magnitude** of a **velocity,** or, put another way, velocity has a direction, speed does not. This is best illustrated by working through the example.

On the first part of the journey sees the truck travel at a constant velocity of 40 mph for 2 hours covering a distance of 80 miles .

B. Slope = 0 so **rate of change** = 0 mph. The truck is stationary for one hour. This could be a lunch break, but as it is only 2 hours since the truck set off it is likely to be the first delivery stop.

C. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{(120-80)\ miles}{(4-3)\ hours}$ = 40 miles per hour. The truck is traveling at 40 mph.

D. Slope = 0 so **rate of change** = 0 mph . The truck is stationary for two hours. It is likely that the driver used these 2 hours for a lunch break plus the second delivery stop. At this point the truck is 120 miles from the start position.

E. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{(0-120)\ miles}{(8-6)\ hours} = \frac{-120\ miles}{2\ hours} = -60$ miles per hour. The truck is traveling at **negative** 60 mph .

Wait, a negative velocity? Does this mean that the truck is reversing? Well, probably not. What it means is that the distance (and don't forget that is the distance measured from the starting position) is decreasing with time. The truck is simply driving in the opposite direction. In this case, back to where it started from. So, the speed of the truck would be 60 mph, but the velocity (which includes direction) is negative because the truck is getting closer to where it started from. The fact that it no longer has two heavy loads means that it travels faster (60 mph as opposed to 40 mph) covering the 120 mile return trip in 2 hours .

Lesson Summary

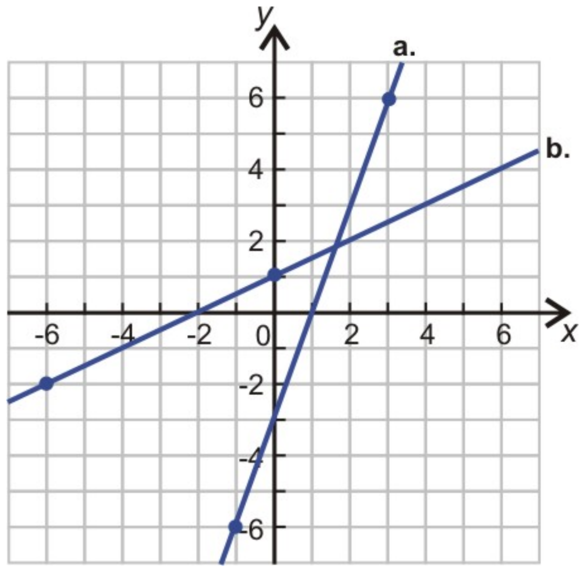
- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- $Slope = \frac{rise}{run}$ or $m = \frac{\Delta y}{\Delta x}$
- The slope between two points (x_1, y_1) and $(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$
- **Horizontal lines** (y = constant) all have a slope of 0.
- **Vertical lines** (x = constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

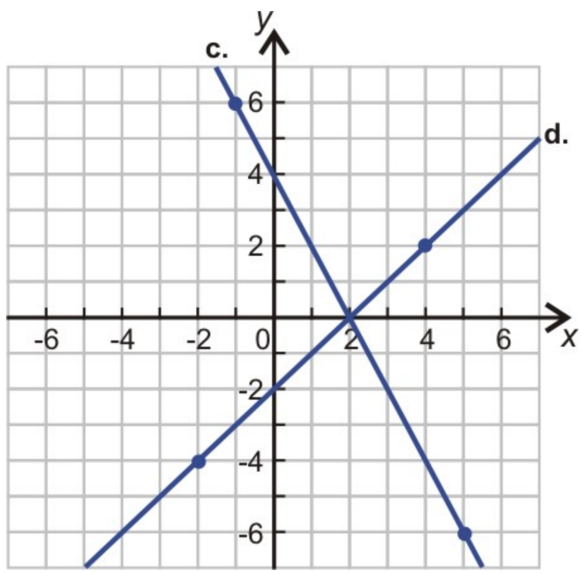
Review Questions

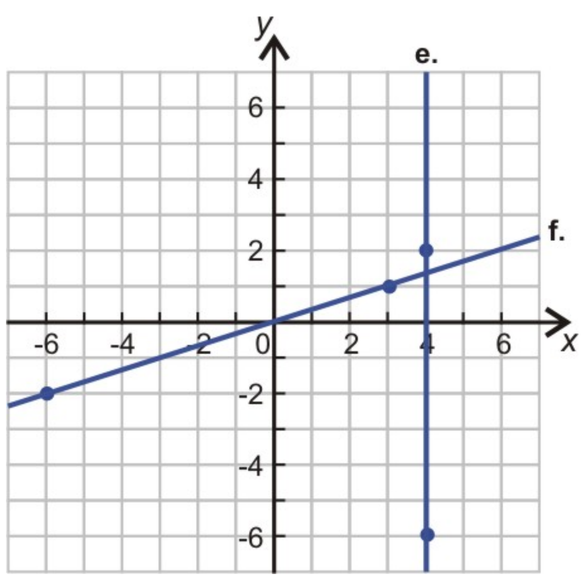
1. Use the slope formula to find the slope of the line that passes through each pair of points.

   (a) $(-5, 7)$ and $(0, 0)$
   (b) $(-3, -5)$ and $(3, 11)$
   (c) $(3, -5)$ and $(-2, 9)$
   (d) $(-5, 7)$ and $(-5, 11)$
   (e) $(9, 9)$ and $(-9, -9)$
   (f) $(3, 5)$ and $(-2, 7)$

2. Use the points indicated on each line of the graphs to determine the slopes of the following lines.

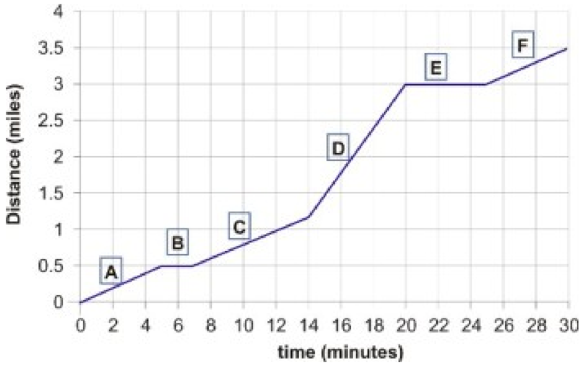






3. The graph below is a distance-time graph for Mark's three and a half mile cycle ride to school.

During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He stopped once at a traffic light and at one point he stopped to mend a tire puncture. Identify each section of the graph accordingly. **Andrew's Distance from Home by Time**



Review Answers

1.
2. (a) -1.4
   (b) 2.67
   (c) -2.8
   (d) undefined
   (e) 1
   (f) -0.4
3.
4. (a) 3
   (b) 0.5
   (c) -2
   (d) 1
   (e) undefined
   (f) $\frac{1}{3}$
5.
6. (a) A. uphill
   (b) B. stopped (traffic light)
   (c) C. uphill
   (d) D. downhill
   (e) E. stopped (puncture)
   (f) F. uphill

4.5 Graphs Using Slope-Intercept Form

Learning Objectives

- Identify the slope and $y-$intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Identify Slope and y-intercept

One of the most common ways of writing linear equations prior to graphing them is called **slope-intercept form.** We have actually seen several slope-intercept equations so far. They take the following form:

$y = mx + b$ where m is the slope and the point $(0, b)$ is the $y-$intercept.

We know that the $y-$intercept is the point at which the line passes through the $y-$axis. The slope is a measure of the steepness of the line. Hopefully, you can see that if we know **one point** on a line and the slope of that line, we know what the line is. Being able to quickly identify the $y-$intercept and slope will aid us in graphing linear functions.

Example 1

Identify the slope and $y-$intercept of the following equations.

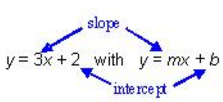
a) $y = 3x + 2$

b) $y = 0.5x - 3$

c) $y = -7x$

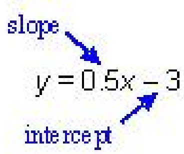
d) $y = -4$

Solution

a) 

Comparing, we see that $m = 3$ and $b = 2$.

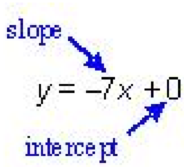
$y = 3x + 2$ **has a slope of** 3 and a $y-$**intercept of** $(0, 2)$

b) 

has a slope of 0.5 *and a y-intercept of* $(0, -3)$.

Note that the $y-$intercept is **negative**. The b term includes the sign of the operator in front of the number. Just remember that $y = 0.5x - 3$ is identical to $y = 0.5x + (-3)$ and is in th eform $y = mx + b$.

c) **At first glance, this does not appear to fit the slope-intercept form. To illustrate how we deal with this, let us rewrite the equation.**



. *We now see that we get a slope of* -7 *and a $y-$intercept of* $(0, 0)$.

Note that the slope is negative. The $(0, 0)$ intercept means that the line passes through origin.

d) **Rewrite as** $y = 0x - 4$, **giving us a slope of 0 and an intercept of** $(0, -4)$.

Remember:

- When $m < 0$ the slope is negative.

For example, $y = -3x + 2$ has a slope of -3.

- When $b < 0$ the intercept is below the x axis.

For example, $y = 4x - 2$ has a y−intercept of $(0, -2)$.

- When $m = 0$ the slope is zero and we have a horizontal line.

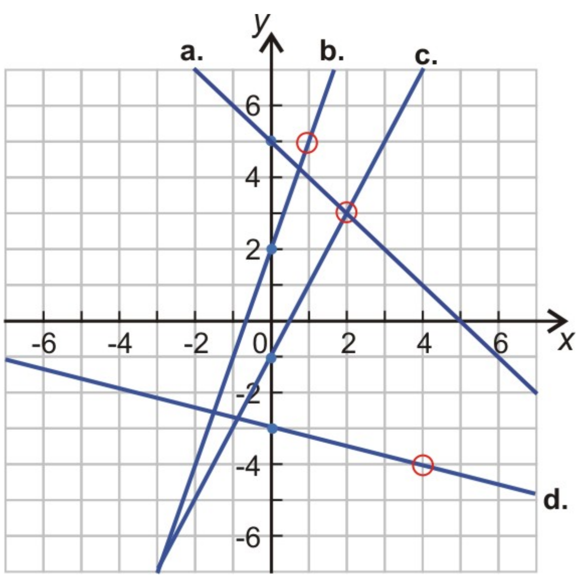
For example, $y = 3$ can be written as $y = 0x + 3$.

- When $b = 0$ the graph passes through the origin.

For example, $y = 4x$ can be written as $= 4x + 0$.

Example 2

Identify the slope and y-intercept of the lines on the graph shown to the right.



The intercepts have been marked, as have a number of lattice points that lines pass through.

a. The y−intercept is $(0, 5)$. The line also passes through $(2, 3)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$$

b. The y−intercept is $(0, 2)$. The line also passes through $(1, 5)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$$

c. The y−intercept is $(0, -1)$. The line also passes through $(2, 3)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$$

d. The y−intercept is $(0, -3)$. The line also passes through $(4, -4)$.

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{-1}{4} = \frac{-1}{4} \text{ or } -0.25$$

Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line it is easy to graph it. Just remember what slope means. Let's look back at this example from Lesson 4.1.

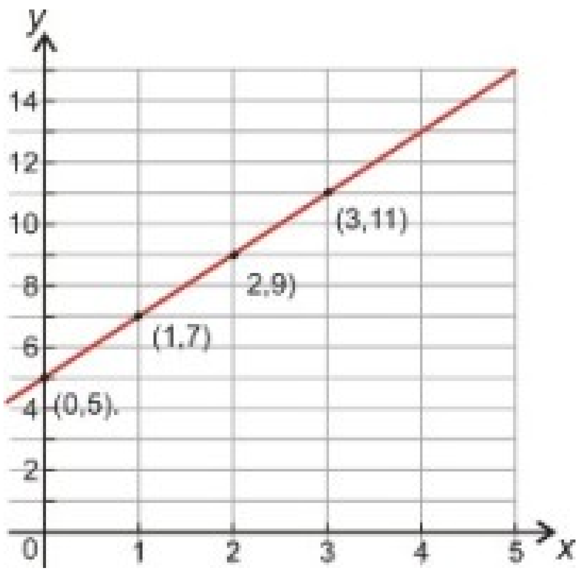
Example 3

Ahiga is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number. Then double it. Then add five to what he got. Ahiga has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the result of applying the rule. His rule is:

$$y = 2x + 5$$

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed the following table of values.



x	y
0	$2.0 + 5 = 0 + 5 = 5$
1	$2.1 + 5 = 2 + 5 = 7$
2	$2.2 + 5 = 4 + 5 = 9$
3	$2.3 + 5 = 6 + 5 = 11$

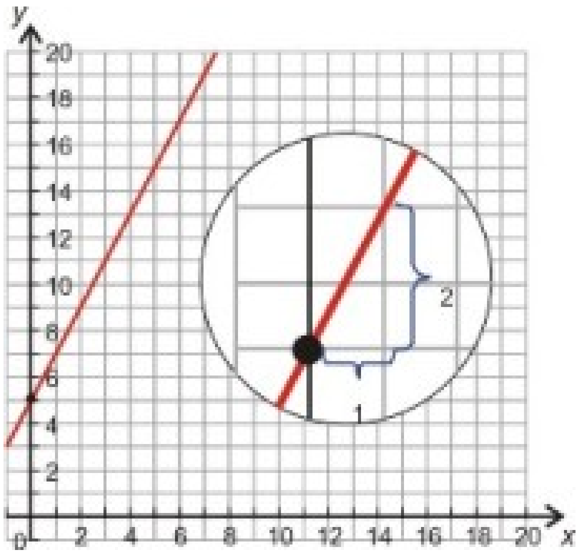
The first entry gave us our y intercept $(0, 5)$. The other points helped us graph the line.

We can now use our equation for slope, and two of the given points.

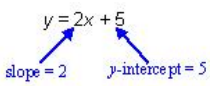
Slope between $(x_1, y_1) = (0, 5)$ and $(x_2, y_2) = (3, 11)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 5}{3 - 0} = \frac{6}{3} = 2$$

Thus confirming that the slope, $m = 2$.



An easier way to graph this function is the slope–intercept method. We can now do this quickly, by identifying the intercept and the slope.

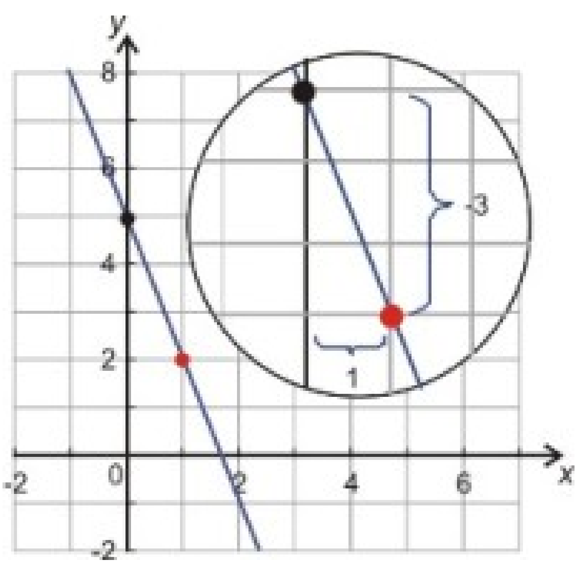


Look at the graph we drew, the line intersects the $y-$axis at 5, and every time we move to the right by one unit, we move up by two units.

So what about plotting a function with a negative slope? Just remember that a negative slope means the function decreases as we increase x.

Example 4

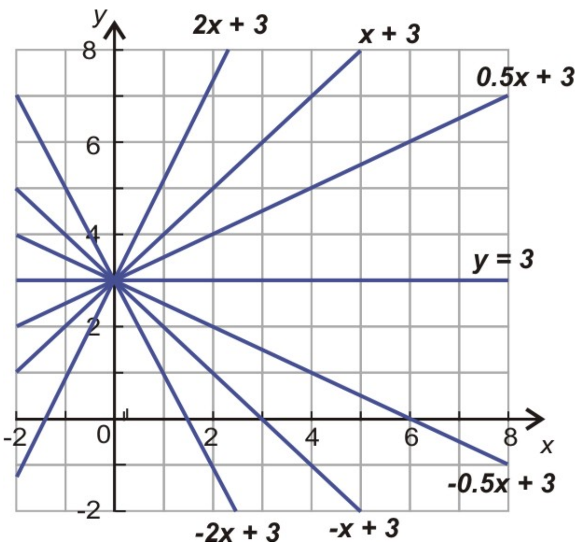
Graph the following function. $y = -3x + 5$



- Identify $y-$intercept $b = 5$
- Plot intercept $(0, 5)$
- Identify slope $m = -3$
- Draw a line through the intercept that has a slope of -3.

To do this last part remember that slope = $\frac{\text{rise}}{\text{run}}$ so for every unit we move to the right the function increases by -3 (in other words, for every square we move right, the function comes **down** by 3).

Changing the Slope of a Line



Look at the graph on the right. It shows a number of lines with different slopes, but all with the same $y-$intercept $(0,3)$.

You can see all the positive slopes increase as we move from left to right while all functions with negative slopes fall as we move from left to right.

Notice that the higher the value of the slope, the steeper the graph.

The graph of $y = 2x+3$ appears as the mirror image of $y = -2x+3$. The two slopes are equal but opposite.

Fractional Slopes and *Rise Over Run*

Look at the graph of $y = 0.5x + 3$. As we increase the x value by 1, the y value increases by 0.5. If we increase the x value by 2, then the y value increases by 1. In fact, if you express any slope as a fraction, you can determine how to plot the graph by looking at the numerator for the *rise* (keep any negative sign included in this term) and the denominator for the *run*.

Example 5

Find integer values for the ***rise*** *and* ***run*** *of following slopes then graph lines with corresponding slopes.*

a. $m = 3$

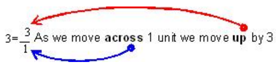
b. $m = -2$

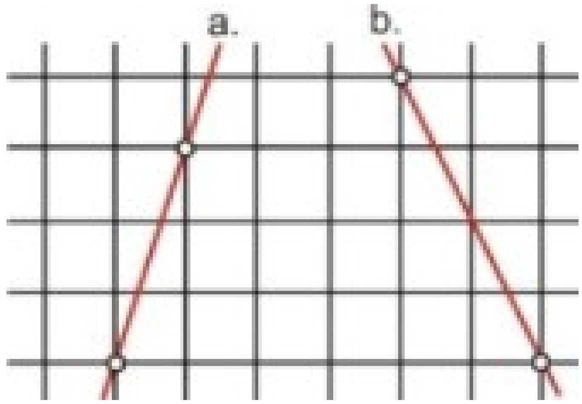
c. $m = 0.75$

d. $m = -0.375$

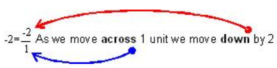
Solution:

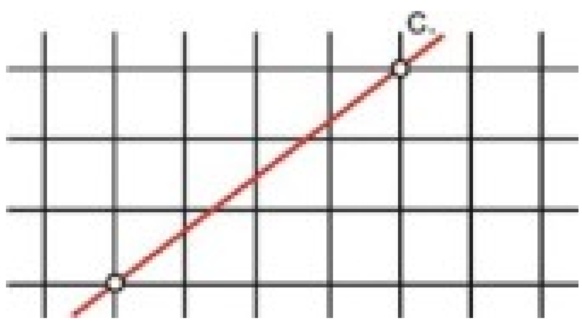
a.


3= 3/1 As we move across 1 unit we move up by 3

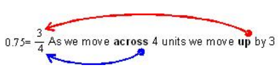

a.
b.

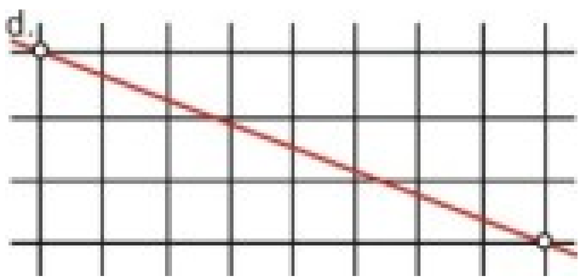
b.


-2= -2/1 As we move across 1 unit we move down by 2


c.

c.

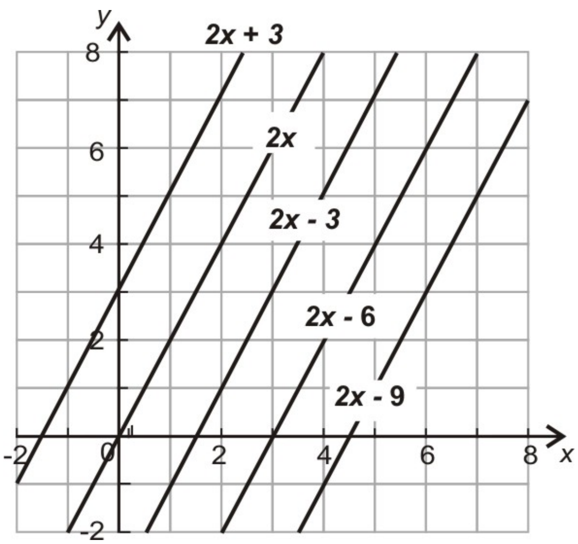

0.75= 3/4 As we move across 4 units we move up by 3


d.

d.


-0.375= -3/8 As we move across 8 units we move down by 3

Changing the Intercept of a Line



When we take an equation (such as $y = 2x$) and change the y intercept (leaving the slope intact) we see the following pattern in the graph on the right.

Notice that changing the intercept simply translates the graph up or down. Take a point on the graph of $y = 2x$, such as $(1, 2)$. The corresponding point on $y = 2x + 3$ would be $(1, 4)$. Similarly the corresponding point on the $y = 2x - 3$ line would be $(1, -1)$.

Will These Lines Ever Cross?

To answer that question, let us take two of the equations $y = 2x$ and $y = 2x + 3$ and solve for values of x and y that satisfy both equations. This will give us the (x, y) coordinates of the point of intersection.

$2x = 2x + 3c$		Subtract $2x$ from both sides.
$0 = 0 + 3$	or	$0 = 3$ This statement is FALSE!

When we get a false statement like this, it means that there are **no** (x, y) values that satisfy both equations simultaneously. The lines will **never** cross, and so they **must be parallel**.

Identify Parallel Lines

In the previous section, when we changed the intercept but left the slope the same, the new line was parallel to the original line. This would be true whatever the slope of the original line, as changing the intercept on a $y = mx + b$ graph does nothing to the slope. This idea can be summed up as follows.

Any two lines with identical slopes are parallel.

Lesson Summary

- A common form of a line (linear equation) is **slope-intercept form:**

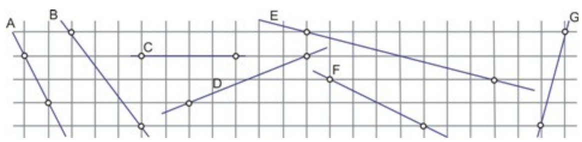
$y = mx + b$ where m is the slope and the point $(0, b)$ is the $y-$intercept

- Graphing a line in slope-intercept form is a matter of first plotting the $y-$intercept $(0, b)$, then plotting more points by moving a step to the right (adding 1 to x) and moving the value of the slope vertically (adding m to y) before plotting each subsequent point.

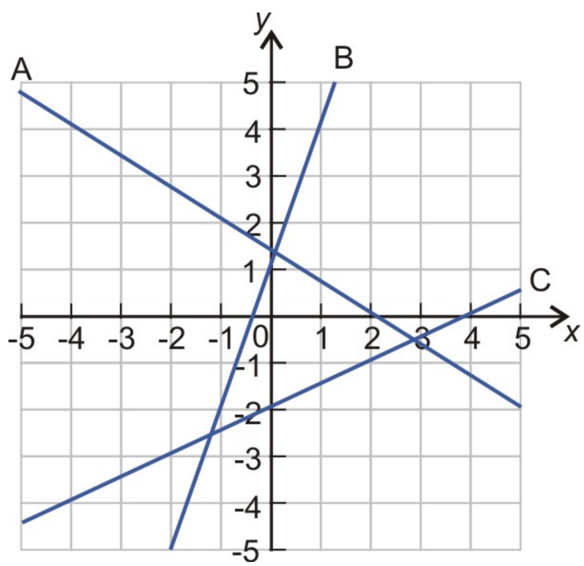
- Any two lines with identical slopes are **parallel.**

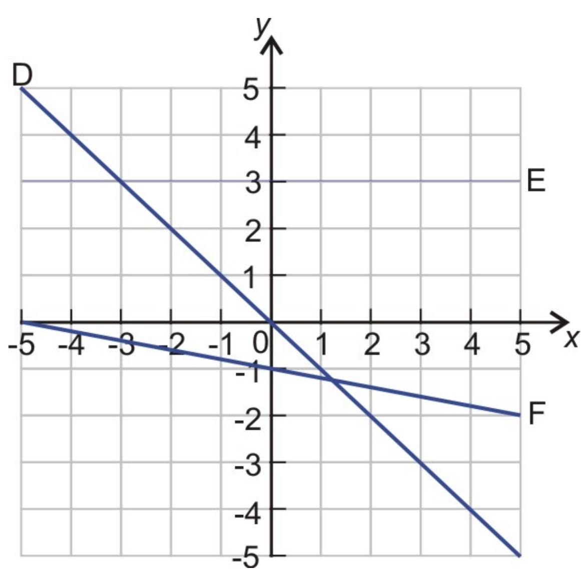
Review Questions

1. Identify the slope and $y-$intercept for the following equations.
   (a) $y = 2x + 5$
   (b) $y = -0.2x + 7$
   (c) $y = x$
   (d) $y = 3.75$
2. Identify the slope of the following lines.



3. Identify the slope and $y-$intercept for the following functions.





4. Plot the following functions on a graph.

   (a) $y = 2x + 5$
   (b) $y = -0.2x + 7$
   (c) $y = -x$
   (d) $y = 3.75$

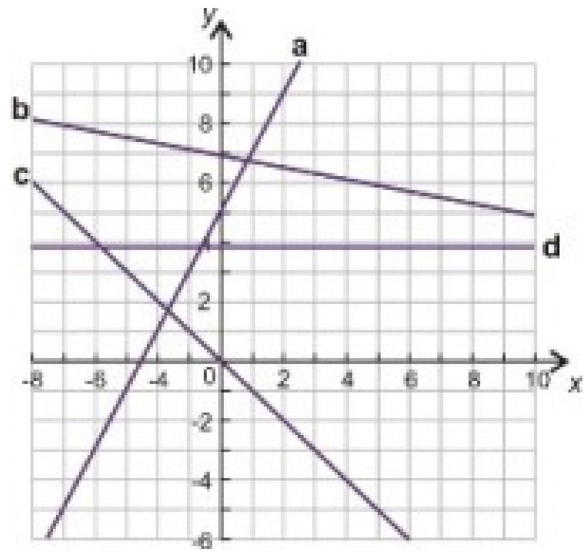
5. Which two of the following lines are parallel?

   (a) $y = 2x + 5$
   (b) $y = -0.2x + 7$
   (c) $y = -x$
   (d) $y = 3.75$
   (e) $y = -\frac{1}{5}x - 11$
   (f) $y = -5x + 5$
   (g) $y = -3x + 11$
   (h) $y = 3x + 3.5$

Review Answers

1.
2. (a) $m = 2, (0, 5)$
   (b) $m = -0.2, (0, 7)$
   (c) $m = 1, (0, 0)$
   (d) $m = 0, (0, 3.75)$
3.
4. (a) A. $m = -2$
   (b) B. $m = -\frac{4}{3}$
   (c) C. $m = 0$
   (d) D. $m = \frac{2}{5}$
   (e) E. $m = -0.25$
   (f) F. $m = -0.5$
   (g) G. $m = 4$
5.
6. (a) A. $y = -\frac{2}{3}x + 1.5$
   (b) B: $y = 3x + 1$
   (c) C: $y = 0.5x - 2$
   (d) D: $y = -x$
   (e) E: $y = 3$
   (f) F: $y = -0.2x - 1$

7.



8. b and e

4.6 Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths of the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries.

$$\frac{2}{5} \times \$12.50 = \$5.00$$

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay $2 \times \$12.50$ and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as:

$$y = k \cdot x$$

k is called the **constant of proportionality**.

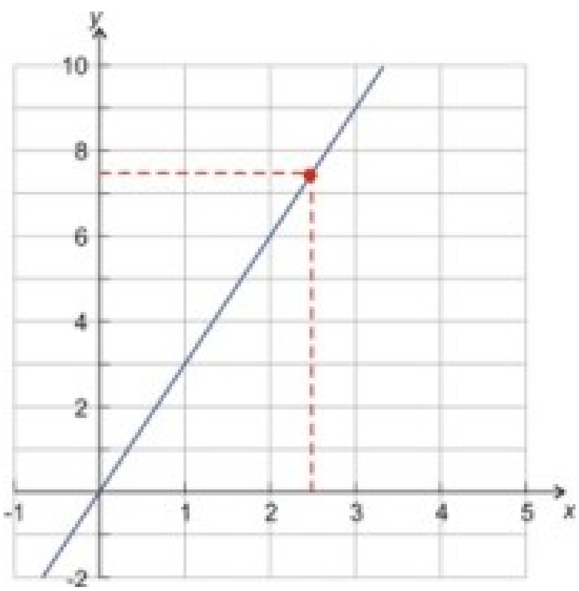
If we were to graph this function you can see that it passes through the origin, because $y = 0$, when $x = 0$ whatever the value of k. So we know that a direct variation, when graphed, has a single intercept at $(0, 0)$.

Example 1

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k.

We can solve for the constant of proportionality using substitution.

Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$



$$7.5 = k(2.5) \qquad \text{Divide both sides by 2.5.}$$

$$\frac{7.5}{2.5} = k = 3$$

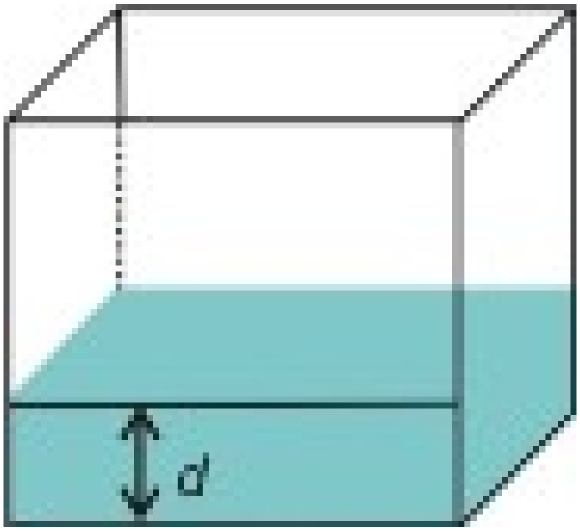
Solution

The constant of proportionality, $k = 3$.

We can graph the relationship quickly, using the intercept $(0, 0)$ and the point $2.5, 7.5)$. The graph is shown right. It is a straight line with a slope $= 3$.

The graph of a direct variation has a slope that is equal to the constant of proportionality, k.

Example 2



The volume of water in a fish-tank, V, varies directly with depth, d. If there are 15 gallons in the tank when the depth is eight inches, calculate how much water is in the tank when the depth is 20 inches.

This is a good example of a direct variation, but for this problem we will need to determine the equation of the variation ourselves. Since the volume, V, depends on depth, d, we will use the previous equation to create new one that is better suited to the content of the new problem.

$$\begin{aligned} y &= k \cdot x && \text{In place of } y \text{ we will use } V \text{ and in place of } x \text{ we will use d.} \\ V &= k \cdot d \end{aligned}$$

We know that when the depth is 8 inches, the volume is 15 gallons. Now we can substitute those values into our equation.

Substitute $V = 15$ and $x = 8$:

$$\begin{aligned} V &= k \cdot d \\ 15 &= k(8) && \text{Divide both sides by8.} \\ \frac{15}{8} &= k = 1.875 \end{aligned}$$

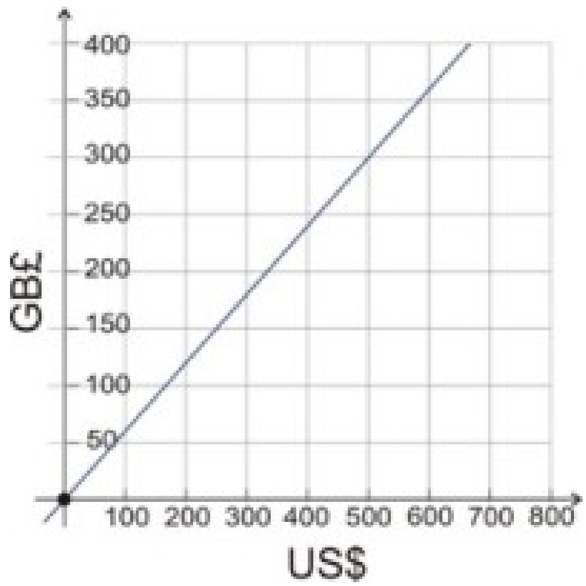
Now to find the volume of water at the final depth we use $V = k \cdot d$ and substitute for our new d.

$$\begin{aligned} V &= k \cdot d \\ V &= 1.875 \times 20 \\ V &= 37.5 \end{aligned}$$

Solution

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Example 3



The graph shown to the right shows a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine the following.

(i) The number of pounds you could buy for \$600.

(ii) The number of dollars it would cost to buy £200.

(iii) The exchange rate in pounds per dollar.

(iv) Is the function continuous or discrete?

Solution

In order to solve (i) and (ii) we could simply read off the graph: it looks as if at $x = 600$ the graph is about one fifth of the way between £350 and £400. So \$600 would buy £360. Similarly, the line $y = 200$ would appear to intersect the graph about a third of the way between \$300 and \$400. We would probably round this to \$330. So it would cost approximately \$330 to buy £200.

To solve for the exchange rate we should note that as this is a direct variation, because the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value. Looking closely at the graph, it is clear that there is one lattice point that the line passes through $(500, 300)$. This will give us the most accurate estimate for the slope (exchange rate).

$$y = k \cdot x \Rightarrow k = \frac{y}{x}$$
$$\text{rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k. Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

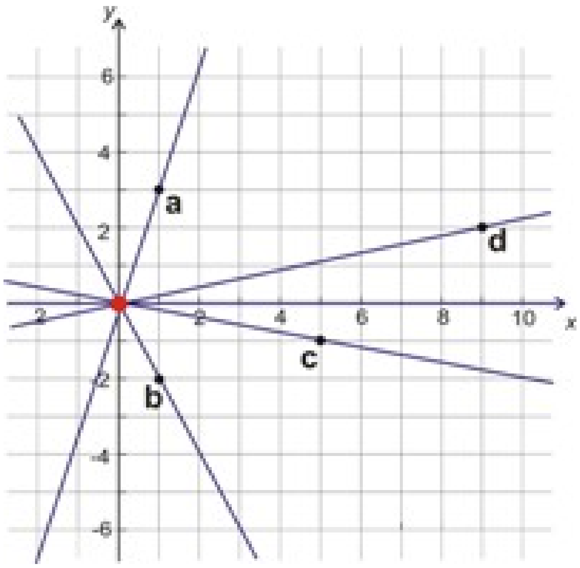
Plot the following direct relations on the same graph.

a. $y = 3x$

b. $y = -2x$

c. $y = -0.2x$

d. $y = \frac{2}{9}x$



Solution

a. The line passes through $(0,0)$. All these functions will pass through this point. It is plotted in red. This function has a slope of 3. When we move across by one unit, the function increases by three units.

b. The line has a slope of -2. When we move across the graph by one unit the function **falls** by two units.

c. The line has a slope of -0.2. As a fraction this is equal to $-\frac{1}{5}$ When we move across by five units, the function **falls** by one unit.

d. The line passes through $(0,0)$ and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time that we have one quantity that doubles when another related quantity doubles, we say that they follow a direct variation.

Newton's Second Law

In 1687, Sir Isaac Newton published the famous *Principea Mathematica*. It contained, among other things, his Second Law of Motion. This law is often written as:

$$F = m \cdot a$$

A force of F (Newtons) applied to a mass of m (kilograms) results in acceleration of a (meterspersecond^2).

Example 5

If a 175 *Newton force causes a heavily loaded shopping cart to accelerate down the aisle with an acceleration of* 2.5 m/s^2, *calculate*

(i) The mass of the shopping cart.

(ii) The force needed to accelerate the same cart at 6 m/s^2 .

Solution

(i) This question is basically asking us to solve for the constant of proportionality. Let us compare the two formulas.

$$y = k \cdot x \qquad \text{The direct variation equation}$$
$$F = m \cdot a \qquad \text{Newton's Second law}$$

We see that the two equations have the same form; y is analogous to force and x analogous to acceleration.

We can solve for m (the mass) by substituting our given values for force and acceleration:

Substitute $F = 175$, $a = 2.5$

$$175 = m(2.5) \qquad \text{Divide both sides by 2.5.}$$
$$70 = m$$

The mass of the shopping cart is 70 kg.

(ii) Once we have solved for the mass we simply substitute that value, plus our required acceleration back into the formula $F = m \cdot a$ and solve for F:

Substitute $m = 70$, $a = 6$

$$F = 70 \times 6 = 420$$

The force needed to accelerate the cart at 6 m/s^2 is 420 Newtons.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship:

$$V = I \cdot R \qquad \text{where R is the resistance (measured in Ohms)}$$

The resistance is considered to be a constant for all values of V and I.

Example 6

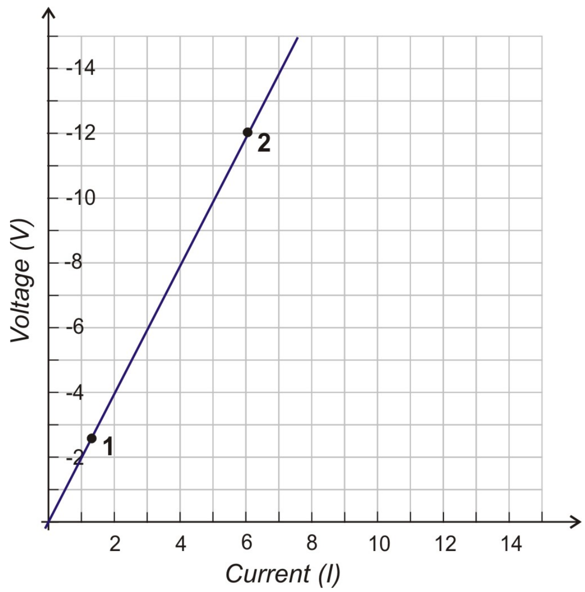
A certain electronics component was found to pass a current of 1.3 *amps at a voltage of* 2.6 *volts. When the voltage was increased to* 12.0 *volts the current was found to be* 6.0 *amps.*

a) Does the component obey Ohms law?

b) What would the current be at 6 *volts?*

Solution

a) Ohm's law is a simple direct proportionality law. Since the resistance R is constant, it acts as our constant of proportionality. In order to know if the component obeys Ohm's law we need to know if it follows a direct proportionality rule. In other words **is V directly proportional to I**?



Method One – Graph It

If we plot our two points on a graph and join them with a line, does the line pass through $(0,0)$?

Point $1 = 2.6$, $I = 1.3$ our point is $(1.3, 2.6)^*$

Point 2 $V = 12.0$, $I = 6.0$ our point is $(6, 12)$

Plotting the points and joining them gives the following graph.

The graph does appear to pass through the origin, so…

Yes, the component obeys Ohms law.

Method Two – Solve for R

We can quickly determine the value of R in each case. It is the ratio of the voltage to the resistance.

Case 1 $R = \frac{V}{I} = \frac{2.6}{1.3} = 2$ Ohms

Case 2 $R = \frac{V}{I} = \frac{12}{6} = 2$ Ohms

The values for R agree! This means that the line that joins point 1 to the origin is the same as the line that joins point 2 to the origin. **The component obeys Ohms law**.

b) To find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$

Substitute $V = 6$, $R = 2$

- In physics, it is customary to plot voltage on the horizontal axis as this is most often the independent variable. In that situation, the slope gives the **conductance, σ**. However, by plotting the current on the horizontal axis, the **slope** is equal to the **resistance, R**.

$$6 = I(2) \quad \text{Divide both sides by 2.}$$
$$3 = I$$

Solution

The current through the component at a voltage of 6 volts is 3 amps.

Lesson Summary

- If a variable y varies ***directly*** with variable x, then we write the relationship as

$$y = k \cdot x$$

Where k is a constant called the **constant of proportionality.**

- **Direct variation** is very common in many areas of science.

Review Questions

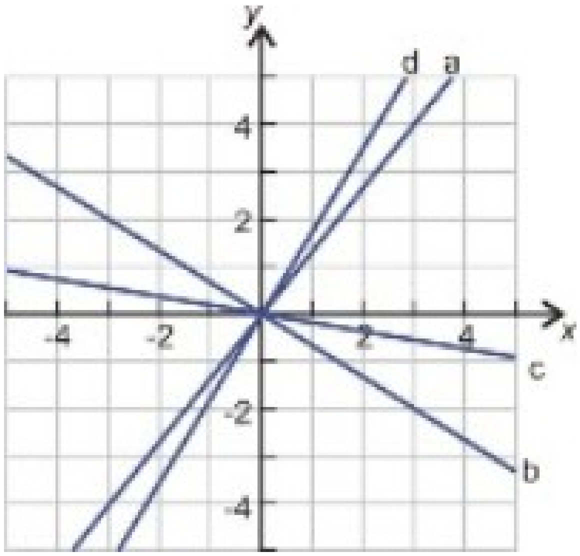
1. Plot the following direct variations on the same graph.

   (a) $y = \frac{4}{3}x$
   (b) $y = -\frac{2}{3}x$
   (c) $y = -\frac{1}{6}x$
   (d) $y = 1.75x$

2. Dasan's mom takes him to the video arcade for his birthday. In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20.00, how long can he keep playing games before his money is gone?
3. The current standard for low-flow showerheads heads is 2.5 gallons per minute. Calculate how long it would take to fill a 30 gallon bathtub using such a showerhead to supply the water.
4. Amen is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 P.M. and leaves it running all night. At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
5. Land in Wisconsin is for sale to property investors. A 232 acre lot is listed for sale for \$200500. Assuming the same price per acre, how much would a 60 acre lot sell for?
6. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:

   (a) The spring constant, k
   (b) The force needed to stretch the spring by 7 cm .
   (c) The distance the spring would stretch with a 23 Newton force.

Review Answers

1. 

2. 57 minutes 8 seconds
3. 12 minutes
4. 12 : 00 Midday
5. \$51, 853
6.
7. (a) $k = 1.2$ N/cm
   (b) 8.4 Newtons
   (c) 19.17 cm

4.7 Linear Function Graphs

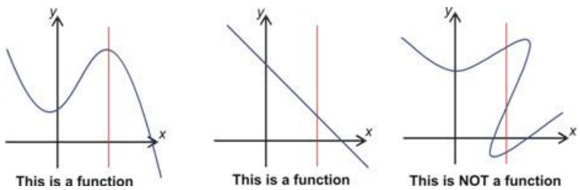
Learning Objectives

- Recognize and use function notation.
- Graph a linear function.
- Change slope and intercepts of function graphs.
- Analyze graphs of real-world functions.

Introduction – Functions

So far we have used the term **function** to describe many of the equations we have been graphing, but the concept of a function is extremely important in mathematics. Not all equations are functions. In order to be a function, the relationship between two variables, x and y, must map each $x-$value to **exactly one** $y-$value.

Visually this means the graph of y versus x must pass the **vertical line test** meaning that a vertical line drawn through the graph of the function must never intersect the graph in more than one place.



Use Function Notation

When we write functions we often use the notation '$f(x) =$' in place of '$y =$'. $f(x) =$ is read "f of x".

Example 1

Rewrite the following equations so that y is a function of x and written $f(x)$.

a. $y = 2x + 5$

b. $y = -0.2x + 7$

c. $x = 4y - 5$

d. $9x + 3y = 6$

Solution

a. Simply replace y with $f(x)$. $f(x) = 2x + 5$

b. $f(x) = -0.2x + 7$

c. Rearrange to isolate y.

$$\begin{aligned} x &= 4y - 5 && \text{Add 5 to both sides.} \\ x + 5 &= 4y && \text{Divide by 4.} \\ \frac{x+5}{4} &= y \\ f(x) &= \frac{x+5}{4} \end{aligned}$$

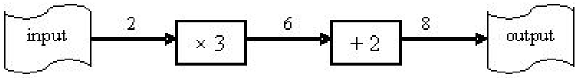
d. Rearrange to isolate y.

$$\begin{aligned} 9x + 3y &= 6 && \text{Subtract } 9x \text{ from both sides.} \\ 3y &= 6 - 9x && \text{Divide by 3.} \\ y &= \frac{6-x}{3} = 2 - 3x \\ f(x) &= 2 - 3x \end{aligned}$$

You can think of a function as a machine made up from a number of separate processes. For example, you can look at the function $3x + 2$ and break it down to the following instructions.

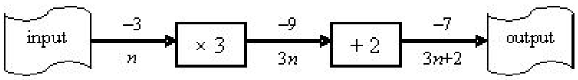
- Take a number
- Multiply it by 3
- Add 2

We can visualize these processes like this:



In this case, the number we chose was 2. Multiplied by 3 it becomes 6. When we add 2 our output is 8.

Let's try that again. This time we will put −3 through our machine to get 7.



On the bottom of this process tree you can see what happens when we put the letter *n* (the variable used to represent ***any*** number) through the function. We can write the results of these processes.

- $f(2) = 8$
- $f(-3) = -7$
- $f(n) = 3n + 2$

Example 2

A function is defined as $f(x) = 6x - 36$. *Evaluate the following:*

a. $f(2)$

b. $f(0)$

c. $f(36)$

d. $f(z)$

e. $f(p)$

Solution

a. Substitute $x = 2$ into the function $f(x)$ $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$

b. Substitute $x = 0$ into the function $f(x)$ $f(0) = 6 \cdot 0 - 36 = 0 - 36 = -36$

c. Substitute $x = 36$ into the function $f(x)$ $f(36) = 6 \cdot 36 - 36 = 216 - 36 = 180$

d. Substitute $x = z$ into the function $f(x)$ $f(z) = 6z + 36$

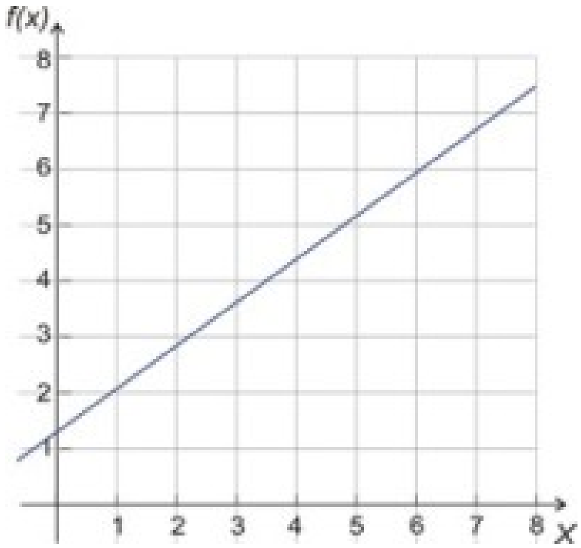
e. Substitute $x = p$ into the function $f(x)$ $f(p) = 6p + 36$

Graph a Linear Function

You can see that the notation '$f(x) =$' and '$y =$' are interchangeable. This means that we can use all the concepts we have learned so far to graph functions.

Example 3

Graph the function $f(x) = \frac{3x+5}{4}$



Solution

We can write this function in ***slope intercept*** form ($y = mx + b$ form).

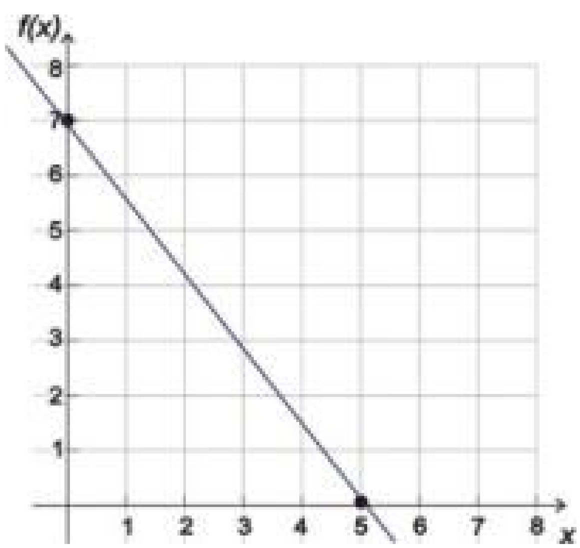
$$f(x) = \frac{3}{4}x + \frac{5}{4} = 0.75x + 1.25$$

So our graph will have a $y-$intercept of $(0, 1.25)$ and a slope of 0.75.

- Remember that this slope **rises** by 3 units for every 4 units we move right.

Example 4

Graph the function $f(x) = \frac{7(5-x)}{5}$



Solution

This time we will solve for the x and y intercepts.

To solve for $y-$intercept substitute $x = 0$.

$$f(0) = \frac{7(5-0)}{5} = \frac{35}{5} = 7$$

To solve for $x-$intercept substitute use $f(x) = 0$.

$$0 = \frac{7(5-x)}{5} \qquad \text{Multiply by 5 and distribute 7.}$$
$$5.0 = 35 - 7x \qquad \text{Add } 7x \text{ to both sides:}$$
$$7x = 35$$
$$x = 5$$

Our graph has intercepts $(0, 7)$ and $(5, 0)$.

Arithmetic Progressions

You may have noticed that with linear functions, when you increase the x value by one unit, the y value increases by a fixed amount. This amount is equal to the slope. For example, if we were to make a table of values for the function $f(x) = 2x + 3$ we might start at $x = 0$ then add one to x for each row.

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

Look at the values for $f(x)$. They go up by two (the slope) each time. When we consider continually adding a fixed value to numbers, we get sequences like $3, 5, 7, 9, 11....$ We call these **arithmetic progressions**. They are characterized by the fact that each number is greater than (or lesser than) than the preceding number by a fixed amount. This amount is called the **common difference**. The common difference can be found by taking two consecutive terms in a sequence and subtracting the first from the second.

Example 5

Find the common difference for the following arithmetic progressions:

a. $7, 11, 15, 19 \ldots$

b. $12, 1, -10, -21 \ldots$

c. $7, \ , 12, \ , 17 \ldots$

Solution

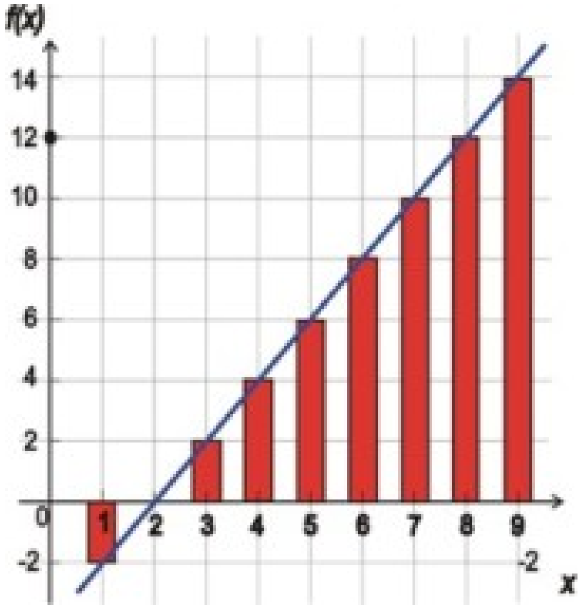
a.

$$11 - 7 = 4$$
$$15 - 11 = 4$$
$$19 - 15 = 4.$$

The common difference is 4.

b. $1 - 12 = -11$. *The common difference is* -11.

c. There are not two consecutive terms here, but we know that to get the term after 7, we would add the common difference. Then to get to 12, we would add the common difference again. Twice the common difference is $12 - 7 = 5$. So the common difference is 2.5.



Arithmetic sequences and linear functions are very closely related. You just learned that to get to the next term in a arithmetic sequence you add the common difference to last term. We have seen that with linear functions the function increases by the value of the slope every time the $x-$value is increased by one. As a result, arithmetic sequences and linear functions look very similar.

The graph to the right shows the arithmetic progression $-2, 0, 2, 4, 6 \ldots$ with the function $y = 2x - 4$. The fundamental difference between the two graphs is that an arithmetic sequence is **discrete** while a linear function is **continuous.**

- **Discrete** means that the sequence has x values only at distinct points (the 1^{st} term, 2^{nd} term, etc). The domain is not all real numbers (often it is whole numbers).
- **Continuous** means that the function has values for all possible values of x, the integers and also all of the numbers in between. The domain is all real numbers.

We can write a formula for an arithmetic progression. We will define the first term as a_1 and d as the common difference. The sequence becomes the following.

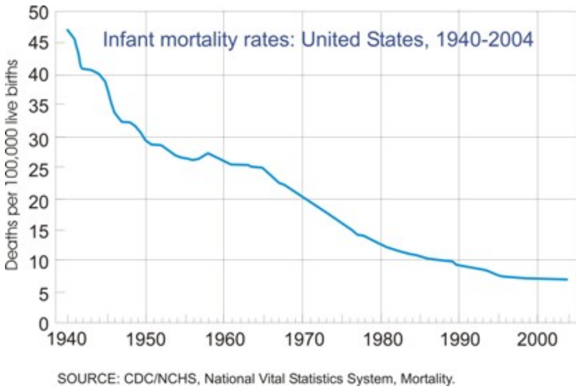
$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots, a_1 + n \cdot d$$

- To find the second term (a_2) we take the first term (a_1) and add d.
- To find the third term (a_3) we take the first term (a_1) and add $2d$.
- To find the nth term (a_n) we take the first term (a_1) and add $(n-1)d$.

Analyze Graphs of Real-World Functions

Example 6

Use the diagram below to determine the three decades since 1940 *in which the infant mortality rate decreased most.*



Let's make a table of the infant mortality rate in the years 1940, 1950, 1960, 1970, 1980, 1990, 2000.

Table 4.3: **Infant Mortality Rates: United States,**

Year	Mortality rate (per 100,000)	change over decade
1940	47	N/A
1950	30	−17
1960	26	−4
1970	20	−6
1980	13	−7
1990	9	−4
2000	7	−2

Solution

The best performing decades were the 1940s (1940 − 1950) with a drop of 17 deaths per 100000. The 1970s (1970 − 1980) with a drop of 7 deaths per 100000. The 1960s (1960 − 1970) with a drop of 6 deaths per 100000.

Lesson Summary

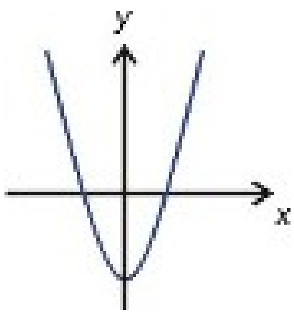
- In order for an equation to be a **function,** the relationship between the two variables, x and y, must map each $x-$value to *exactly* one $y-$value, or $y = f(x)$.
- The graph of a function of y versus x must pass the **vertical line test**. Any vertical line will only cross the graph of the function in one place.
- The sequence of $f(x)$ values for a linear function form an arithmetic progression. Each number is greater than (or less than) the preceding number by a fixed amount, or **common difference.**

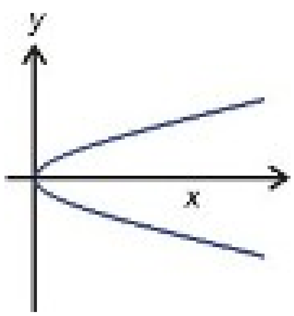
Review Questions

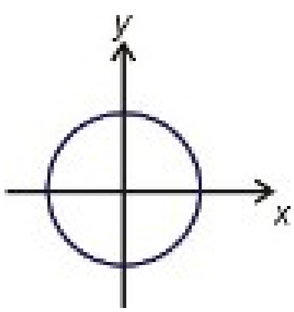
1. When an object falls under gravity, it gains speed at a constant rate of 9.8 m/s every second. An item dropped from the top of the Eiffel Tower, which is 300 meters tall, takes 7.8 seconds to hit the ground. How fast is it moving on impact?
2. A prepaid phone card comes with $20 worth of calls on it. Calls cost a flat rate of $0.16 per minute. Write the value of the card as a function of minutes per calls. Use a function to determine the number of minutes you can make with the card.
3. For each of the following functions evaluate:

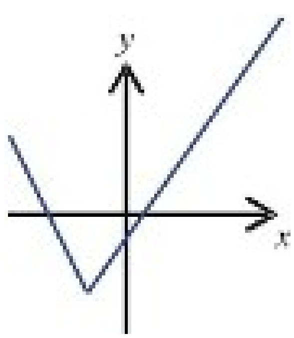
(a) $f(x) = -2x + 3$
(b) $f(x) = 0.7x + 3.2$
(c) $f(x) = \frac{5(2-x)}{11}$
   i. $f(-3)$
   ii. $f(7)$
   iii. $f(0)$
   iv. $f(z)$

4. Determine whether the following could be graphs of **functions**.

(a)


(b)


(c)


(d)


5. The roasting guide for a turkey suggests cooking for 100 minutes plus an additional 8 minutes per pound.

(a) Write a function for the roasting time the given the turkey weight in pounds (x).
(b) Determine the time needed to roast a 10 lb turkey.

(c) Determine the time needed to roast a 27 lb turkey.
(d) Determine the maximum size turkey you could roast in $4\frac{1}{2}$ hours .

6. Determine the missing terms in the following arithmetic progressions.

(a) $\{-11, 17, \ , 73\}$
(b) $\{2, \ , -4\}$
(c) $\{13, \ , \ , \ , 0\}$

Review Answers

1. 76.44 m/s
2. $f(x) = 2000 - 16x125$ minutes
3.
4. (a) i. 9
   ii. -11
   iii. 3
   iv. $f(z) = -2z + 3$
   (b) i. 1.1
   ii. 8.1
   iii. 3.2
   iv. $f(z) = 0.7z + 3.2$
   (c) i. 2.27
   ii. -2.27
   iii. 0.909
   iv. $f(z) = \frac{10}{11} - \frac{5}{11}z$
5.
6. (a) yes
   (b) no
   (c) no
   (d) yes
7.
8. (a) $f(x) = 8x + 100$
   (b) 180 min = 3 hrs
   (c) 316 min = 5 hrs 16 min
   (d) 21.25 lbs.
9.
10. (a) 45
    (b) -1
    (c) $9.75, 6.5, 3.25$

4.8 Problem-Solving Strategies - Graphs

Learning Objectives

- Read and understand given problem situations.
- Use the strategy: read a graph.
- Develop and apply the strategy: make a graph.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we have been solving problems where quantities are linearly related to each other. In this section, we will look at a few examples of linear relationships that occur in real-world problems. Remember back to our Problem Solving Plan.

Step 1:

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2:

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3:

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4:

Look – Check and Interpret

Check to see if your answer makes sense.

Let's look at an example that investigates a geometrical relationship.

Example 1

A cell phone company is offering its costumers the following deal. You can buy a new cell phone for $60 *and pay a monthly flat rate of* $40 *per month for unlimited calls. How much money will this deal cost you after* 9 *months?*

Solution

Let's follow the problem solving plan.

Step 1:

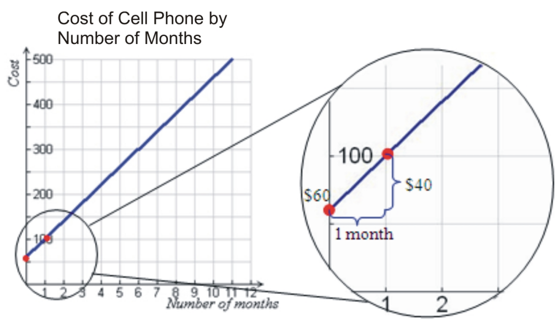
$$\text{cell phone} = \$60, \text{calling plan} = \$40 \text{per month}$$
$$\text{Let} x = \text{number of months}$$
$$\text{Let} y => \text{cost in dollars}$$

Step 2: Let's solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

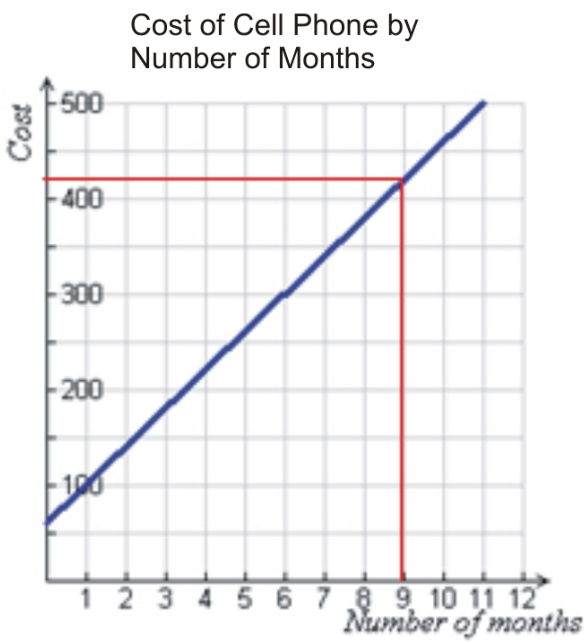
Since you pay \$60 for the phone when you get the phone, then the $y-$intercept is $(0, 60)$.

You pay \$40 for each month, so the cost rises by \$40 for one month, so the slope $= 40$.

We can graph this line using the slope-intercept method.



Step 3: The question was: *"How much will this deal cost after* 9 *months?"*



We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.

We see that after 9 months **you pay approximately** \$420.

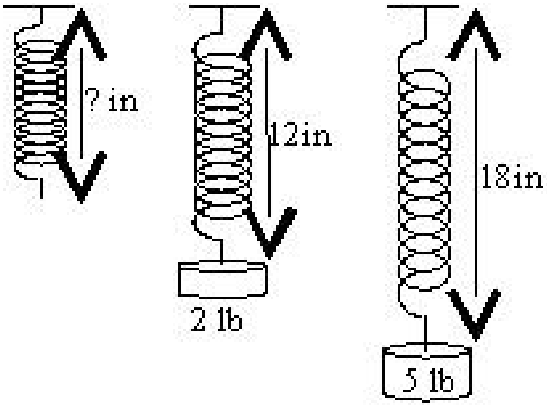
Step 4: To check if this is correct, let's think of the deal again. Originally, you pay \$60 and then \$40 for 9 months.

$$\begin{aligned}
\text{Phone} &= \$60 \\
\text{Calling plan} &= \$40 \times 9 = \$360 \\
\text{Total cost} &= \$420.
\end{aligned}$$

The answer checks out.

Example 2

A stretched spring has a length of 12 inches *when a weight of* 2 *lbs is attached to the spring. The same* *spring has a length of* 18 *inches when a weight of* 5 *lbs is attached to the spring. It is known from physics that within certain weight limits, the function that describes how much a spring stretches with different weights is a linear function. What is the length of the spring when no weights are attached?*



Solution

Let's apply problem solving techniques to our problem.

Step 1:

We know: the length of the spring = 12 inches when weight = 2 lbs.

the length of the spring = 18 inches when weight = 5 lbs.

We want: the length of the spring when the weight = 0 lbs.

Let x = the weight attached to the spring.

Let y = the length of the spring

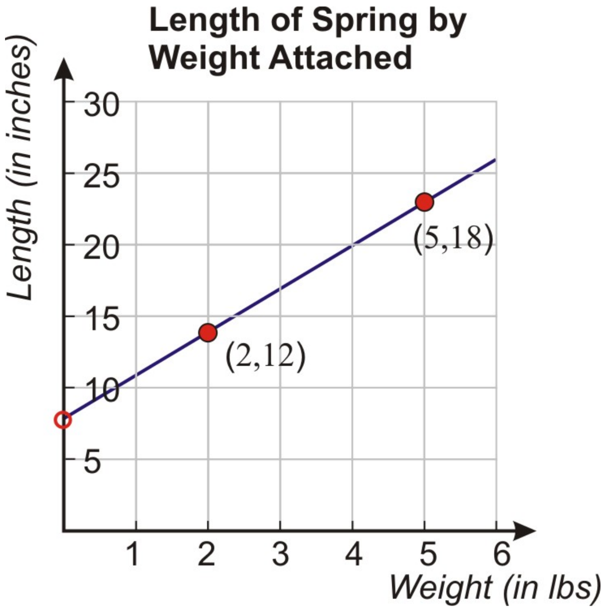
Step 2

Let's solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph.

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point $(2, 12)$.

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point $(5, 18)$.



If we join these two points by a line and extend it in both directions we get the relationship between weight and length of the spring.

Step 3

The question was: *"What is the length of the spring when no weights are attached?*

We can answer this question by reading the graph we just made. When there is no weight on the spring, the x value equals to zero, so we are just looking for the $y-$ intercept of the graph. Looking at the graph we see that the $y-$ intercept is **approximately** 8 inches .

Step 4

To check if this correct, let's think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs, so the slope of the line is $\frac{6\ inches}{3\ lbs} = 2\ inches/lb$.

To find the length of the spring when there is no weight attached, we look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. Since we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is 12 inches -4 inches $=8$ inches.

The answer checks out.

Example 3

Christine took one hour to read 22 *pages of* Harry Potter and the Order of the Phoenix. *She has* 100 *pages left to read in order to finish the book. Assuming that she reads at a constant rate of pages per hour, how much time should she expect to spend reading in order to finish the book?*

Solution: Let's apply the problem solving techniques:

Step 1

We know that it takes Christine takes 1 hour to read 22 pages.

We want to know how much time it takes her to read 100 pages.

Let $x =$ the time expressed in hours.

Let $y =$ the number of pages.

Step 2

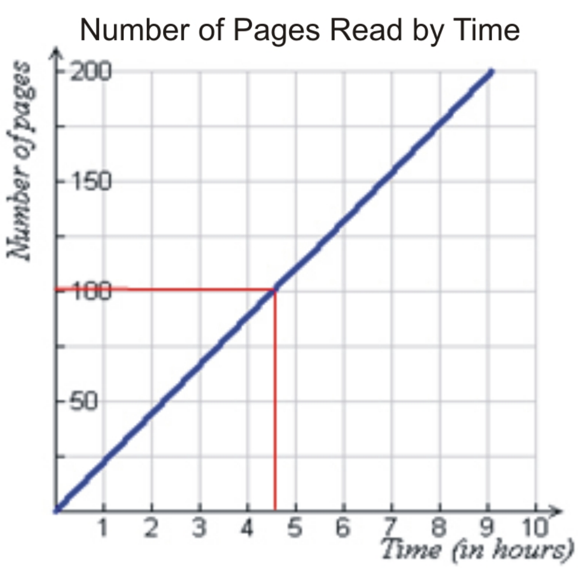
Let's solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph.

Christine takes one hour to read 22 pages. This gives point $(1, 22)$.

A second point is not given but we know that Christine takes 0 hours to read 0 pages. This gives the point $(0, 0)$.

If we join these two points by a line and extend it in both directions we get the relationship between the amount of time spent reading and the number of pages read.



Step 3

The question was: *"How much time should Christine expect to spend reading* 100 *pages?"*

We find the answer from reading the graph – we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately** 4.5 hours to read the remaining 100 pages.

Step 4

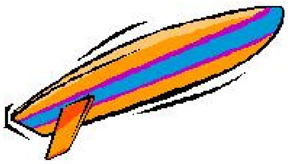
To check if this correct, let's think of the problem again.

We know that Christine reads 22 pages per hour. This is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \ pages}{22 \ pages \ per \ hour} = 4.54 \ hours$. This is very close to what we gathered from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs $249. He was given a birthday present of $50 and he has a summer job that pays him $6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?



Solution

Let's apply the problem solving techniques.

Step 1

We know – Surfboard costs $249.

He has $50.

His job pays $6.50 per hour.

We want – How many hours does Aatif need to work to buy the surfboard?

Let $x =$ the time expressed in hours

Let $y =$ Aatif's earnings

Step 2

Let's solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Peter has $50 at the beginning. This is the $y-$intercept of $(0, 50)$.

He earns $6.50 per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the $y-$ intercept of $(0, 50)$ and we know that for each unit in the horizontal direction the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.

Step 3

The question was *"How many hours does Aatif need to work in order to buy the surfboard?"*

We find the answer from reading the graph. Since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately** 31 **hours** to earn the money.

Step 4

To check if this correct, let's think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - 50 = 199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide $\frac{\$199}{\$6.50 \text{ per hour}} = 30.6$ hours. This is very close to the result we obtained from reading the graph.

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** are:

1. **Understand the problem**
2. **Devise a plan – Translate.** Build a graph.
3. **Carry out the plan – Solve.** Use the graph to answer the question asked.
4. **Look – Check and Interpret**

Review Questions

Solve the following problems by making a graph and reading a graph.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39. How much will this membership cost a member by the end of the year?

2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit. What was the original length of the candle?
3. Tali is trying to find the width of a page of his telephone book. In order to do this, he takes a measurement and finds out that 550 pages measures 1.25 inches . What is the width of one page of the phone book?
4. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make $25. How many glasses of lemonade must they sell to break even?

Review Answers

1. $668
2. 5.67 inches
3. 0.0023 inches
4. 56 glasses

Chapter 5

Writing Linear Equations

5.1 Linear Equations in Slope-Intercept Form

Learning Objectives

- Write an equation given slope and $y-$intercept.
- Write an equation given the slope and a point.
- Write an equation given two points.
- Write a linear function in slope-intercept form.
- Solve real-world problems using linear models in slope-intercept form.

Introduction

We saw in the last chapter that linear graphs and equations are used to describe a variety of real-life situations. In mathematics, we want to find equations that explain a situation as presented in a problem. In this way, we can determine the rule that describes the relationship between the variables in the problem. Knowing the equation or rule is very important since it allows us to find the values for the variables. There are different ways to find an equation that describes the problem. The methods are based on the information you can gather from the problem. In graphing these equations, we will assume that the domain is all real numbers.

Write an Equation Given Slope and $y-$intercept

Let's start by learning how to write an equation in slope–intercept form $y = mx + b$.

b is the $y-$intercept (*the value of* y *when* $x = 0$. *This is the point where the line crosses the* $y-$*axis)*.

m is the slope *how the quantity* y *changes with each one unit of* x.

If you are given the slope and $y-$intercept of a line:

1. Start with the slope–intercept form of the line $y = mx + b$.
2. Substitute the given values of m and b into the equation.

Example 1

a) *Write an equation with a* slope $= 4$ *and a* $y-$*intercept* $= -3$.

b) *Write an equation with a* slope $= -2$ *and a* $y-$*intercept* $= 7$.

c) *Write an equation with a* slope $= \frac{2}{3}$ *and a* $y-$intercept $= \frac{4}{5}$.

a) **Solution**

We are given $m = 4$ and $b = -3$. Plug these values into the slope–intercept form $y = mx + b$.

$$y = 4x - 3$$

b) **Solution**

We are given $m = -2$ and $b = 7$. Plug these values into the slope–intercept form $y = mx + b$..

$$y = -2x + 7$$

c) **Solution**

We are given $m = \frac{2}{3}$ and $b = \frac{4}{5}$. Plug these values into the slope–intercept form $y = mx + b$.

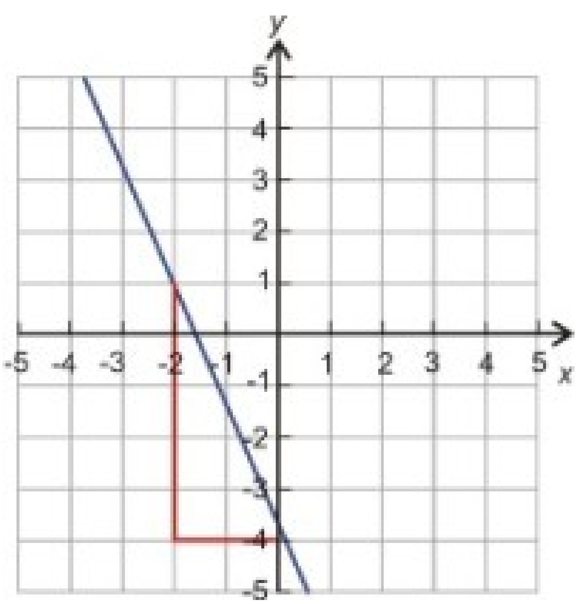
$$y = \frac{2}{3}x + \frac{4}{5}$$

You can also write an equation in slope-intercept form if you are given the graph of the line.

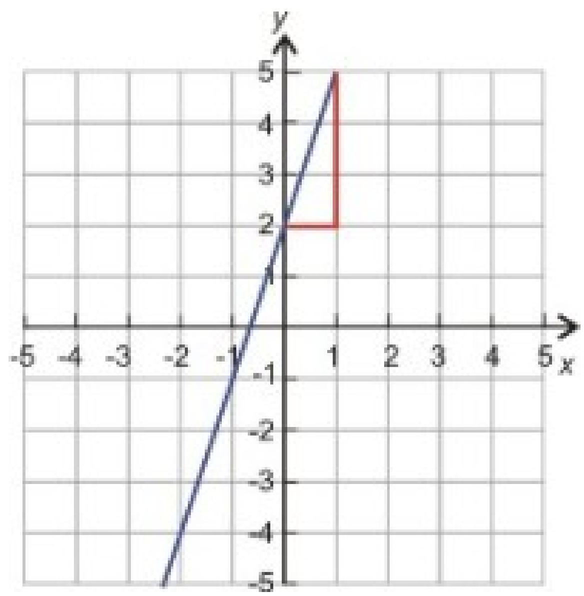
Example 2

Write the equation of each line in slope–intercept form.

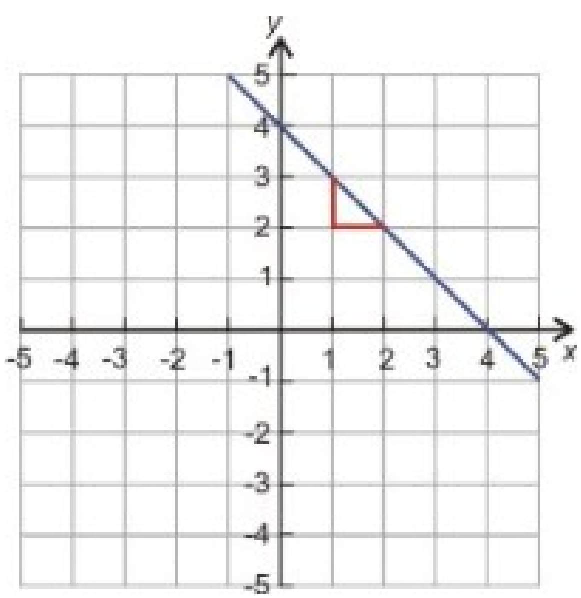
a)



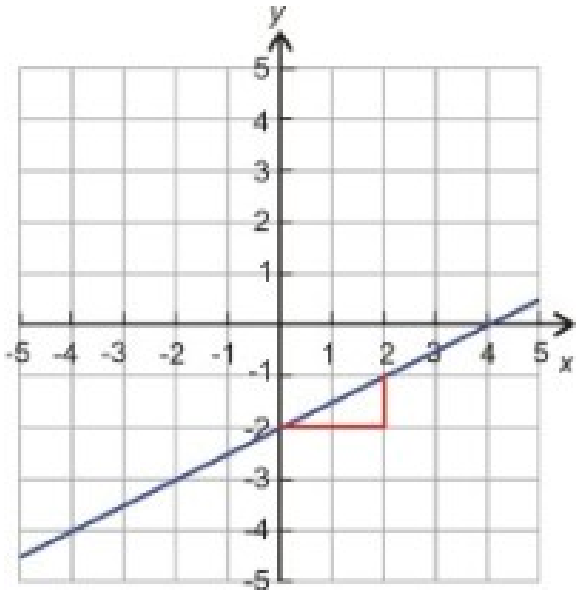
b)



c)



d)



a) The $y-$intercept $= -4$ and the slope $= -\frac{5}{2}$. Plug these values into the slope–intercept form $y = mx + b$.

Solution

$$-\frac{5}{2}x - 4$$

b) The $y-$intercept $= 2$ and the slope $=\frac{3}{1}$. Plug these values into the slope–intercept form $y = mx + b$.

Solution

$$y = 3x + 2$$

c) The $y-$intercept $= 4$ and the slope $= -\frac{1}{1}$. Plug these values into the slope–intercept form $y = mx + b$.

Solution

$$y = -x + 4$$

d) The $y-$intercept $= -2$ and the slope $= \frac{1}{2}$. Plug these values into the slope–intercept form $y = mx + b$.

Solution

$$y = \frac{1}{2}x - 2.$$

Write an Equation Given the Slope and a Point

Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case we can still use the slope–intercept form to find the equation of the line.

For example, we are told that the slope of a line is two and that the line passes through the point $(1, 4)$. To find the equation of the line, we start with the slope–intercept form of a line.

$$y = mx + b$$

Plug in the value of the slope.

We don't know the value of b but we know that the slope is two, and that point $(1,4)$ is on this line. Where the x value is one , and the y value is four. We plug this point in the equation and solve for b.

$$\begin{aligned}4 &= 2(1) + b\\ 4 &= 2 + b\\ -2 &= -2\\ 2 &= b\end{aligned}$$

Therefore the equation of this line is $y = 2x + 2$.

If you are given the slope and a point on the line:

1. Start with the slope–intercept form of the line $y = mx + b$.
2. Plug in the given value of m into the equation.
3. Plug the x and y values of the given point and solve for b.
4. Plug the value of b into the equation.

Example 3

Write the equation of the line in slope–intercept form.

a) The slope of the line is 4 and the line contains point $(-1,5)$.

b) The slope of the line is $-\frac{2}{3}$ and the line contains point $(2,-2)$.

c) The slope of the line is –3 and the line contains point $(3,-5)$.

Solution

a)

$$\begin{aligned}\text{Start with the slope–intercept form of the line} \qquad & y = mx + b\\ \text{Plug in the slope.} \qquad & y = 4x + b\\ \text{Plug point } (-1,5) \text{ into the equation.} \qquad & 5 = 4(-1) + b \Rightarrow b = 9\\ \text{Plug the value of b into the equation.} \qquad & y = 4x + 9\end{aligned}$$

b)

$$\begin{aligned}\text{Start with the slope–intercept form of the line} \qquad & y = mx + b\\ \text{Plug in the slope.} \qquad & y = -\frac{2}{3}x + b\\ \text{Plug point } (2,-2) \text{ into the equation.} \qquad & -2 = -\frac{2}{3}(2) + b \Rightarrow b = -2 + \frac{4}{3} = -\frac{2}{3}\\ \text{Plug the value of b into the equation.} \qquad & y = -\frac{2}{3}x - \frac{2}{3}\end{aligned}$$

c)

$$\begin{aligned}\text{Start with the slope–intercept form of the line} \qquad y &= mx + b\\ \text{Plug in the slope.} \qquad y &= -3x + b\\ \text{Plug point } (3,-5) \text{ into the equation.} \qquad -5 &= -3(3) + b \Rightarrow b = 4\\ \text{Plug the value of b into the equation.} \qquad y &= -3x + 4\end{aligned}$$

Write an Equation Given Two Points

One last case is when we are just given two points on the line and we are asked to write the line of the equation in slope–intercept form.

For example, we are told that the line passes through the points $(-2, 3)$ and $(5, 2)$. To find the equation of the line we start with the slope–intercept form of a line

$$y = mx + b$$

Since we don't know the slope, we find it using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Now substitute the x_1 and x_2 and the y_1 and y_2 values into the slope formula to solve for the slope.

$$m = \frac{2 - 3}{5 - (-2)} = -\frac{1}{7}$$

We plug the value of the slope into the slope–intercept form $y = -\frac{1}{7}x + b$

We don't know the value of b but we know two points on the line. We can plug either point into the equation and solve for b. Let's use point $(-2, 3)$.

Therefore, the equation of this line is $y = -\frac{1}{7}x + \frac{19}{7}$.

If you are given two points on the line:

1. Start with the slope–intercept form of the line $y = mx + b$
2. Use the two points to find the slope using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.
3. Plug the given value of m into the equation.
4. Plug the x and y values of one of the given points into the equation and solve for b.
5. Plug the value of b into the equation.
6. Plug the other point into the equation to check the values of m and b.

Example 4

Write the equations of each line in slope–intercept form.

a) The line contains the points $(3, 2)$ and $(-2, 4)$.

b) The line contains the points $(-4, 1)$ and $(-2, 3)$.

Solution:

a)

1. Start with the slope–intercept form of the line $y = mx + b$.
2. Find the slope of the line. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{-2 - 3} = -\frac{2}{5}$
3. Plug in the value of the slope. $y = -\frac{2}{5}x + b$
4. Plug point $(3, 2)$ into the equation. $2 = -\frac{2}{5}(3) + b \Rightarrow b = 2 + \frac{6}{5} = \frac{16}{5}$
5. Plug the value of b into the equation. $y = -\frac{2}{5}x + \frac{16}{5}$
6. Plug point $(-2, 4)$ into the equation to check. $4 = -\frac{2}{5}(-2) + \frac{16}{5} = \frac{4}{5} + \frac{16}{5} = \frac{20}{5} = 4$

b)

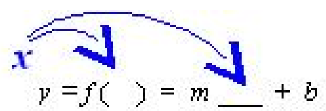
1. Start with the slope–intercept form of the line $y = mx + b$.

2. Find the slope of the line. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-2-(-4)} = \frac{2}{2} = 1$
3. Plug in the value of the slope. $y = x + b$
4. Plug point $(-2, 3)$ into the equation. $3 = -2 + b \Rightarrow b = 5$
5. Plug the value of b into the equation. $y = x + 5$
6. Plug point $(-4, 1)$ into the equation to check. $1 = -4 + 5 = 1$

Write a Linear Function in Slope-Intercept Form

Remember that you write a linear function in the form $f(x) = mx + b$. Here $f(x)$ represents the y values of the equation or the graph. So $y = f(x)$ and they are often used interchangeably. Using the functional notation in an equation gives us more information.

For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you **plug in** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y.



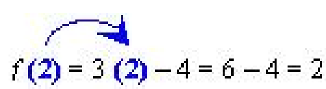
In this case when you plug x into the function, the function tells you to multiply it by m and then add b to the result. This generates all the values of y you need.

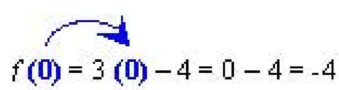
Example 5

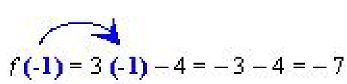
Consider the linear function $f(x) = 3x - 4$. *Find* $f(2)$, $f(0)$ *and* $f(-1)$.

Solution

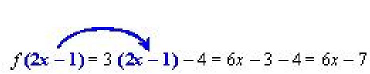
All the numbers in the parentheses are the values of x that you need to plug into the equation of the function.







When you plug values into a function, it is best to plug in the whole parenthesis, not just the value inside the parenthesis. We often plug expressions into the function instead of numbers, and it is important to keep the expression inside the parenthesis in order to perform the correct order of operations. For example, we want to find $f(2x - 1)$ for the same function we used before.

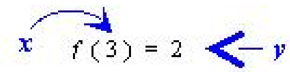


Functional notation is a very compact way of giving information. For example you are told that $f(3) = 2$.

To read this information, remember a few things.

The value inside the parentheses is the $x-$value.

The value equal to the function is the dependent value (i.e. the $y-$value for lines).



So, $f(3) = 2$ tells you that $x = 3$ and $y = 2$ or that point $(3,2)$ is on the line.

We will now use functional notation to write equations of lines in slope–intercept form.

Example 6

Find the equation of the following lines in slope–intercept form

a) $m = -2$ *and* $f(0) = 5$.

b) $m = 3.5$ *and* $f(-2) = 1$.

c) $f(-1) = 1$ *and* $f(1) = -1$.

Solution

a) We are told that $m = -2$ and the line contains point $(0,5)$, so $b = 5$.

Plug the values of m and b into the slope–intercept form $f(x) = mx + b$.

$$f(x) = -2x + 5.$$

b) We are told that $m = 3.5$ and line contains point $(-2,1)$.

Start with slope–intercept form.	$f(x) = mx + b$
Plug in the value of the slope.	$f(x) = 3.5x + b$
Plug in the point $(-2,1)$.	$1 = 3.5(-2) + \Rightarrow b = 1 + 7 = 8$
Plug the value of b in the equation.	$f(x) = 3.5x + 8$

c) We are told that the line contains the points $(-1,1)$ and $(1,-1)$.

Start with slope–intercept form.	$f(x) = mx + b$
Find the slope.	$m = \dfrac{-1-1}{1-(-1)} = \dfrac{-2}{2} = -1$
Plug in the value of the slope.	$f(x) = -1x + b$
Plug in the point	$(-1,1).1 = -1(-1) + b \Rightarrow b = 0$
Plug the value of b in the equation.	$f(x) = -x$

Solve Real-World Problems Using Linear Models in Slope-Intercept Form

Let's apply the methods we just learned to a few application problems that can be modeled using a linear relationship.

Example 7

Nadia has \$200 *in her savings account. She gets a job that pays* \$7.50 *per hour and she deposits all her earnings in her savings account. Write the equation describing this problem in slope–intercept form. How many hours would Nadia need to work to have* \$500 *in her account?*

Let's define our variables

$y =$ amount of money in Nadia's savings account

$x =$ number of hours

You can see that the problem gives us the y–intercept and the slope of the equation.

We are told that Nadia has \$200 in her savings account, so $b = 200$.

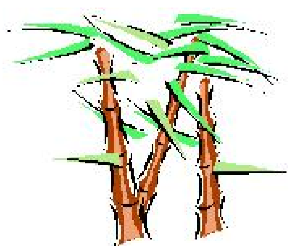
We are told that Nadia has a job that pays \$7.50 per hour, so $m = 7.50$.

If we plug these values in the slope–intercept form $y = mx + b$ we obtain $y = 7.5x + 200$.

To answer the question, we plug in $y = 500$ and solve for x. $500 = 7.5x + 200 \Rightarrow 7.5x = 300 \Rightarrow x = 40$ hours.

Solution

Nadia must work 40 hours if she is to have \$500 in her account.



Example 8

A stalk of bamboo of the family Phyllostachys nigra grows at steady rate of 12 *inches per day and achieves its full height of* 720 *inches in* 60 *days. Write the equation describing this problem in slope–intercept form.*

How tall is the bamboo 12 *days after it started growing?*

Let's define our variables

$y =$ the height of the bamboo plant in inches

$x =$ number of days

You can see that the problem gives us the slope of the equation and a point on the line.

We are told that the bamboo grows at a rate of 12 inches per day, so $m = 12$.

We are told that the plant grows to 720 inches in 60 days, so we have the point $(60, 720)$.

Start with the slope–intercept form of the line	$y = mx + b$
Plug in the slope.	$y = 12x + b$
Plug in point $(60, 720)$.	$720 = 12(60) + b \Rightarrow b = 0$
Plug the value of b back into the equation.	$y = 12x$

To answer the question, plug in $x = 12$ to obtain $y = 12(12) = 144$ inches.

Solution

The bamboo is 144 inches (12 feet!) tall 12 days after it started growing.

Example 9

Petra is testing a bungee cord. She ties one end of the bungee cord to the top of a bridge and to the other end she ties different weights and measures how far the bungee stretches. She finds that for a weight of 100 lb, *the bungee stretches to* 265 *feet and for a weight of* 120 lb, *the bungee stretches to* 275 *feet . Physics tells us that in a certain range of values, including the ones given here, the amount of stretch is a linear function of the weight. Write the equation describing this problem in slope–intercept form. What should we expect the stretched length of the cord to be for a weight of* 150 lbs*?*

Let's define our variables

$y =$ the stretched length of the bungee cord in feet in feet

$x =$ the weight attached to the bungee cord in pounds

You can see that the problem gives us two points on the line.

We are told that for a weight of 100 lbs the cord stretches to 265 feet, so we have point $(100, 265)$.

We are told that for a weight of 200 lbs the cord stretches to 275 feet, so we have point $(120, 270)$.

Start with the slope–intercept form of the line	$y = mx + b$
Find the slope of the line.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{270 - 265}{120 - 100} = \frac{5}{20} = \frac{1}{4}$
Plug in the value of the slope.	$y = \frac{1}{4}x + b$
Plug point $(100, 265)$ into the equation.	$265 = \frac{1}{4}(100) + b \Rightarrow b = 265 - 25 = 240$
Plug the value of b into the equation.	$y = \frac{1}{4}x + 240$

To answer the question, we plug in $x = 150$. $y = \frac{1}{4}(150) + 240 \Rightarrow y = 37.5 + 240 = 277.5$ feet

Solution

For a weight of 150 lbs we expect the stretched length of the cord to be 277.5 feet.

Lesson Summary

- The equation of a line in **slope-intercept** form is $y = mx + b$.

Where m is the slope and $(0, b)$ is the $y-$intercept).

- If you are **given the slope and $y-$intercept** of a line:

1. Simply plug m and b into the equation.

- If you are **given the slope and a point** on the line:

1. Plug in the given value of m into the equation.
2. Plug the x and y values of the given point and solve for b.
3. Plug the value of b into the equation.

- If you are given two points on the line:

1. Use the two points to find the slope using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.
2. Plug the value of m into the equation.
3. Plug the x and y values of one of the given points and solve for b.
4. Plug the value of b into the equation.
5. Plug the other point into the equation to check the values of m and b.

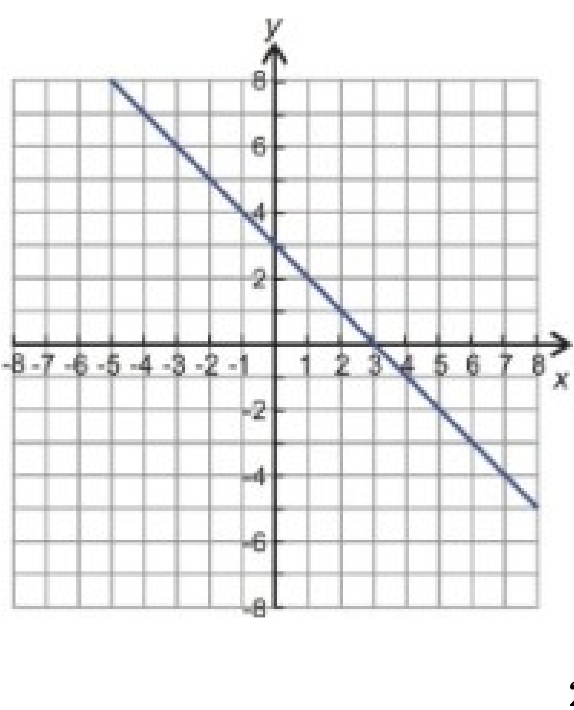
Review Questions

Find the equation of the line in slope–intercept form.

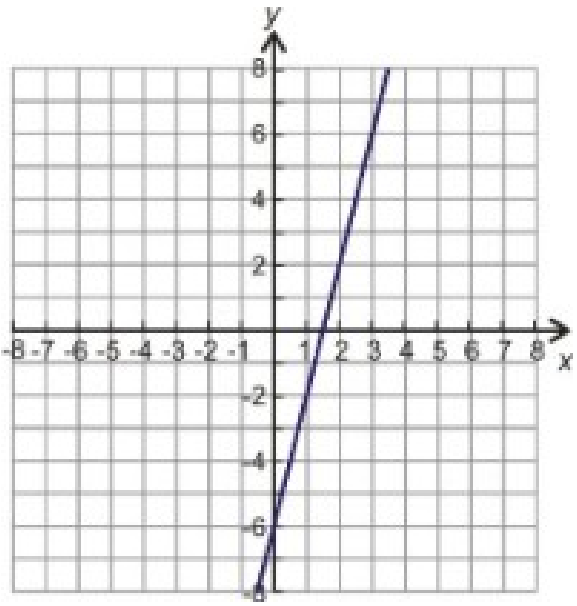
1. The line has slope of 7 and y–intercept of -2.
2. The line has slope of -5 and y–intercept of 6.
3. The line has slope of $-\frac{1}{4}$ and contains point $(4, -1)$.
4. The line has slope of $\frac{2}{3}$ and contains point $\left(\frac{1}{2}, 1\right)$.
5. The line has slope of -1 and contains point $\left(\frac{4}{5}, 0\right)$.
6. The line contains points $(2, 6)$ and $(5, 0)$.
7. The line contains points $(5, -2)$ and $(8, 4)$.
8. The line contains points $(3, 5)$ and $(-3, 0)$.
9. The line contains points $(10, 15)$ and $(12, 20)$.

Write the equation of each line in slope-intercept form.

10.



11.



Find the equation of the linear function in slope–intercept form.

12. $m = 5$, $f(0) = -3$
13. $m = -7$, $f(2) = -1$
14. $m = \frac{1}{3}$, $f(-1) = \frac{2}{3}$
15. $m = 4.2$, $f(-3) = 7.1$
16. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
17. $f(1.5) = -3$, $f(-1) = 2$
18. To buy a car, Andrew puts a down payment of $1500 and pays $350 per month in installments. Write an equation describing this problem in slope-intercept form. How much money has Andrew paid at the end of one year?
19. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, write an equation describing this problem in slope-intercept form. What was the height of the rose when Anne planted it?
20. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs. and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs., hangs from it?

Review Answers

1. $y = 7x - 2$
2. $y = -5x + 6$
3. $y = -\frac{1}{4}x$
4. $y = \frac{2}{3}x + \frac{2}{3}$
5. $y = -1x + \frac{4}{5}$
6. $y = -2x + 10$
7. $y = 2x - 12$
8. $y = \frac{5}{6}x + \frac{5}{2}$

9. $y = \frac{5}{2}x - 10$
10. $y = -x + 3$
11. $y = 4x - 6$
12. $f(x) = 5x - 3$
13. $f(x) = -7x + 13$
14. $f(x) = \frac{1}{3}x + 1$
15. $f(x) = 4.2x + 19.7$
16. $f(x) = -2x + \frac{5}{4}$
17. $f(x) = -2x$
18. $y = 350x + 1500$; $y = \$5700$
19. $y = 0.5x + 8.5$; $y = 8.5$ inches
20. $y = .025x + 1$ or $y = \frac{1}{40}x + 1$; $y = 4.5$ m

5.2 Linear Equations in Point-Slope Form

Learning Objectives

- Write an equation in point-slope form.
- Graph an equation in point-slope form.
- Write a linear function in point-slope form.
- Solve real-world problems using linear models in point-slope form.

Introduction

In the last lesson, we saw how to write the equation of a straight line in slope-intercept form. We can rewrite this equation in another way that sometimes makes solving the problem easier. The equation of a straight line that we are going to talk about is called **point-slope form.**

$$y - y_0 = m(x - x_0)$$

Here m is the slope and (x_0, y_0) is a point on the line. Let's see how we can use this form of the equation in the three cases that we talked about in the last section.

Case 1: You know the slope of the line and the $y-$intercept.
Case 2: You know the slope of the line and a point on the line.
Case 3: You know two points on the line.

Write an Equation in Point-Slope Form

Case 1 You know the slope and the $y-$intercept.

1. Start with the equation in point-slope form $y - y_0 = m(x - x_0)$.
2. Plug in the value of the slope.
3. Plug in 0 for x_0 and b for y_0.

Example 1

Write the equation of the line in point-slope form, given that the slope $= -5$ and the $y-$intercept $= 4$.

Solution:

1. Start with the equation in point-slope form. $y - y_0 = m(x - x_0)$
2. Plug in the value of the slope. $y - y_0 = -5(x - x_0)$
3. Plug in 0 for x_0 and 4 for y_0. $y - (-4) = -5(x - (0))$

Therefore, the equation is $y + 4 = -5x$

Case 2 You know the slope and a point on the line.

1. Start with the equation in point-slope form $y - y_0 = m(x - x_0)$.
2. Plug in the value of the slope.
3. Plug in the x and y values in place of x_0 and y_0.

Example 2

Write the equation of the line in point-slope form, given that the slope $= \frac{3}{5}$ *and the point* $(2, 6)$ *is on the line.*

Solution:

1. Start with the equation in point-slope form. $y - y_0 = m(x - x_0)$
2. Plug in the value of the slope. $y - y_0 = \frac{3}{5}(x - x_0)$
3. Plug in 2 for x_0 and 6 for y_0. $y - (6) = \frac{3}{5}(x - (2))$

The equation is $y - 6 = \frac{3}{5}(x - 2)$

Notice that the equation in point-slope form is not solved for y.

Case 3 You know two points on the line.

1. Start with the equation in point-slope form $y - y_0 = m(x - x_0)$.
2. Find the slope using the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$
3. Plug in the value of the slope.
4. Plug in the x and y values of one of the given points in place of x_0 and y_0

Example 3

Write the equation of the line in point-slope form, given that the line contains points $(-4, -2)$ *and* $(8, 12)$.

Solution

1. Start with the equation in point-slope form. $y - y_0 = m(x - x_0)$

2. Find the slope using the slope formula. $m = \frac{12 - (-2)}{8 - (-4)} = \frac{14}{12} = \frac{7}{6}$

3. Plug in the value of the slope. $y - y_0 = \frac{7}{6}(x - x_0)$

4. Plug in -4 for x_0 and -2 for y_0. $y - (-2) = \frac{7}{6}(x - (-4))$

Therefore, the equation is $y + 2 = \frac{7}{6}(x + 4)$ **Answer 1**

In the last example, you were told that for the last step you could choose either of the points you were given to plug in for the point (x_0, y_0) but it might not seem like you would get the same answer if you plug the second point in instead of the first. Let's redo Step 4.

4. Plug in 8 for x_0 and 12 for y_0 $y - 12 = \frac{7}{6}(x - 8)$ **Answer 2**

This certainly does not see like the same answer as we got by plugging in the first point. What is going on?

Notice that the equation in point-slope form is not solved for y. Let's change both answers into slope-intercept form by solving for y.

Answer 1	Answer 2
$y+2=\frac{7}{6}(x+4)$	$y-12=\frac{7}{6}(x-8)$
$y+2=\frac{7}{6}x+\frac{28}{6}$	$y-12=\frac{7}{6}x-\frac{56}{6}$
$y=\frac{7}{6}x+\frac{14}{3}-2$	$y=\frac{7}{6}x-\frac{28}{3}+12$
$y=\frac{7}{6}x+\frac{8}{3}$	$y=\frac{7}{6}x+\frac{8}{3}$

Now that the two answers are solved for y, you can see that they simplify to the same thing. In point-slope form you can get an infinite number of right answers, because there are an infinite number of points on a line. The slope of the line will always be the same but the answer will look different because you can substitute any point on the line for (x_0, y_0). However, regardless of the point you pick, the point-slope form should always simplify to the same slope-intercept equation for points that are on the same line.

In the last example you saw that sometimes we need to change between different forms of the equation. To change from point-slope form to slope-intercept form, we just solve for y.

Example 4

Re-write the following equations in slope-intercept form.

a) $y-5=3(x-2)$

b) $y+7=-(x+4)$

Solution

a) To re-write in slope-intercept form, solve for y.

$$\begin{aligned} y-5 &= 3(x-2) \\ -5 &= 3x-6 \\ y &= 3x-1 \end{aligned}$$

b) To re-write in slope-intercept form, solve for y.

$$\begin{aligned} y+7 &= -(x+4) \\ y+7 &= -x-4 \\ y &= -x-11 \end{aligned}$$

Graph an Equation in Point-Slope Form

If you are given an equation in point-slope form, it is not necessary to re-write it in slope-intercept form in order to graph it. The point-slope form of the equation gives you enough information so you can graph the line $y-y_0=m(x-x_0)$. From this equation, we know a point on the line (x_0, y_0) and the slope of the line.

To graph the line, you first plot the point (x_0, y_0). Then the slope tells you how many units you should go up or down and how many units you should go to the right to get to the next point on the line. Let's demonstrate this method with an example.

Example 5

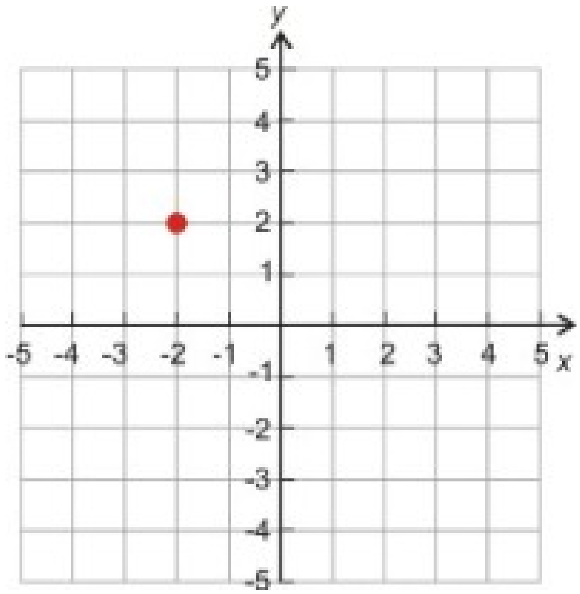
Make a graph of the line given by the equation $y-2=\frac{2}{3}(x+2)$

Solution

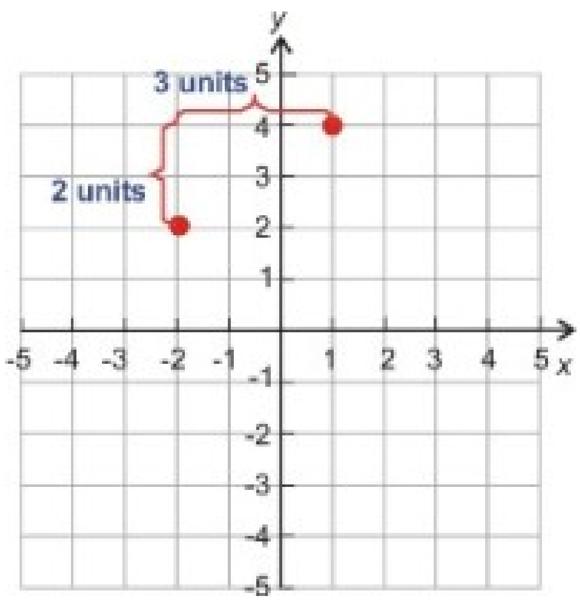
Let's rewrite the equation $y-(2)=\frac{2}{3}(x+2)$.

Now we see that point $(-2,2)$ is on the line and that the slope $=\frac{2}{3}$.

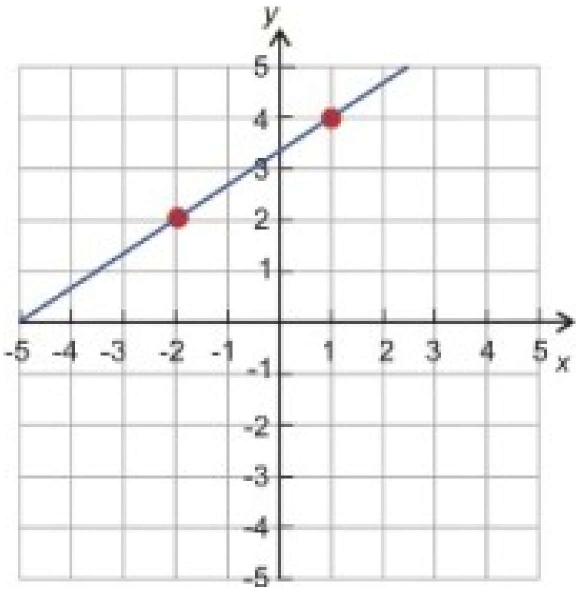
First plot point $(-2,2)$ on the graph.



A slope of $\frac{2}{3}$ tells you that from your point you should move. 2 units up and 3 units to the right and draw another point.



Now draw a line through the two points and extend the line in both directions.



Write a Linear Function in Point-Slope Form

The functional notation for the point-slope form of the equation of a line is:

$$f(x) - f(x_0) = m(x - x_0)$$

Note that we replaced the each y with the $f(x)$

$y = f(x)$ and $y_0 = f(x_0)$

That tells us more clearly that we find values of y by plugging in values of x into the function defined by the equation of the line. Let's use the functional notation to solve some examples.

Example 6

Write the equation of the following linear functions in point-slope form.

a) $m = 25$ and $f(0) = 250$

b) $m = 9.8$ and $f(5.5) = 12.5$

c) $f(32) = 0$ and $f(77) = 25$

a) Here we are given the slope $= 25$ and the point on the line gives $x_0 = 0$, $f(x_0) = 250$

1. Start with the equation in point-slope form. $f(x) - f(x_0) = m(x - x_0)$
2. Plug in the value of the slope. $f(x) - f(x_0) = 25(x - x_0)$
3. Plug in 0 for x_0 and 250 for $f(x_0)$. $f(x) - 250 = 25(x - 0)$

Solution

The linear function is $f(x) - 250 = 25x$.

b) Here we are given that slope $= 9.8$ and the point on the line gives $x_0 = 5.5$, $f(x_0) = 12.5$

1. Start with the equation in point-slope form. $f(x) - f(x_0) = m(x - x_0)$
2. Plug in the value of the slope. $f(x) - f(x_0) = 9.8(x - x_0)$

3. Plug in 5.5 for x_0 and 12.5 for $f(x_0)$. $f(x) - 12.5 = 9.8(x - 5.5)$

Solution

The linear function is $f(x) - 12.5 = 9.8(x - 5.5)$.

c) Here we are given two points $(32, 0)$ and $(77, 25)$.

1. Start with the equation in point-slope form. $f(x) - f(x_0) = m(x - x_0)$
2. Find the value of the slope. $m = \frac{25-0}{77-32} = \frac{25}{45} = \frac{5}{9}$
3. Plug in the value of the slope. $f(x) - f(x_0) = \frac{5}{9}(x - x_0)$
4. Plug in 32 for and 0 for $f(x_0)$. $f(x) - 0 = \frac{5}{9}(x - 32)$

Solution

The linear function is $f(x) - 0 = \frac{5}{9}(x - 32)$.

Solve Real-World Problems Using Linear Models in Point-Slope Form

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example 7

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges \$40 *per day and some amount of cents per mile. Marciel drives* 46 *miles and the final amount of the bill (before tax) is* \$63 . *What is the amount per mile the truck rental company charges per day? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove* 220 *miles?*

Let's define our variables:

$x =$ distance in miles

$y =$ cost of the rental truck in dollars

We see that we are given the $y-$intercept and the point $(46, 63)$.

Peter pays a flat fee of \$40 for the day. This is the y–intercept.

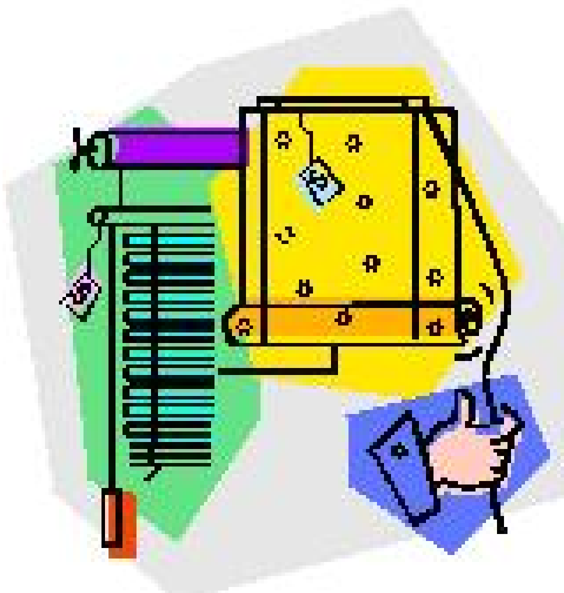
He pays \$63 for 46 miles –this is the coordinate point $(46, 63)$.

Start with the point-slope form of the line.	$(y - y_0) = m(x - x_0)$
Plug in the coordinate point.	$63 - y_0 = m(46 - x_0)$
Plug in point $(0, 40)$.	$63 - 40 = m(46 - 0)$
Solve for the slope.	$23 = m(46) \rightarrow m = \frac{23}{46} = 0.5$
The slope is :	0.5 dollars per mile
So, the truck company charges 50 cents per mile.	(\$0.5 = 50 cents)Equation of line is : $y \quad = 0.5x + 40$

To answer the question of 220 miles we plug in $x = 220$.

Solution

$y - 40 = 0.5(220) \Rightarrow y = \$150$



Example 8

Anne got a job selling window shades. She receives a monthly base salary and a \$6 commission for each window shade she sells. At the end of the month, she adds up her sales and she figures out that she sold 200 window shades and made \$2500 . Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Let's define our variables

$x =$ number of window shades sold

$y =$ Anne's monthly salary in dollars

We see that we are given the slope and a point on the line:

Anne gets \$6 for each shade, so the slope = 6 dollars/shade.

She sold 200 shades and made \$2500, so the point is $(200, 2500)$.

Start with the point-slope form of the line.	$y - y_0 = m(x - x_0)$
Plug in the coordinate point.	$y - y_0 = 6(x - x_0)$
Plug in point $(200, 2500)$.	$y - 2500 = 6(x - 200)$

Anne's base salary is found by plugging in $x = 0$. We obtain $y - 2500 = -1200 \Rightarrow y = \$1300$

Solution

Anne's monthly base salary is $1300.

Lesson Summary

- The **point-slope form** of an equation for a line is: $y - y_0 = m(x - x_0)$.
- If you are **given the slope and a point** on the line:
  1. Simply plug the point and the slope into the equation.
- If you are **given the slope and $y-$intercept** of a line:
  1. Plug the value of m into the equation
  2. Plug the $y-$intercept point into the equation $y_0 = y-$intercept and $x_0 = 0$.
- If you are **given two points** on the line:
  1. Use the two points to find the slope using the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.
  2. Plug the value of m into the equation.
  3. Plug either of the points into the equation as (x_0, y_0).
- The **functional notation of point-slope form** is $f(x) - f(x_0) = m(x - x_0)$.

Review Questions

Write the equation of the line in point-slope form.

1. The line has slope $-\frac{1}{10}$ and goes through point $(10, 2)$.
2. The line has slope -75 and goes through point $(0, 125)$.
3. The line has slope 10 and goes through point $(8, -2)$.
4. The line goes through the points $(-2, 3)$ and $(-1, -2)$.
5. The line contains points $(10, 12)$ and $(5, 25)$.
6. The line goes through points $(2, 3)$ and $(0, 3)$.
7. The line has a slope $\frac{3}{5}$ and a y–intercept -3.
8. The line has a slope -6 and a $y-$intercept 0.5.

Write the equation of the linear function in point-slope form.

9. $m = -\frac{1}{5}$ and $f(0) = 7$
10. $m = -12$ and $f(-2) = 5$
11. $f(-7) = 5$ and $f(3) = -4$
12. $f(6) = 0$ and $f(0) = 6$
13. $m = 3$ and $f(2) = -9$
14. $m = -\frac{9}{5}$ and $f(0) = 32$
15. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm. Write an equation in point-slope form that describes this situation. What is the unstretched length of the spring?
16. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet feet to a depth of 50 feet . Write an equation in point-slope form that describes this situation. What was the submarine's depth five minutes after it started surfacing?

Review Answers

1. $y - 2 = -\frac{1}{10}(x - 10)$
2. $y - 125 = -75x$
3. $y + 2 = 10(x - 8)$
4. $y + 2 = -5(x + 1)$ or $y - 3 = -5(x + 2)$
5. $y - 25 = -\frac{13}{5}(x - 5)$ or $y - 12 = -\frac{13}{5}(x - 10)$
6. $y - 3 = 0$
7. $y + 3 = \frac{3}{5}x$
8. $y - 0.5 = -6x$
9. $f(x) - 7 = -\frac{1}{5}x$
10. $f(x) - 5 = -12(x + 2)$
11. $f(x) - 5 = -\frac{9}{10}(x + 7)$or $f(x) + 4 = -\frac{9}{10})x - 3)$
12. $f(x) = -x(x - 6)$ or $f(x) - 6 = -x$
13. $f(x) + 9 = 3(x - 2)$
14. $f(x) - 32 = \frac{9}{5}x$
15. $y - 20 = \frac{1}{40}(x - 100)$ unstretched length = 17.5 cm
16. $y - 50 = -17.5(x - 20)$ or $y - 400 = -17.5x$ depth = 312.5 feet

5.3 Linear Equations in Standard Form

Learning Objectives

- Write equivalent equations in standard form.
- Find the slope and $y-$intercept from an equation in standard form.
- Write equations in standard form from a graph.
- Solve real-world problems using linear models in standard form.

Introduction

In this section, we are going to talk about the standard form for the equation of a straight line. The following linear equation is said to be in standard form.

$$ax + by = c$$

Here a, b and c are constants that have no factors in common and the constant a is a non-negative value. Notice that the b in the standard form is different than the b in the slope-intercept form. There are a few reasons why standard form is useful and we will talk about these in this section. The first reason is that standard form allows us to write equations for vertical lines which is not possible in slope-intercept form.

For example, let's find the equation of the line that passes through points $(2, 6)$ and $(2, 9)$.

Let's try the slope-intercept form $y = mx + b$

We need to find the slope $m = \frac{9-6}{2-2} = \frac{3}{0}$. The slope is undefined because we cannot divide by zero.

The point-slope form $y - y_0 = m(x - x_0)$ also needs the slope, so we cannot write an equation for this line in either the slope-intercept or the point-slope form.

Since we have two points in a plane, we know that a line passes through these two points, but how do we find the equation of that line? It turns out that this line has no y value in it. Notice that the value of x in both points is two for the different values of y, so we can say that it does not matter what y is because x will always equal two. Here is the equation in standard form.

$1 \cdot x + 0 \cdot y = 2$ or $x = 2$

The line passing through point $(2, 6)$ and $(2, 9)$ is a **vertical line** passing through $x = 2$. Note that the equation of a horizontal line would have no x variable, since y would always be the same regardless of the value of x. For example, a **horizontal line** passing through point $(0, 5)$ has this equation in standard form.

$0 \cdot x + 1 \cdot y = 5$ or $y = 5$

Write Equivalent Equations in Standard Form

So far you have learned how to write equations of lines in slope-intercept form and point-slope form. Now you will see how to rewrite equations in standard form.

Example 1

Rewrite the following equations in standard form.

a) $y = 5x - 7$

b) $y - 2 = -3(x + 3)$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Solution

We need to rewrite the equations so that all the variables are on one side of the equation and the coefficient of x is not negative.

a) $y = 5x - 7$

Subtract y from both sides.	$0 = 5x - y - 7$
Add 7 to both sides.	$7 = 5x - y$
The equation in standard form is :	$5x - y = 7$

b) $y - 2 = -3(x + 3)$

Distribute the -3 on the right-hand-side.	$y - 2 = -3x - 9$
Add $3x$ to both sides.	$y + 3x - 2 = -9$
Add 2 to both sides.	$y + 3x = -7$
The equation in standard form is :	$y + 3x = -7$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Find the common denominator for all terms in the equation. In this case, the common denominator equals 6.

Multiply all terms in the equation by 6.	$6\left(y = \frac{2}{3}x + \frac{1}{2}\right) \Rightarrow 6y = 4x + 3$
Subtract $6y$ from both sides.	$0 = 4x - 6y + 3$
Subtract 3 from both sides.	$-3 = 4x - 6y$
The equation in standard form is :	$4x - 6y = -3$

Find the Slope and y–intercept From an Equation in Standard Form

The slope-intercept form and the point-slope form of the equation for a straight line both contain the slope of the equation explicitly, but the standard form does not. Since the slope is such an important feature of a line, it is useful to figure out how you would find the slope if you were given the equation of the line in standard form.

$$ax + by = c$$

Let's rewrite this equation in slope-intercept form by solving the equation for y.

$$\begin{aligned} \text{Subtract ax from both sides.} \qquad & by = -ax + c \\ \text{Divide all terms by b.} \qquad & y = -\frac{a}{b}x + \frac{c}{b} \end{aligned}$$

If we compare with the slope-intercept form $y = mx + b$, we see that the slope, $m = -\frac{a}{b}$ and the y–intercept $= \frac{c}{b}$. Again, notice that the b in the standard form is different than the b in the slope-intercept form.

Example 2

Find the slope and the y–intercept of the following equations written in standard form:

a) $3x + 5y = 6$

b) $2x - 3y = -8$

c) $x - 5y = 10$

Solution

The slope $m = -\frac{a}{b}$ and the y–intercept $= \frac{c}{b}$.

a) $3x + 5y = 6$ $m = -\frac{3}{5}$ and y–intercept $= \frac{6}{5}$

b) $2x - 3y = -8$ $m = \frac{2}{3}$ and $y - -$intercept $= \frac{8}{3}$

c) $x - 5y = 10$ $m = \frac{1}{5}$ and y–intercept $= \frac{10}{-5} = -2$

Write Equations in Standard Form From a Graph

If we are given a graph of a straight line, it is fairly simple to write the equation in slope-intercept form by reading the slope and y–intercept from the graph. Let's now see how to write the equation of the line in standard form if we are given the graph of the line.

First, remember that to graph an equation from standard form we can use the cover-up method to find the intercepts of the line. For example, let's graph the line given by the equation $3x - 2y = 6$.

To find the x–intercept, cover up the y term (remember, x–intercept is where $y = 0$).

$$3x = 6 \Rightarrow x = 2$$

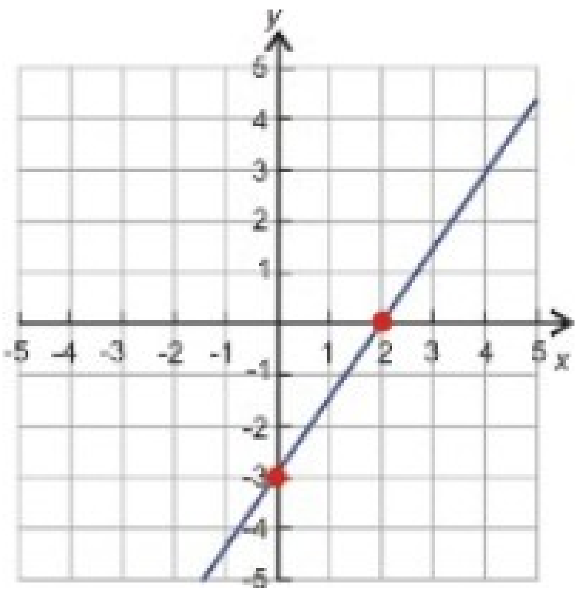
The x–intercept is $(2, 0)$

To find the y–intercept, cover up the x term (remember, y–intercept is where $x = 0$).

$$3y = 6 \Rightarrow y = -3$$

The $y-$intercept is $(0, -3)$

We plot the intercepts and draw a line through them that extends in both directions.

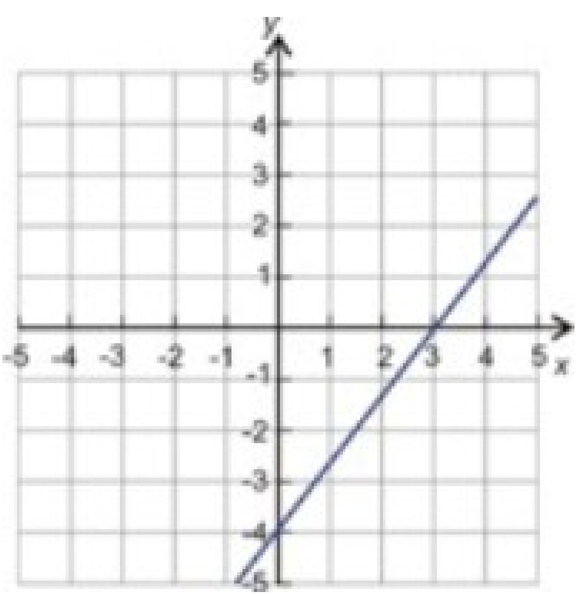


Now we want to apply this process in reverse. If we have the graph of the line, we want to write the equation of the line in standard form.

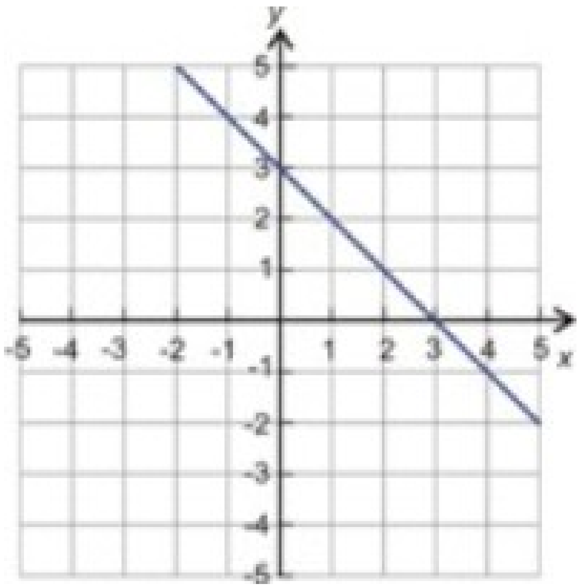
Example 3

Find the equation of the line and write in standard form.

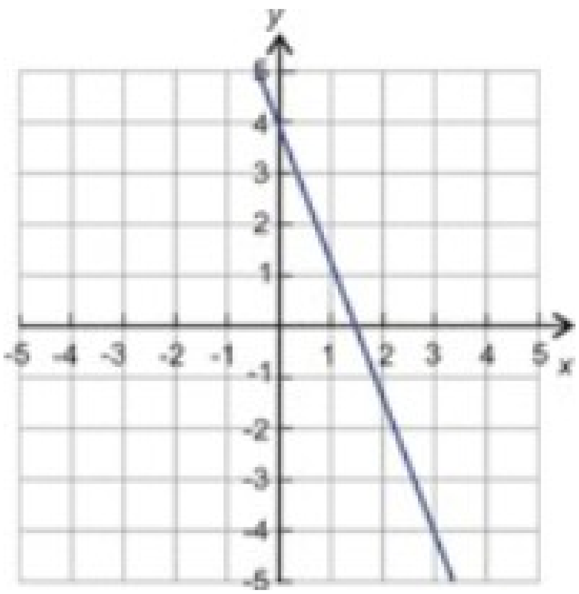
a)



b)



c)



Solution

a) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,-4) \Rightarrow y = -4$.

We saw that in standard form $ax + by = c$,

if we "cover up" the y term, we get $ax = c$

if we "cover up" the x term, we get $by = c$

We need to find the numbers that when multiplied with the intercepts give the same answer in both cases. In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by -3 gives the same result.

$(x = 3) \times 4 \Rightarrow 4x = 12$ and $(y = -4) \times (-3) \Rightarrow -3y = 12$

Therefore, $a = 4$, $b = -3$ and $c = 12$ and the standard form is:

$$4x - 3y = 12$$

b) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,3) \Rightarrow y = 3$.

The values of the intercept equations are already the same, so $a = 1$, $b = 1$ and $c = 3$. The standard form is:

$$x + y = 3$$

c) We see that the $x-$intercept is $(\frac{3}{3}, 0) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0, 4) \Rightarrow y = 4$.

Let's multiply the $x-$intercept equation by $2 \Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3.

$\Rightarrow 8x = 12$ and $3y = 12$

The standard form is $8x + 3y = 12$.

Solve Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of the equation is useful.

Example 4

Nimitha buys fruit at her local farmer's market. This Saturday, oranges cost \$2 *per pound and cherries cost* \$3 *per pound. She has* \$12 *to spend on fruit. Write an equation in standard form that describes this situation. If she buys* 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables

$x =$ pounds of oranges

$y =$pounds of cherries

The equation that describes this situation is: $2x + 3y = 12$

If she buys 4 pounds of oranges, we plug $x = 4$ in the equation and solve for y.

$$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$$

Nimitha can buy $1\frac{1}{3}$ pounds of cherries.

Example 5

Jethro skateboards part of the way to school and walks for the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If Jethro skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables.

$x =$ hours Jethro skateboards

$y =$ hours Jethro walks

The equation that describes this situation is $7x + 3y = 6$

If Jethro skateboards $\frac{1}{2}$ an hour, we plug $x = 0.5$ in the equation and solve for y.

$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$

Jethro must walk $\frac{5}{6}$ of an hour.

Lesson Summary

- A linear equation in the form $ax + by = c$ is said to be in **standard form.** Where a, b and c are constants (b is different than the $y-$intercept b) and a is non-negative.
- Given an equation in standard form, $ax + by = c$, the **slope**, $a = -\frac{a}{b}$, and the $y-$**intercept** $= \frac{c}{b}$.
- The **cover-up method** is useful for graphing an equation in standard form. To find the $y-$intercept, cover up the x term and solve the remaining equation for y. Likewise, to find the $x-$intercept, cover up the y term and solve the remaining equation for x.

Review Questions

Rewrite the following equations in standard form.

1. $y = 3x - 8$
2. $y - 7 = -5(x - 12)$
3. $2y = 6x + 9$
4. $y = \frac{9}{4}x + \frac{1}{4}$

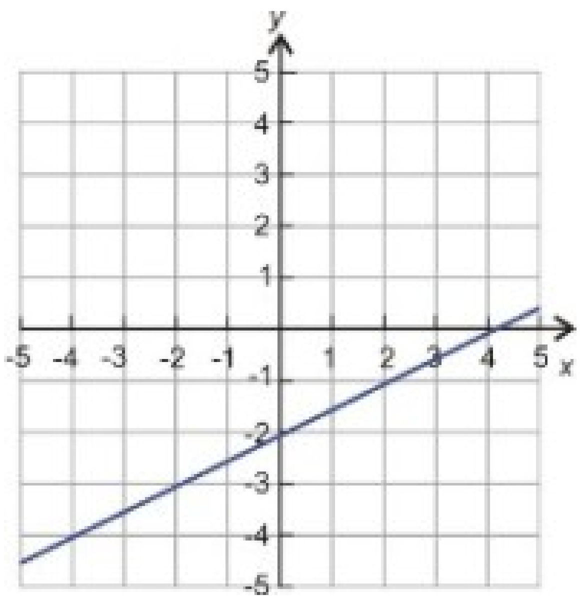
5. $y + \frac{3}{5} = \frac{2}{3}(x - 2)$
6. $3y + 5 = 4(x - 9)$

Find the slope and $y-$intercept of the following lines.

7. $5x - 2y = 15$
8. $3x + 6y = 25$
9. $x - 8y = 12$
10. $3x - 7y = 20$
11. $9x - 9y = 4$
12. $6x + y = 3$

Find the equation of each line and write it in standard form.

13.



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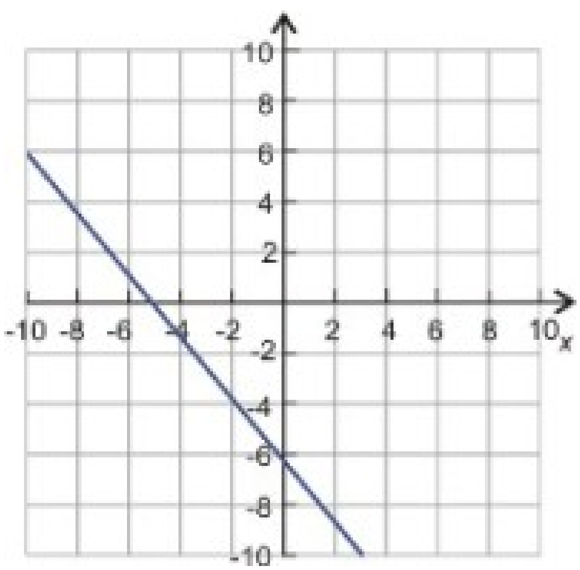

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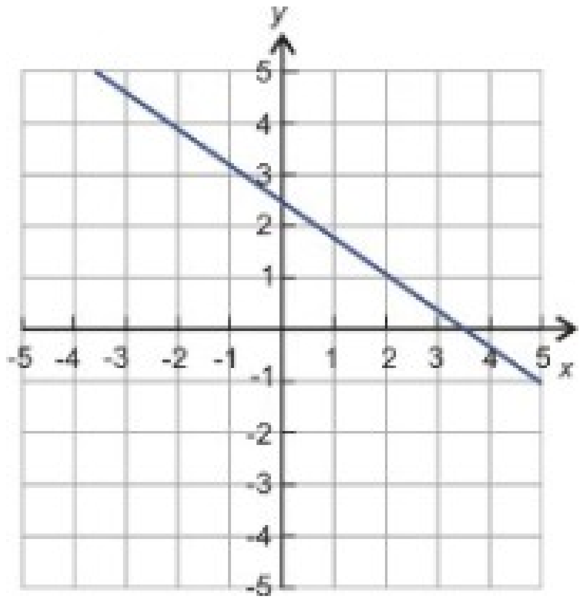

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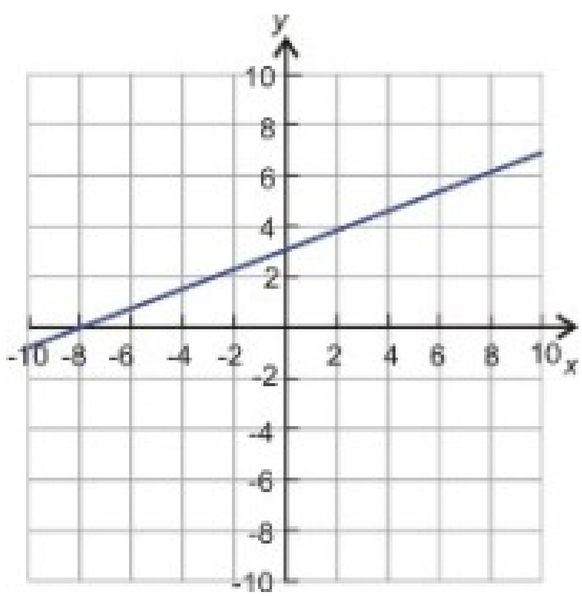


17. Andrew has two part time jobs. One pays \$6 per hour and the other pays \$10 per hour. He wants to make \$366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hour per week at the \$10 per hour job, how many hours does he need to work per week in his \$6 per hour job in order to achieve his goal?
18. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than \$400 in interest per year. Write an equation in standard form that describes this problem. If she invests \$5000 in the 5% interest account, how much money does she need to invest in the other account?

Review Answers

1. $3x - y = 8$
2. $5x + y = 67$
3. $6x - 2y = -9$
4. $9x - 4y = -1$

5. $10x - 15y = 29$
6. $4x - 3y = 41$
7. $m = (5/2), b = -15/2$
8. $m = -(1/2), b = 25/6$
9. $m = (1/8), b = -3/2$
10. $m = (3/7), b = -20/7$
11. $m = 1, b = -4/9$
12. $m = -6, b = 3$
13. $x - 2y = 4$
14. $6x + 5y = -30$
15. $10x + 14y = 35$
16. $3x - 8y = -24$
17. $x =$ number of hours per week worked at \$6 per hour job $y =$ number of hours per week worked at \$10 per hour job Equation $6x + 10y = 366$ Answer 36 hours
18. $x =$ amount of money invested at 5% annual interest $y =$ amount of money invested at 7% annual interest Equation $5x + 7y = 40000$ Answer \$2142.86

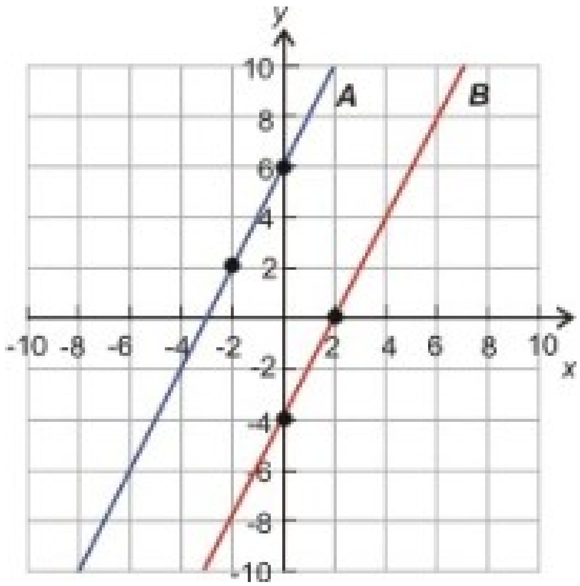
5.4 Equations of Parallel and Perpendicular Lines

Learning Objectives

- Determine whether lines are parallel or perpendicular.
- Write equations of perpendicular lines.
- Write equations of parallel lines.
- Investigate families of lines.

Introduction

In this section, you will learn how **parallel lines** are related to each other on the coordinate plane. You will also learn how **perpendicular lines** are related to each other. Let's start by looking at a graph of two parallel lines.



The two lines will never meet because they are parallel. We can clearly see that the two lines have different $y-$intercepts, more specifically 6 and -4.

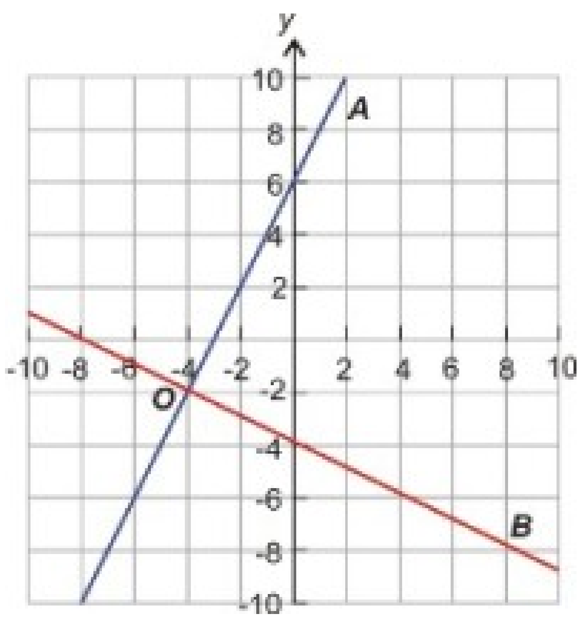
How about the slopes of the lines? Are they related in any way? Because the lines never meet, they must rise at the same rate. This means that the slopes of the two lines are the same.

Indeed, if we calculate the slopes of the lines, we find the following results.

Line $\boldsymbol{A}$: $m = \frac{6-2}{0-(-2)} = \frac{4}{2} = 2$

Line $\boldsymbol{B}$: $m = \frac{0-(-4)}{2-0} = \frac{4}{2} = 2$

For Parallel Lines: the slopes are the same, $m_1 = m_2$, and the $y-$intercepts are different.

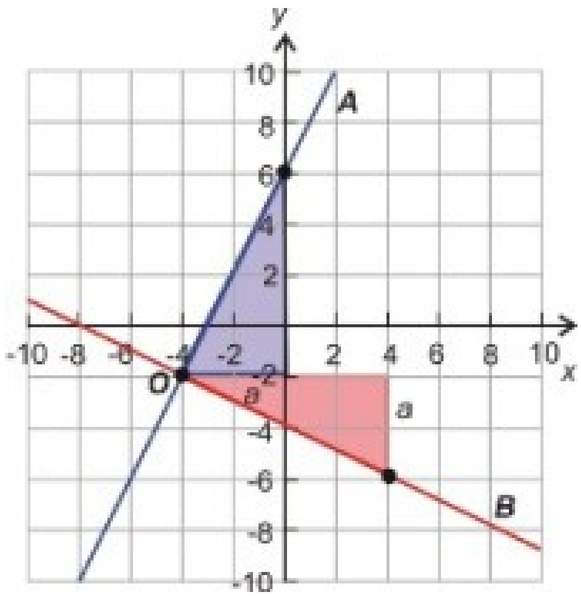


Now let's look at a graph of two perpendicular lines.

We find that we can't say anything about the $y-$intercepts. In this example, they are different, but they would be the same if the lines intersected at the $y-$intercept.

Now we want to figure out if there is any relationship between the slopes of the two lines.

First of all we see that the *slopes must have opposite signs, one negative and one positive.*



To find the slope of line ***A,*** we pick two points on the line and draw the blue (upper) right triangle. The legs of the triangle represent the rise and the run.

Looking at the figure $m_1 = \frac{b}{a}$

To find the slope of line ***B,*** we pick two points on the line and draw the red (lower) right triangle. If we look at the figure, we see that the two triangles are identical, only rotated by 90°.

Looking at the diagram $m_2 = -\frac{a}{b}$

For Perpendicular Lines: the slopes are negative reciprocals of each other. $m_1 = -\frac{1}{m_2}$ or $m_1 m_2 = -1$

Determine Whether Lines are Parallel or Perpendicular

You can find whether lines are parallel or perpendicular by comparing the slopes of the lines. If you are given points on the line you can find the slope using the formula. If you are given the equations of the lines, rewrite each equation in a form so that it is easy to read the slope, such as the slope-intercept form.

Example 1

Determine whether the lines are parallel or perpendicular or neither.

a) One line passes through points $(2, 11)$ and $(-1, 2)$; another line passes through points $(0, -4)$ and $(-2, -10)$.

b) One line passes through points $(-2, -7)$ and $(1, 5)$; another line passes through points $(4, 1)$ and $(-8, 4)$.

c) One lines passes through points $(3, 1)$ and $(-2, -2)$; another line passes through points $(5, 5)$ and $(4, -6)$.

Solution

Find the slope of each line and compare them.

a) $m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3$ and $m_2 = \frac{-10-(-4)}{-2-0} = \frac{-6}{-2} = 3$

The slopes are equal, so the lines are parallel.

b) $m_1 = \frac{5-(-7)}{1-(-2)} = \frac{12}{3} = 4$ and $m_2 = \frac{4-1}{-8-4} = \frac{3}{-12} = -\frac{1}{4}$

The slopes are negative reciprocals of each other, so the lines are perpendicular.

c) $m_1 = \frac{-2-1}{-2-3} = \frac{-3}{-5} = \frac{3}{5}$ and $m_2 = \frac{-6-5}{4-5} = \frac{-13}{-1} = 13$

The slopes are not the same or negative reciprocals of each other, so the lines are neither parallel nor perpendicular.

Example 2

Determine whether the lines are parallel or perpendicular or neither.

a) Line 1: $3x + 4y = 2$ Line 2: $8x - 6y = 5$

b) Line 1: $2x = y - 10$ Line 2: $y = -2x + 5$

c) Line 1: $7y + 1 = 7x$ Line 2: $x + 5 = y$

Solution

Write each equation in slope-intercept form.

a) Line 1 $3x + 4y = 2 \Rightarrow 4y = -3x + 2 \Rightarrow y = -\frac{3}{4}x + \frac{1}{2} \Rightarrow \text{slope} = -\frac{3}{4}$

Line 2: $8x - 6y = 5 \Rightarrow 8x - 5 = 6y \Rightarrow y = \frac{8}{6}x - \frac{5}{6} \Rightarrow y = \frac{4}{3}x - \frac{5}{6} \Rightarrow \text{slope} = \frac{4}{3}$

The slopes are negative reciprocals of each other, so the lines are perpendicular to each other.

b) Line 1 $2x = y - 10 \Rightarrow y = 2x + 10 \Rightarrow \text{slope} = 2$

Line 2 $y = -2x + 5 \Rightarrow \text{slope} = -2$

The slopes are not the same or negative reciprocals of each other, so the lines are neither parallel nor perpendicular.

c) Line 1 $7y + 1 = 7x \Rightarrow 7y = 7x - 1 \Rightarrow y = x - \frac{1}{7} \Rightarrow \text{slope} = 1$

Line 2: $x + 5 = y \Rightarrow y = x + 5 \Rightarrow \text{slope} = 1$

The slopes are the same so the lines are parallel.

Write Equations of Perpendicular Lines

We can use the properties of perpendicular lines to write an equation of a line perpendicular to a given line. You will be given the equation of a line and asked to find the equation of the perpendicular line passing through a specific point. Here is the general method for solving a problem like this.

- Find the slope of the given line from its equation. You might need to rewrite the equation in a form such as the slope-intercept form.
- Find the slope of the perpendicular line by writing the negative reciprocal of the slope of the given line.
- Use the slope and the point to write the equation of the perpendicular line in point-slope form.

Example 3

Find the equation perpendicular to the line $y = -3x + 5$ *that passes through point* $(2, 6)$.

Solution

Find the slope of the given line $y = -3x + 5$ has a slope $= -3$.

The slope of the perpendicular line is the negative reciprocal $m = \frac{1}{3}$

Now, we are trying to find the equation of a line with slope $m = \frac{1}{3}$ that passes through point $(2, 6)$.

Use the point-slope form with the slope and the point $y - 6 = \frac{1}{3}(x - 2)$

The equation of the line could also be written as $y = \frac{1}{3}x + \frac{16}{3}$

Example 4

Find the equation of the line perpendicular to $x-5y=15$ *that passes through the point* $(-2,5)$.

Solution

Rewrite the equation in slope-intercept form $x-5y=15 \Rightarrow -5y=-x+15 \Rightarrow y=\frac{1}{5}x-3$

The slope of the given line is $m=\frac{1}{5}$ and the slope of the perpendicular is the negative reciprocal or $m=-5$. We are looking for a line with a slope $m=-5$ that passes through the point $(-2,5)$.

Use the point-slope form with the slope and the point $y-5=-5(x+2)$

The equation of the line could also be written as $y=-5x-5$

Example 5

Find the equation of the line perpendicular to $y=-2$ *that passes through the point* $(4,-2)$.

Solution

The equation is already in slope intercept form but it has an x term of 0 $y=0x-2$. This means the slope is $m=0$.

We'd like a line with slope that is the negative reciprocal of 0. The reciprocal of 0 is $m=\frac{1}{0}=$ *undefined*. Hmmm... It seems like we have a problem. But, look again at the desired slope in terms of the definition of slope $m=\frac{rise}{run}=\frac{1}{0}$. So our desired line will move 0 units in x for every 1 unit it rises in y. This is a vertical line, so the solution is the vertical line that passes through $(4,-2)$. This is a line with an x coordinate of 4 at every point along it.

The equation of the line is: $x=4$

Write Equations of Parallel Lines

We can use the properties of parallel lines to write an equation of a line parallel to a given line. You will be given the equation of a line and asked to find the equation of the parallel line passing through a specific point. Here is the general method for solving a problem like this.

- Find the slope of the given line from its equation. You might need to rewrite the equation in a form such as the slope-intercept form.
- The slope of the parallel line is the same as that of the given line.
- Use the slope and the point to write the equation of the perpendicular line in slope-intercept form or point-slope form.

Example 6

Find the equation parallel to the line $y=6x-9$ *that passes through point* $(-1,4)$.

Solution

Find the slope of the given line $y=6x-9$ has a slope $=6$.

Since parallel lines have the same slope, we are trying to find the equation of a line with slope $m=6$ that passes through point $(-1,4)$.

Start with the slope-intercept form.	$y=mx+b$
Plug in the slope	$y=6x+b$
Plug in point $(-1,4)$.	$4=6(-1)+b \Rightarrow b=4+6 \Rightarrow b=10$

The equation of the line is $y = 6x + 10$.

Example 7

Find the equation of the line parallel to $7 - 4y = 0$ *that passes through the* point $(9, 2)$.

Solution

Rewrite the equation in slope-intercept form.

$$7 - 4y = 0 \Rightarrow 4y - 7 = 0 \Rightarrow 4y = 7 \Rightarrow y = \frac{7}{4} \Rightarrow y = 0x + \frac{7}{4}$$

The slope of the given line is $m = 0$. This is a horizontal line.

Since the slopes of parallel lines are the same, we are looking for a line with slope $m = 0$ that passes through the point $(9, 2)$.

$$\begin{aligned} \text{Start with the slope-intercept form.} \qquad & y = 0x + b \\ \text{Plug in the slope.} \qquad & y = 0x + b \\ \text{Plug in point } (9, 2). \qquad & 2 = 0(9) + b \\ & \Rightarrow b = 2 \end{aligned}$$

The equation of the line is $y = 2$.

Example 8

Find the equation of the line parallel to $6x - 5y = 12$ *that passes through the point* $(-5, -3)$.

Solution

Rewrite the equation in slope-intercept form.

$$3x - 5y = 12 \Rightarrow 5y = 6x - 12 \Rightarrow y = \frac{6}{5}x - \frac{12}{5}$$

The slope of the given line is $m = \frac{6}{5}$.

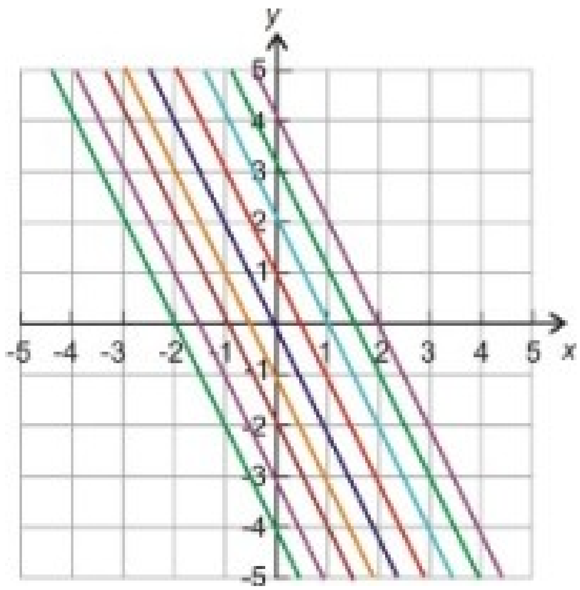
Since the slopes of parallel lines are the same, we are looking for a line with slope $m = \frac{6}{5}$ that passes through the point $(-5, -3)$.

$$\begin{aligned} \text{Start with the slope-intercept form.} \qquad & y = mx + b \\ \text{Plug in the slope.} \qquad & y = \frac{6}{5}x + b \\ \text{Plug in point } (-5, -3). \qquad & -3 = \frac{6}{5}(-5) + b \Rightarrow -3 = -6 + b \Rightarrow b = 3 \end{aligned}$$

The equation of the line is: $y = \frac{6}{5}x + 3$

Investigate Families of Lines

A straight line has two very important properties, its **slope** and its **$y-$intercept.** The slope tells us how steeply the line rises or falls, and the $y-$intercept tells us where the line intersects the $y-$axis. In this section, we will look at two families of lines. A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families. One where the slope is the same and one where the $y-$ intercept is the same.



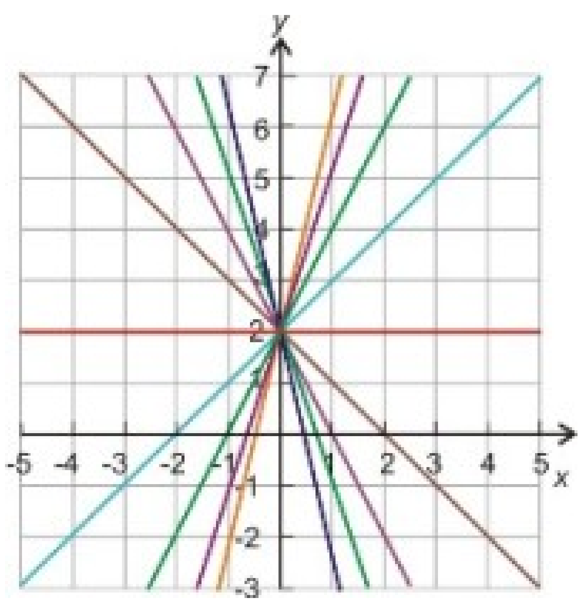
Family 1

Keep slope unchanged and vary the $y-$intercept.

The figure to the right shows the family of lines $y = -2x + b$.

All the lines have a slope of -2 but the value of b is different for each of the lines.

Notice that in such a family all the lines are parallel. All the lines look the same but they are shifted up and down the $y-$axis. As b gets larger the line rises on the $y-$axis and as b gets smaller the line goes lower on the $y-$axis. This behavior is often called a **vertical shift.**



Family 2

Keep the $y-$intercept unchanged and vary the slope.

The figure to the right shows the family of lines $y = mx + 2$.

All lines have a $y-$intercept of two but the value of the slope is different for each of the lines. The lines "start" with $y = 2$ (red line) which has a slope of zero. They get steeper as the slope increases until it gets to the line $x = 0$ (purple line) which has an undefined slope.

Example 9

Write the equation of the family of lines satisfying the given condition:

a) Parallel to the $x-$axis

b) Through the point $(0, -1)$

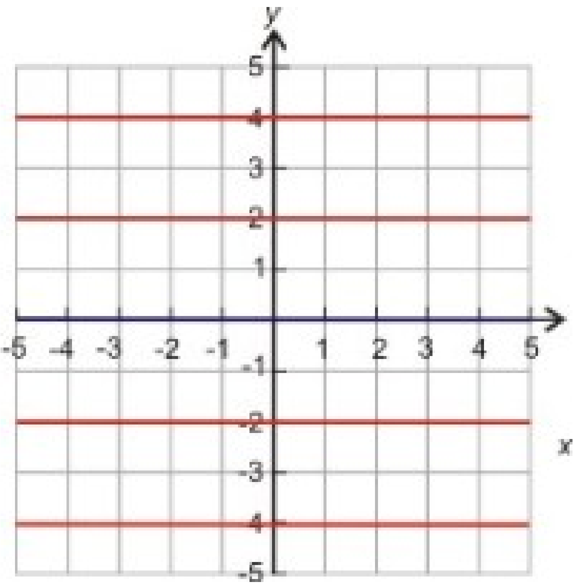
c) Perpendicular to $2x + 7y - 9 = 0$

d) Parallel to $x + 4y - 12 = 0$

Solution

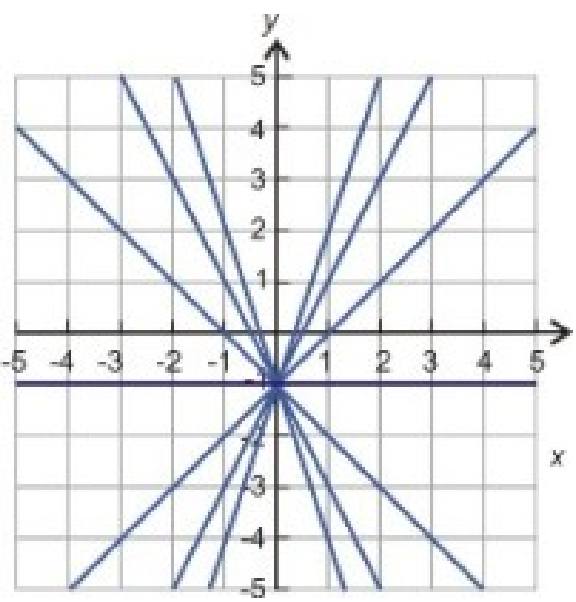
a) All lines parallel to the $x-$axis will have a slope of zero. It does not matter what the $y-$intercept is.

The family of lines is $y = 0 \cdot x + b$ or $y = b$.



b) All lines passing through the point $(0, -1)$ have the same $y-$intercept, $b = -1$.

The family of lines is $y = mx - 1$.



c) First we need to find the slope of the given line.

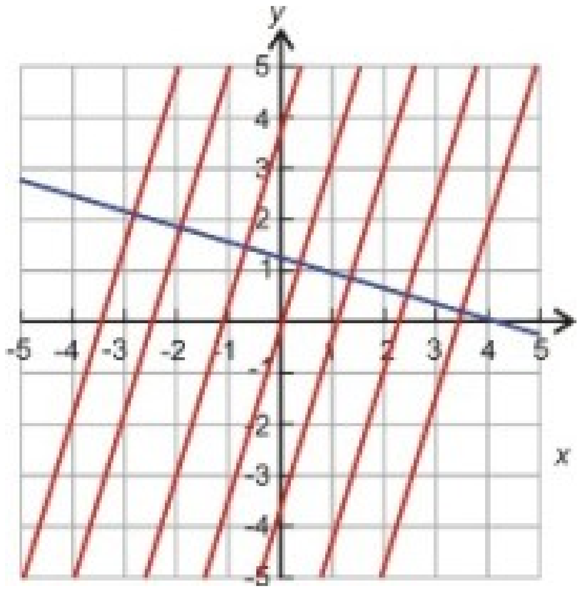
Rewrite $2x+7y-9=0$ in slope-intercept form $y=-\frac{2}{7}x+\frac{9}{7}$.

The slope is $-\frac{2}{7}$.

The slope of our family of lines is the negative reciprocal of the given slope $m=\frac{7}{2}$.

All the lines in this family have a slope of $m=\frac{7}{2}$ but different $y-$intercepts.

The family of lines is $y=\frac{7}{2}x+b$.



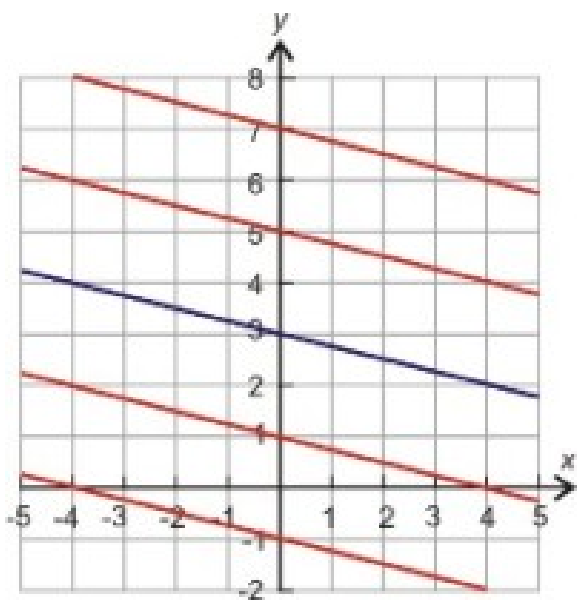
d) First we need to find the slope of the given line.

Rewrite $x+4y-12=0$ in slope-intercept form $y=-\frac{1}{4}x+3$.

The slope is $m=-\frac{1}{4}$.

All the lines in the family have a slope of $m=-\frac{1}{4}$ but different $y-$intercepts.

The family of lines is $y=-\frac{1}{4}x+b$.



Lesson Summary

- **Parallel lines** have the same slopes, $m_1 = m_2$, but different $y-$intercepts.
- **Perpendicular lines** have slopes which are the negative reciprocals of each other.

$$m_1 = -\frac{1}{m_2} \ or \ m_1 m_2 = -1$$

- **To find the line parallel (or perpendicular)** to a specific line which passes through a given point:

1. Find the slope of the given line from its equation.
2. Compute the slope parallel (or perpendicular) to the line.
3. Use the computed slope and the specified point to write the equation of the new line in point-slope form.
4. Transform from point-slope form to another form if required.

- **A family of lines** is a set of lines that have something in common with each other. There are two types of line families. One where the slope is the same and one where the $y-$intercept is the same.

Review Questions

Determine whether the lines are parallel, perpendicular or neither.

1. One line passes through points $(-1, 4)$ and $(2, 6)$; another line passes through points $(2, -3)$ and $(8, 1)$.
2. One line passes through points $(4, -3)$ and $(-8, 0)$; another line passes through points $(-1, -1)$ and $(-2, 6)$.
3. One line passes through points $(-3, 14)$ and $(1, -2)$; another line passes through points $(0, -3)$ and $(-2, 5)$.
4. One line passes through points $(3, 3)$ and $(-6, -3)$; another line passes through points $(2, -8)$ and $(-6, 4)$.
5. Line 1: $4y + x = 8$ Line 2: $12y + 3x = 1$
6. Line 1: $5y + 3x + 1$ Line 2: $6y + 10x = -3$
7. Line 1: $2y - 3x + 5 = 0$ Line 2: $y + 6x = -3$
8. Find the equation of the line parallel to $5x - 2y = 2$ that passes through point $(3, -2)$.
9. Find the equation of the line perpendicular to $y = -\frac{2}{5}x - 3$ that passes through point $(2, 8)$.
10. Find the equation of the line parallel to $7y + 2x - 10 = 0$ that passes through the point $(2, 2)$.
11. Find the equation of the line perpendicular to $y + 5 = 3(x - 2)$ that passes through the point $(6, 2)$. Write the equation of the family of lines satisfying the given condition.
12. All lines pass through point $(0, 4)$.
13. All lines are perpendicular to $4x + 3y - 1 = 0$.
14. All lines are parallel to $y - 3 = 4x + 2$.
15. All lines pass through point $(0, -1)$.

Review Answers

1. parallel
2. neither
3. parallel

4. perpendicular
5. parallel
6. perpendicular
7. neither
8. $y = \frac{5}{2}x - \frac{19}{2}$
9. $y = \frac{5}{2}x + 3$
10. $y = -\frac{2}{7}x + \frac{18}{7}$
11. $y = -\frac{1}{3}x + 4$
12. $y = mx + 4$
13. $y = \frac{3}{4}x + b$
14. $y = 4x + b$
15. $y = mx - 1$

5.5 Fitting a Line to Data

Learning Objectives

- Make a scatter plot.
- Fit a line to data and write an equation for that line.
- Perform linear regression with a graphing calculator.
- Solve real-world problems using linear models of scattered data.

Introduction

Often in application problems, the relationship between our dependent and independent variables is linear. That means that the graph of the dependent variable vs. independent variable will be a straight line. In many cases we don't know the equation of the line but we have data points that were collected from measurements or experiments. The goal of this section is to show how we can find an equation of a line from data points collected from experimental measurements.

Make a Scatter Plot

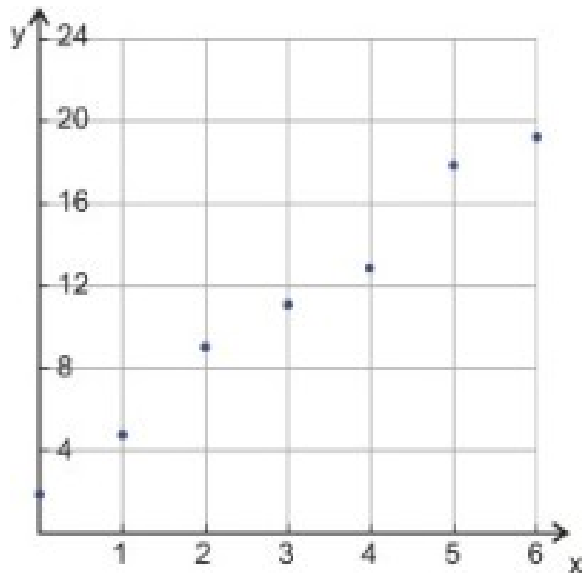
A **scatter plot** is a plot of all the ordered pairs in the table. This means that a scatter plot is a relation, and not necessarily a function. Also, the scatter plot is discrete, as it is a set of distinct points. Even when we expect the relationship we are analyzing to be linear, we should not expect that all the points would fit perfectly on a straight line. Rather, the points will be "scattered" about a straight line. There are many reasons why the data does not fall perfectly on a line such as **measurement error** and **outliers.**

Measurement error is always present as no measurement device is perfectly accurate. In measuring length, for example, a ruler with millimeter markings will be more accurate than a ruler with just centimeter markings.

An **outlier** is an accurate measurement that does not fit with the general pattern of the data. It is a statistical fluctuation like rolling a die ten times and getting the six side all ten times. It can and will happen, but not very often.

Example 1

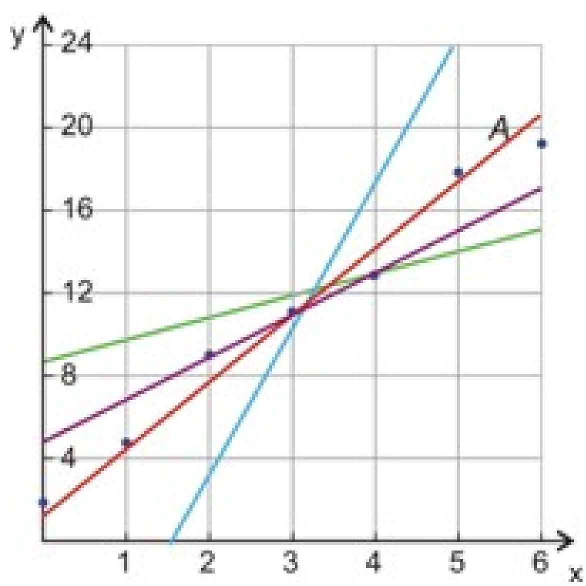
Make a scatter plot of the following ordered pairs: $(0,2),(1,4.5),(2,9),(3,11),(4,13),(5,18),(6,19.5)$



Solution

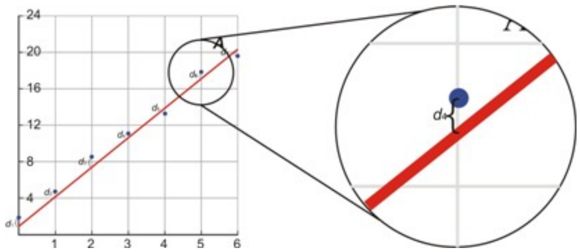
We make a scatter plot by graphing all the ordered pairs on the coordinate axis.

Fit a Line to Data



Notice that the points look like they might be part of a straight line, although they would not fit perfectly on a straight line. If the points were perfectly lined up it would be quite easy to draw a line through all of them and find the equation of that line. However, if the points are "scattered", we try to find a line that best fits the data.

You see that we can draw many lines through the points in our data set. These lines have equations that are very different from each other. We want to use the line that is closest to **all** the points on the graph. The best candidate in our graph is the red line ***A.*** We want to minimize the sum of the distances from the point to the line of fit as you can see in the figure below.



Finding this line mathematically is a complex process and is not usually done by hand. We usually "eye-ball" the line or find it exactly by using a graphing calculator or computer software such as Excel. The line in the graph above is "eye-balled," which means we drew a line that comes closest to all the points in the scatter plot.

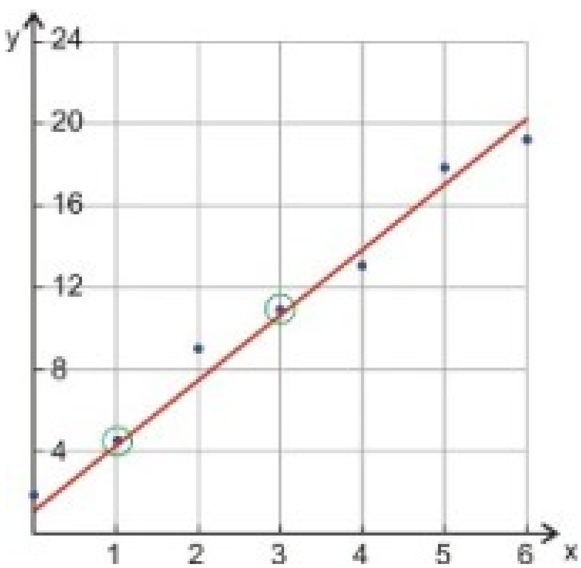
When we use the line of best fit we are assuming that there is a continuous linear function that will approximate the discrete values of the scatter plot. We can use this to interpret unknown values.

Write an Equation for a Line of Best Fit

Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression.**

Caution: Make sure you don't get caught making a common mistake. In many instances the line of best fit will not pass through many or any of the points in the original data set. This means that you can't just use two random points from the data set. **You need to use two points that are on the line.**

We see that two of the data points are very close to the line of best fit, so we can use these points to find the equation of the line $(1, 4.5)$ and $(3, 11)$.



Start with the slope-intercept form of a line $y = mx + b$.

Find the slope $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25$

Then $y = 3.25x + b$

Plug $(3, 11)$ into the equation. $11 = 3.25(3) + b \Rightarrow b = 1.25$

The equation for the line that fits the data best is $y = 3.25x + 1.25$.

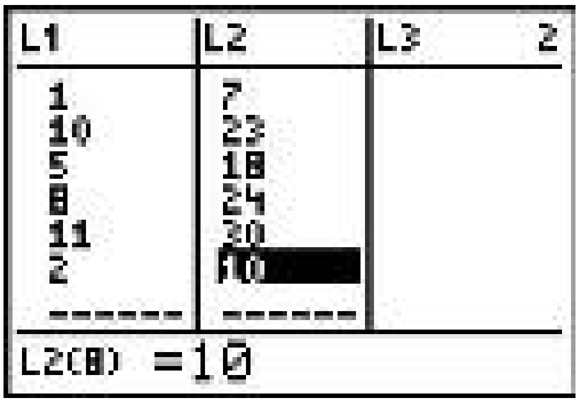
Perform Linear Regression with a Graphing Calculator

Drawing a line of fit can be a good approximation but you can't be sure that you are getting the best results because you are guessing where to draw the line. Two people working with the same data might get two different equations because they would be drawing different lines. To get the most accurate equation for the line, we can use a graphing calculator. The calculator uses a mathematical algorithm to find the line that minimizes the sum of the squares.

Example 2

Use a graphing calculator to find the equation of the line of best fit for the following data $(3,12),(8,20),(1,7),(10,23),(5,18),(8,24),(11,30),(2,10)$.

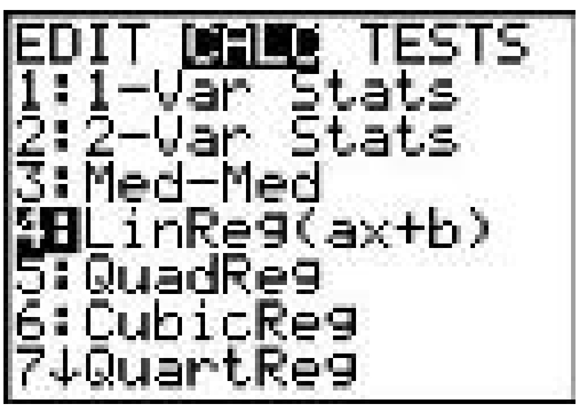
Solution



Step 1 *Input the data in your calculator.*

Press [**STAT**] and choose the [**EDIT**] option.

Input the data into the table by entering the *x* values in the first column and the *y* values in the second column.



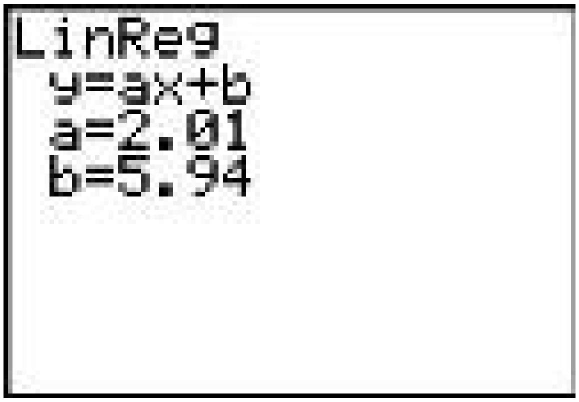
Step 2 *Find the equation of the line of best fit.*

Press [**STAT**] again use right arrow to select [**CALC**] at the top of the screen.

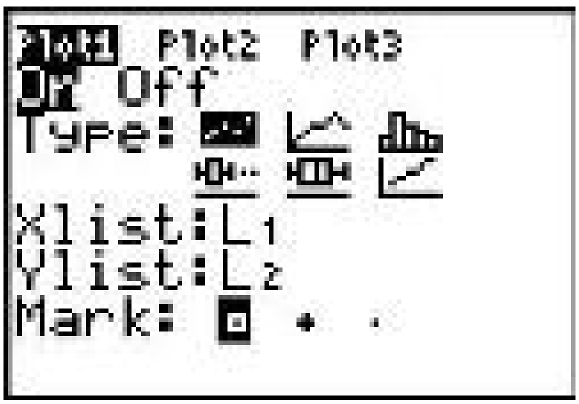
Chose option number 4: $\mathit{LinReg}(ax+b)$ and press [**ENTER**]

The calculator will display $\mathit{LinReg}(ax+b)$

Press [**ENTER**] and you will be given the *a* and *b* values.



Here a represents the slope and b represents the y-intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

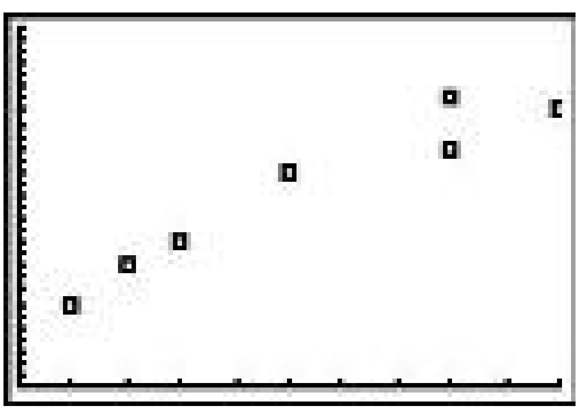


Step 3 *Draw the scatter plot.*

To draw the scatter plot press [**STATPLOT**] [**2nd**] [**Y=**].

Choose Plot 1 and press [**ENTER**.

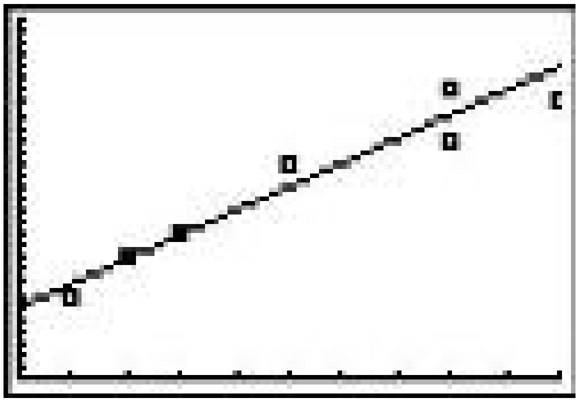
Press the On option and choose the Type as scatter plot (the one highlighted in black).



Make sure that the X list and Y list names match the names of the columns of the table in Step 1.

Choose the box or plus as the mark since the simple dot may make it difficult to see the points.

Press [**GRAPH**] and adjust the window size so you can see all the points in the scatter plot.



Step 4 *Draw the line of best fit through the scatter plot.*

Press [**Y=**]

Enter the equation of the line of best fit that you just found $Y_1 = 2.01X + 5.94$

Press [**GRAPH**].

Solve Real-World Problems Using Linear Models of Scattered Data

In a real-world problem, we use a data set to find the equation of the line of best fit. We can then use the equation to predict values of the dependent or independent variables. The usual procedure is as follows.

1. Make a scatter plot of the given data.
2. Draw a line of best fit.
3. Find an equation of a line either using two points on the line or the TI-83/84 calculator.
4. Use the equation to answer the questions asked in the problem.

Example 3

Gal is training for a 5 K *race (a total of* 5000 meters , *or about* 3.1 miles *). The following table shows her times for each month of her training program. Assume here that her times will decrease in a straight line with time (does that seem like a good assumption?) Find an equation of a line of fit. Predict her running time if her race is in August.*

Table 5.1:

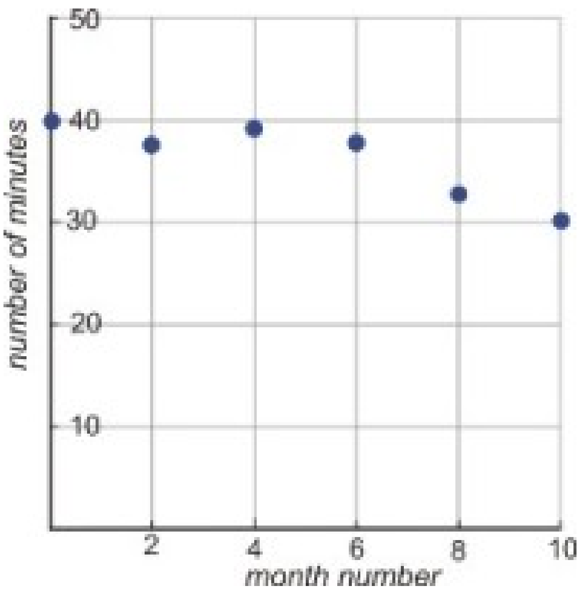
Month	Month number	Average time (minutes)
January	0	40

Table 5.1: (continued)

Month	Month number	Average time (minutes)
February	1	38
March	2	39
April	3	38
May	4	33
June	5	30

Solution

Let's make a scatter plot of Gal's running times. The independent variable, x, is the month number and the dependent variable, y, is the running time in minutes. We plot all the points in the table on the coordinate plane.

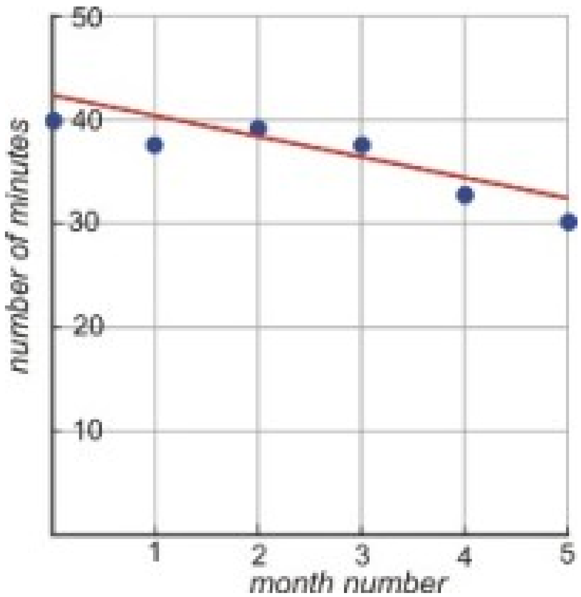


Draw a line of fit.

Choose two points on the line $(0, 41)$ and $(4, 34)$.

Find the equation of the line.

$$m = \frac{34 - 41}{4 - 0} = -\frac{7}{4} = -1\frac{3}{4}$$
$$y = -\frac{7}{4}x + b$$
$$41 = -\frac{7}{2}(0) + b \Rightarrow b = 41$$
$$y = -\frac{7}{4}x + 41$$



In a real-world problem, the slope and y–intercept have a physical significance.

$$\text{Slope} = \frac{number\ of\ minutes}{month}$$

Since the slope is negative, the number of minutes Gal spends running a 5K race decreased as the months pass. The slope tells us that Gal's running time decreases by $\frac{7}{4}$ or 1.75 minutes per month.

The y–intercept tells us that when Gal started training, she ran a distance of 5K in 41 minutes , which is just an estimate, since the actual time was 40 minutes .

The problem asks us to predict Gal's running time in August. Since June is assigned to month number five, then August will be month number seven. We plug $x = 7$ into the equation of the line of best fit.

$$y = -\frac{7}{4}(7) + 41 = -\frac{49}{4} + 41 = -\frac{49}{4} + \frac{164}{4} = \frac{115}{4} = 28\frac{3}{4}$$

The equation predicts that Gal will be running the 5K race in 28.75 minutes .

In this solution, we eye-balled a line of best fit. Using a graphing calculator, we found this equation for a line of fit $y = -2.2x + 43.7$.

If we plug $x = 7$ in this equation, we get $y = -2.2(7) + 43.7 = 28.3$. This means that Gal ran her race in 28.3 minutes . You see that the graphing calculator gives a different equation and a different answer to the question. The graphing calculator result is more accurate but the line we drew by hand still gives a good approximation to the result.

Example 4

Baris is testing the burning time of "BriteGlo" candles. The following table shows how long it takes to burn candles of different weights. Assume it's a linear relation and we can then use a line to fit the data. If a candle burns for 95 hours , *what must be its weight in ounces?*

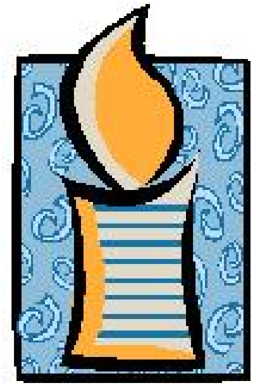
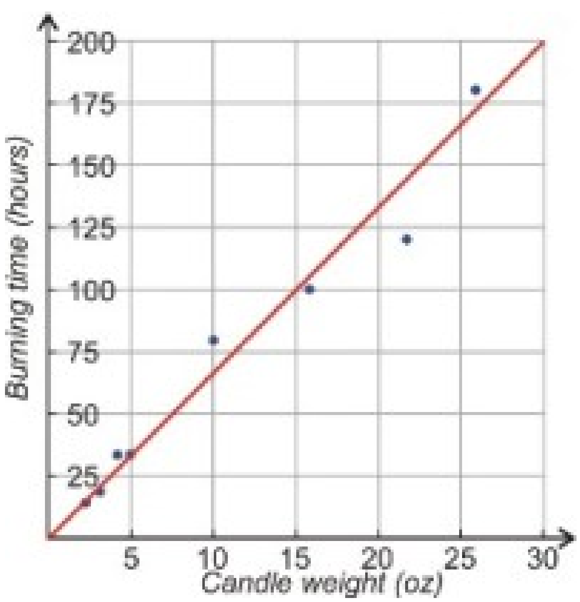


Table 5.2: **Candle Burning Time Based on Candle Weight**

Candle weight (oz)	Time (hours)
2	15
3	20
4	35
5	36
10	80
16	100
22	120
26	180

Solution

Let's make a scatter plot of the data. The independent variable, x, is the candle weight in ounces and the dependent variable, y, is the time in hours it takes the candle to burn. We plot all the points in the table on the coordinate plane.



Then we draw the line of best fit.

Now pick two points on the line $(0,0)$ and $(30,200)$.

Find the equation of the line:

$$\begin{aligned}
m &= \frac{200}{30} = \frac{20}{3} \\
y &= \frac{20}{3}x + b \\
0 &= \frac{20}{3}(0) + b \Rightarrow b = 0 \\
y &= \frac{20}{3}x
\end{aligned}$$

In this problem the slope is burning time divided by candle weight. A slope of $\frac{20}{3} = 6\frac{2}{3}$ tells us for each extra ounce of candle weight, the burning time increases by $6\frac{2}{3}$ hours .

A y–intercept of zero tells us that a candle of weight 0 oz will burn for 0 hours .

The problem asks for the weight of a candle that burns 95 hours . We are given the value of $y = 95$. We need to use the equation to find the corresponding value of x.

$$y = \frac{20}{3}x \Rightarrow = \frac{20}{3}x \Rightarrow x = \frac{285}{20} = \frac{57}{4} = 14\frac{1}{4}$$

A candle that burns 95 hours weighs 14.25 oz.

The graphing calculator gives the linear regression equation as $y = 6.1x + 5.9$ and a result of 14.6 oz.

Notice that we can use the line of best fit to estimate the burning time for a candle of any weight.

Lesson Summary

- A **scatter plot** is a plot of all ordered pairs of experimental measurements.
- **Measurement error** arises from inaccuracies in the measurement device. All measurements of continuous values contain measurement error.
- An **outlier** is an experimental measurement that does not fit with the general pattern of the data.
- For experimental measurements with a linear relationship, you can draw a **line of best fit** which minimizes the distance of each point to the line. Finding the line of best fit is called **linear regression.** A statistics class can teach you the math behind linear regression. For now, you can estimate it visually or use a graphing calculator.

Review Questions

For each data set, draw the scatter plot and find the equation of the line of best fit for the data set by hand.

1. $(57,45)$ $(65,61)$ $(34,30)$ $(87,78)$ $(42,41)$ $(35,36)$ $(59,35)$ $(61,57)$ $(25,23)$ $(35,34)$
2. $(32,43)$ $(54,61)$ $(89,94)$ $(25,34)$ $(43,56)$ $(58,67)$ $(38,46)$ $(47,56)$ $(39,48)$
3. $(12,18)$ $(5,24)$ $(15,16)$ $(11,19)$ $(9,12)$ $(7,13)$ $(6,17)$ $(12,14)$
4. $(3,12)$ $(8,20)$ $(1,7)$ $(10,23)$ $(5,18)$ $(8,24)$ $(2,10)$

For each data set, use a graphing calculator to find the equation of the line of best fit.

5. $(57,45)$ $(65,61)$ $(34,30)$ $(87,78)$ $(42,41)$ $(35,36)$ $(59,35)$ $(61,57)$ $(25,23)$ $(35,34)$

6. $(32,43)$ $(54,61)$ $(89,94)$ $(25,34)$ $(43,56)$ $(58,67)$ $(38,46)$ $(47,56)$ $(95,105)$ $(39,48)$
7. $(12,18)$ $(3,26)$ $(5,24)$ $(15,16)$ $(11,19)$ $(0,27)$ $(9,12)$ $(7,13)$ $(6,17)$ $(12,14)$
8. Shiva is trying to beat the samosa eating record. The current record is 53.5 samosas in 12 minutes. The following table shows how many samosas he eats during his daily practice for the first week of his training. Will he be ready for the contest if it occurs two weeks from the day he started training? What are the meanings of the slope and the y –intercept in this problem?

Table 5.3:

Day	No. of Samosas
1	30
2	34
3	36
4	36
5	40
6	43
7	45

9. Nitisha is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the resulting bounce. The table below shows her data. Draw a scatter plot and find the equation. What is the initial height if the bounce height is 65 cm? What are the meanings of the slope and the y–intercept in this problem?

Table 5.4:

Initial height (cm)	Bounce height (cm)
30	22
35	26
40	29
45	34
50	38
55	40
60	45
65	50
70	52

10. The following table shows the median California family income from 1995 to 2002 as reported by the US Census Bureau. Draw a scatter plot and find the equation. What would you expect the median annual income of a Californian family to be in year 2010? What are the meanings of the slope and the y–intercept in this problem?

Table 5.5:

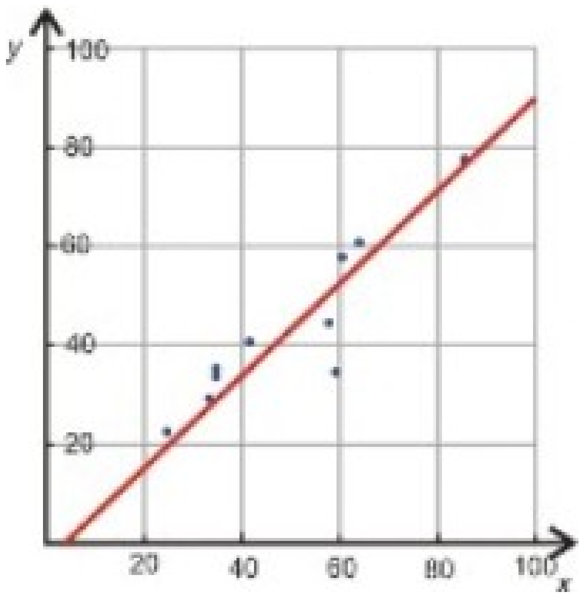
Year	Income
1995	53807
1996	55217

Table 5.5: (continued)

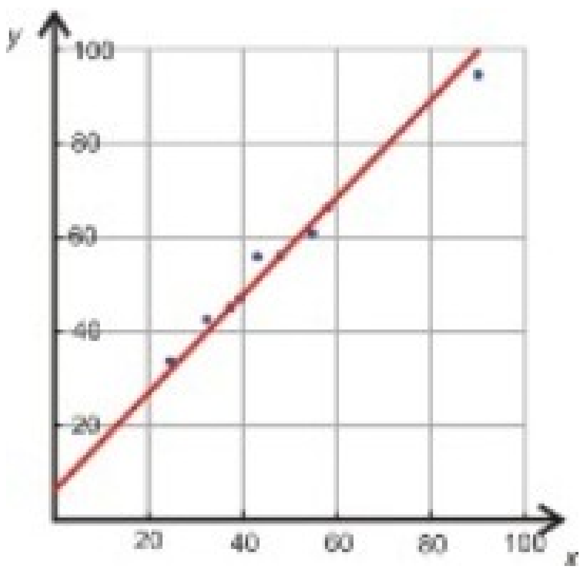
Year	Income
1997	55209
1998	55415
1999	63100
2000	63206
2001	63761
2002	65766

Review Answers

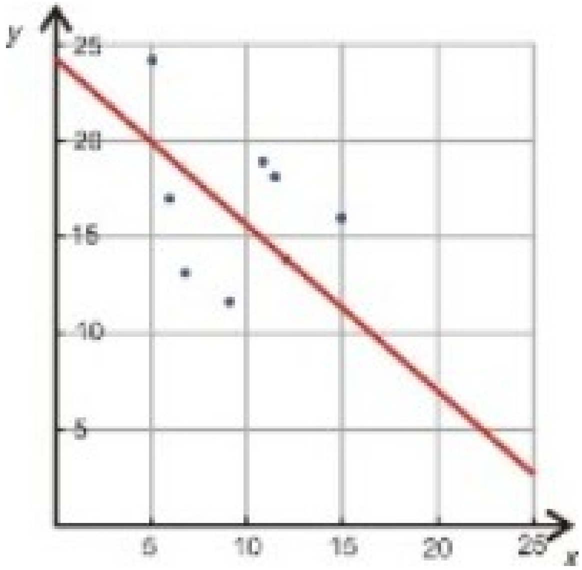
1. $y = 0.9x - 0.8$



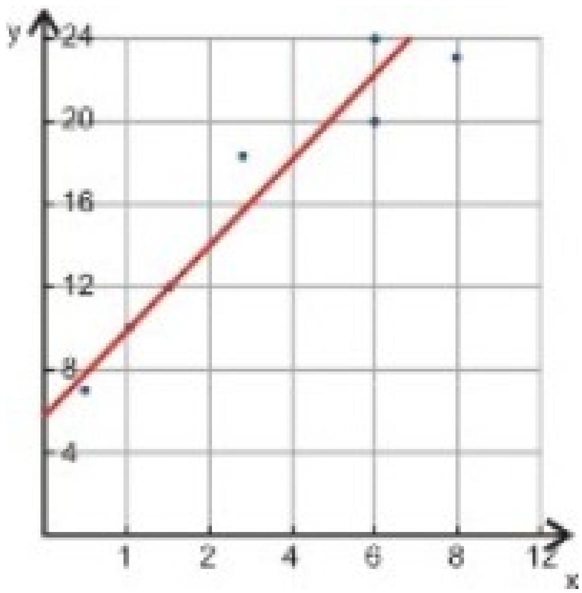
2. $y = 1.05x + 6.1$



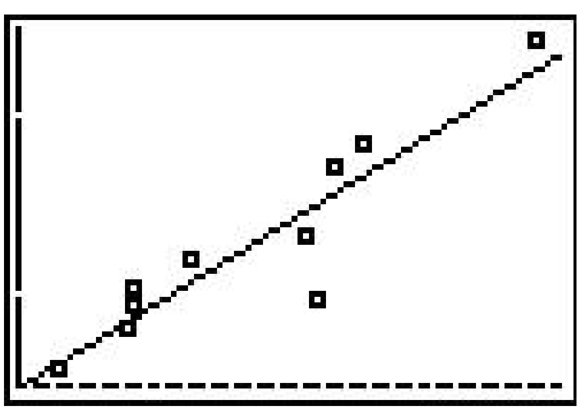
3. $y = -0.86x + 24.3$



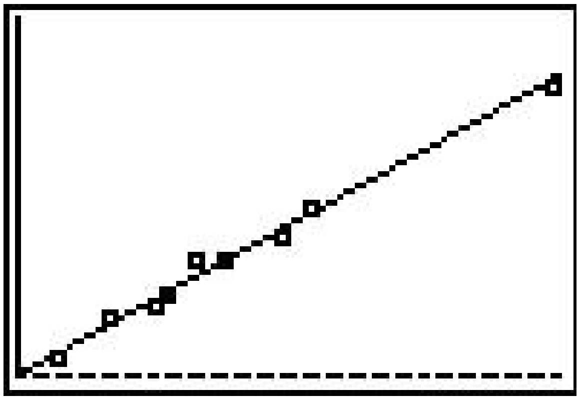
4. $y = 2x + 6$



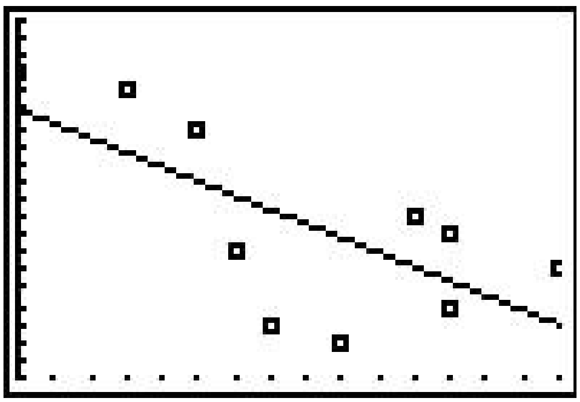
5. $y = .8x + 3.5$



6. $y = .96x + 10.83$

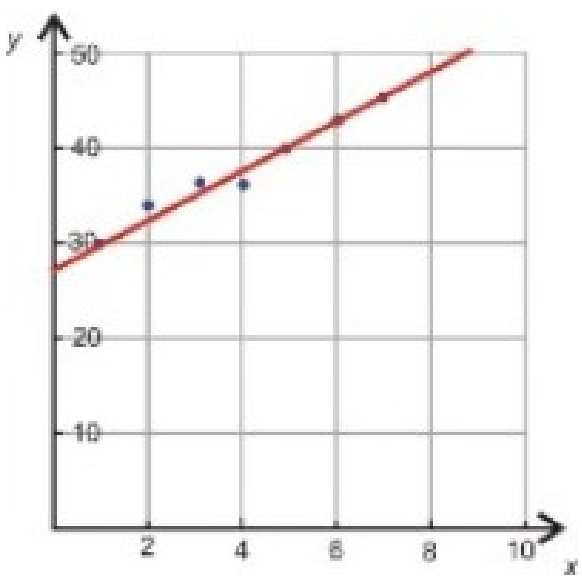


7. $y = -.8x + 25$



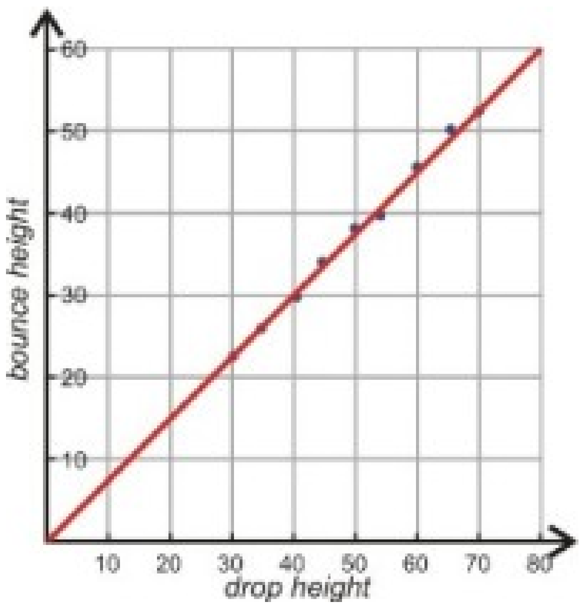
8. $y = 2.5x + 27.5$

Solution



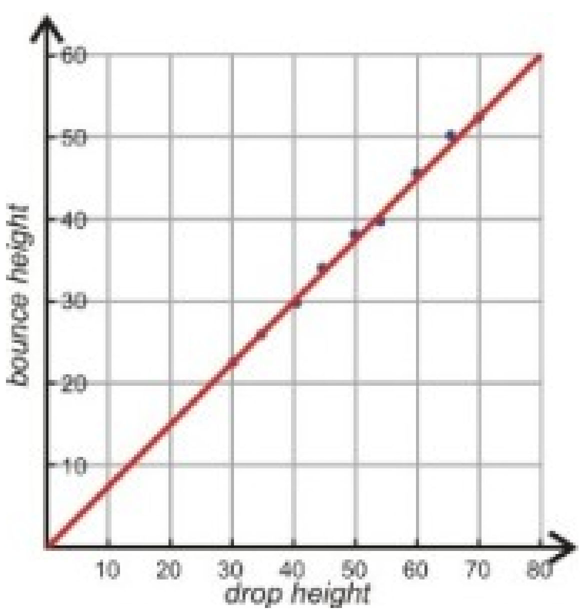
$y = 57.5$. Shiva will beat the record.

9. $y = 0.75x - 0.5$



slope = the ratio of bounce height to drop height y-intercept = how far the ball bounces if it's dropped from a height of zero. The line is the best *fit* to the data. We know that dropping it from height of zero should give a bounce of zero and −0.5 cm is pretty close to zero. Drop height = 83.3 cm when bounce height = 65 cm

10. $y = 1.75x + 53.8$



x = years since 1995 y = Income in thousands of dollars slope = increase in income per year (in thousands) y-intercept = income in 1995 (in thousands) Income in 2010 is $80050.

5.6 Predicting with Linear Models

Learning Objectives

- Collect and organize data.
- Interpolate using an equation.
- Extrapolate using an equation.

- Predict using an equation.

Introduction

Numerical information appears in all areas of life. You can find it in newspapers, magazines, journals, on the television or on the internet. In the last section, we saw how to find the equation of a line of best fit and how to use this equation to make predictions. The line of 'best fit' is a good method if the relationship between the dependent and the independent variables is linear. In this section, you will learn other methods that help us estimate data values. These methods are useful in linear and non-linear relationships equally. The methods you will learn are **linear interpolation** which is useful if the information you are looking for is between two known points and **linear extrapolation** which is useful for estimating a value that is either less than or greater than the known values.

Collect and Organize Data

Data can be collected through **surveys** or **experimental measurements.**

Surveys are used to collect information about a population. Surveys of the population are common in political polling, health, social science and marketing research. A survey may focus on opinions or factual information depending on its purpose.

Experimental measurements are data sets that are collected during experiments.

The information collected by the US Census Bureau (www.census.gov) or the Center for Disease Control (www.cdc.gov) are examples of data gathered using surveys. The US Census Bureau collects information about many aspects of the US population. The census takes place every ten years and it polls the population of the United States.

Let's say we are interested in how the median age for first marriages has changed during the 20^{th} century.

Example 1

Median age at first marriage

The US Census gives the following information about the median age at first marriage for males and females.

In 1890, the median age for males was 26.1 and for females it was 22.0.

In 1900, the median age for males was 25.9 and for females it was 21.9.

In 1910, the median age for males was 25.1 and for females it was 21.6.

In 1920, the median age for males was 24.6 and for females it was 21.2.

In 1930, the median age for males was 24.3 and for females it was 21.3.

In 1940, the median age for males was 24.3 and for females it was 21.5.

In 1950, the median age for males was 22.8 and for females it was 20.3.

In 1960, the median age for males was 22.8 and for females it was 20.3.

In 1970, the median age for males was 23.2 and for females it was 20.8.

In 1980, the median age for males was 24.7 and for females it was 22.0.

In 1990, the median age for males was 26.1 and for females it was 23.9.

In 2000, the median age for males was 26.8 and for females it was 25.1.

This is not a very efficient or clear way to display this information. Some better options are organizing the data in a table or a scatter plot.

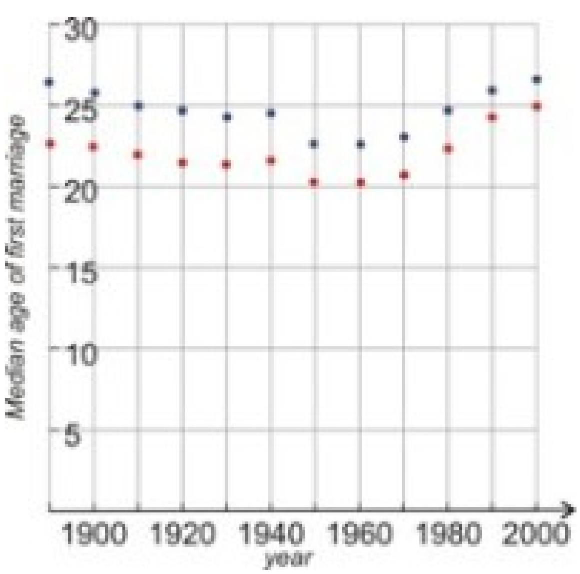
A table of the data would look like this.

Table 5.6: **Median Age of Males and Females at First Marriage by Year**

Year	Median Age of Males	Median Age of Females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

A scatter plot of the data would look like this.

Median Age of Males and Females at First Marriage by Year



The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The next example shows the percent of women that smoke during pregnancy.

Example 2

Pregnant women and smoking

The CDC has the following information.

In the year 1990, 18.4 percent of pregnant women smoked.

In the year 1991, 17.7 percent of pregnant women smoked.

In the year 1992, 16.9 percent of pregnant women smoked.

In the year 1993, 15.8 percent of pregnant women smoked.

In the year 1994, 14.6 percent of pregnant women smoked.

In the year 1995, 13.9 percent of pregnant women smoked.

In the year 1996, 13.6 percent of pregnant women smoked.

In the year 2000, 12.2 percent of pregnant women smoked.

In the year 2002, 11.4 percent of pregnant women smoked.

In the year 2003, 10.4 percent of pregnant women smoked.

In the year 2004, 10.2 percent of pregnant women smoked.

Let's organize this data more clearly in a table and in a scatter plot.

Here is a table of the data.

Table 5.7: **Percent of Pregnant Women Smokers by Year**

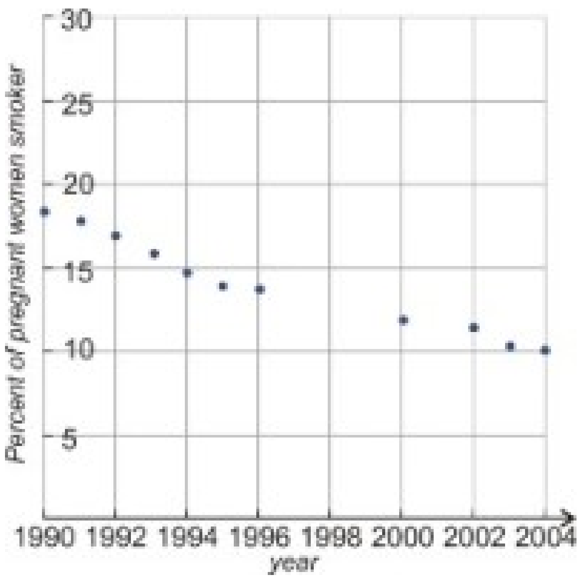
Year	Percent of pregnant women smokers
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6

Table 5.7: (continued)

Year	Percent of pregnant women smokers
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Here is a scatter plot of the data.

Percent of Pregnant Women Smokers by Year



Interpolate Using an Equation

Linear interpolation is often used to fill the gaps in a table. Example one shows the median age of males and females at the time of their first marriage. However, the information is only available at ten year intervals. We know the median age of marriage every ten years from 1890 to 2000, but we would like to estimate the median age of marriage for the years in between. Example two gave us the percentage of women smoking while pregnant. But, there is no information collected for 1997, 1998, 1999 and 2001 and we would like to estimate the percentage for these years. Linear interpolation gives you an easy way to do this.

The strategy for linear interpolation is to use a straight line to connect the known data points (we are assuming that the data would be continuous between the two points) on either side of the unknown point. Then we use that equation to estimate the value we are looking for.

Example 3

Estimate the median age for the first marriage of a male in the year 1946.

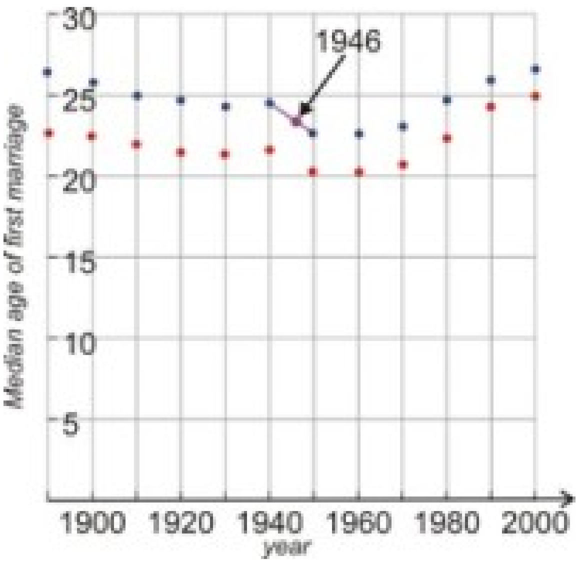
Table 5.8: **Median Age of Males and Females at First Marriage by Year (excerpt)**

Year	Median age of males	Median age of females
...	...	...

Table 5.8: (continued)

Year	Median age of males	Median age of females
1940	24.3	21.5
1950	22.8	20.3
...	...	...

The table to the left shows only the data for the years 1950 and 1960 because we want to estimate a data point between these two years.



We connect the two points on either side of 1946 with a straight line and find its equation.

$$\text{Slope} \qquad m = \frac{22.8 - 24.3}{1950 - 1940} = \frac{-1.5}{10} = -0.15$$

$$y = -0.15x + b$$

$$24.3 = -0.15(1940) + b$$

$$b = 315.3$$

$$\text{Equation} \qquad y = -0.15x + 315.3$$

To estimate the median age of marriage of males in year 1946 we plug $x = 1946$ in the equation.

$$y = -0.15(1946) + 315.3 = 23.4 \textit{ years old}$$

Example 4

Estimate the percentage of pregnant women that were smoking in the year 1998.

Table 5.9: **Percent of Pregnant Women Smokers by Year (excerpt)**

Year	Percent of Pregnant Women Smokers
...	...
1996	13.6
2000	12.2
...	...

The table to the left shows only the data for year 1996 and 2000 because we want to estimate a data point between these two years.

Connect the points on either side of 1998 with a straight line and find the equation of that line.

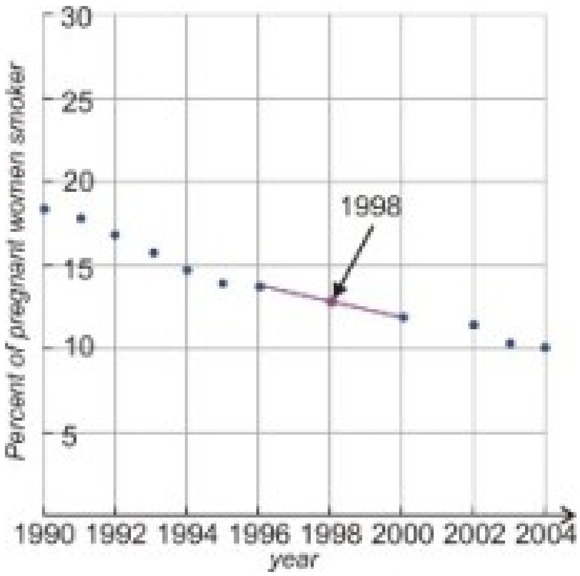
$$\text{Slope} \qquad m = \frac{12.2 - 13.6}{2000 - 1996} = \frac{-1.4}{4} = -0.35$$

$$y = -0.35x + b$$

$$12.2 = -0.35(2000) + b$$

$$b = 712.2$$

$$\text{Equation} \qquad y = -0.35x + 712.2$$



To estimate the percentage of pregnant women who smoked in year 1998 we plug $x = 1998$ into the equation.

$$y = -0.35(1998) + 712.2 = 12.9\%$$

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval in which you are interested, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation** which uses a curve instead of a straight line to estimate values between points.

Extrapolating: How to Use it and When Not to Use it

Linear extrapolation can help us estimate values that are either higher or lower than the range of values of our data set. The strategy is similar to linear interpolation. However you only use a subset of the data, rather than all of the data. For linear data, you are ALWAYS more accurate by using the best fit line method of the previous section. For non-linear data, it is sometimes useful to extrapolate using the last two or three data points in order to estimate a $y-$value that is higher than the data range. To estimate a value that is higher than the points in the data set, we connect the last two data points with a straight line and find its equation. Then we can use this equation to estimate the value we are trying to find. To estimate a value that is lower than the points in the data set, we follow the same procedure. But we use the first two points of our data instead.

Example 5

Winning Times

The winning times for the women's 100 meter race are given in the following table[3]. Estimate the winning time in the year 2010. Is this a good estimate?

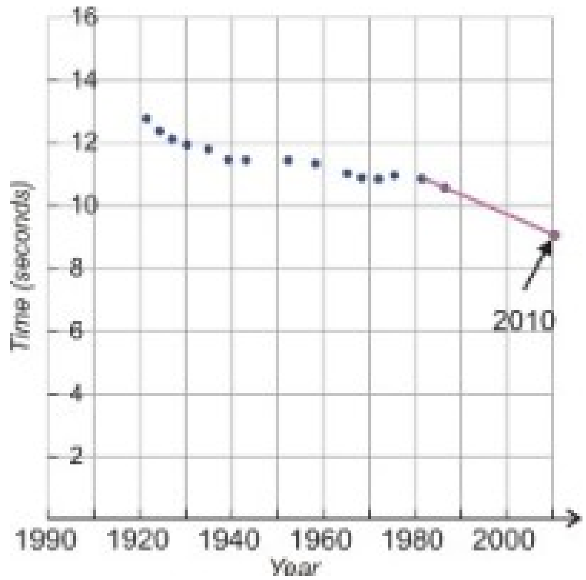
Table 5.10:

Winner	Country	Year	Time (seconds)
Mary Lines	UK	1922	12.8
Leni Schmidt	Germany	1925	12.4
Gerturd Glasitsch	Germany	1927	12.1
Tollien Schuurman	Netherlands	1930	12.0
Helen Stephens	USA	1935	11.8
Lulu Mae Hymes	USA	1939	11.5
Fanny Blankers-Koen	Netherlands	1943	11.5
Marjorie Jackson	Australia	1952	11.4
Vera Krepkina	Soviet Union	1958	11.3
Wyomia Tyus	USA	1964	11.2
Barbara Ferrell	USA	1968	11.1
Ellen Strophal	East Germany	1972	11.0
Inge Helten	West Germany	1976	11.0
Marlies Gohr	East Germany	1982	10.9
Florence Griffith Joyner	USA	1988	10.5

Solution

We start by making a scatter plot of the data. Connect the last two points on the graph and find the equation of the line.

Winning Times for the Women's 100 meter Race by Year



Slope

$$m = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$$
$$y = -0.067x + b$$
$$10.5 = -0.067(1988) + b$$
$$b = 143.7$$

Equation $y = -0.067x + 143.7$

The winning time in year 2010 is estimated to be:

$$y = -0.067(2010) + 143.7 = \underline{9.03 \text{ seconds}}$$

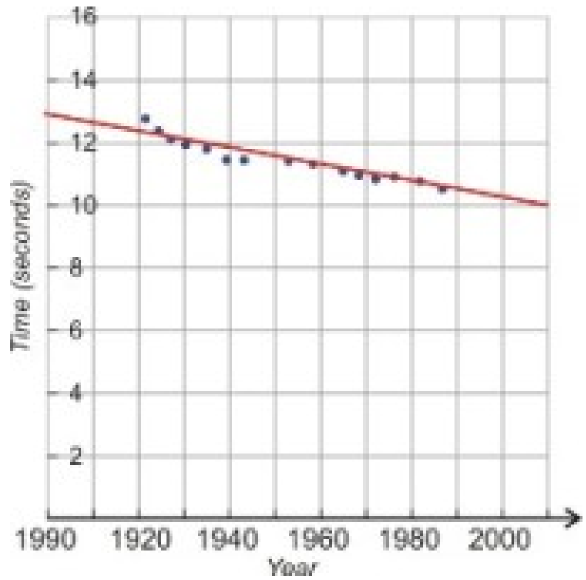
[3] Source: `http://en.wikipedia.org/wiki/World_Record_progression_100_m_women.`

How accurate is this estimate? It is likely that it's not very accurate because 2010 is a long time from 1988. This example demonstrates the weakness of linear extrapolation. Estimates given by linear extrapolation are never as good as using the equation from the best fit line method. In this particular example, the last data point clearly does not fit in with the general trend of the data so the slope of the extrapolation line is much steeper than it should be. As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race, she was accused of using performance-enhancing drugs but this fact was never proven. In addition, there is a question about the accuracy of the timing because some officials said that the tail wind was not accounted for in this race even though all the other races of the day were impacted by a strong wind.

Predict Using an Equation

Linear extrapolation was not a good method to use in the last example. A better method for estimating the winning time in 2010 would be the use of linear regression (i.e. *best fit line method*) that we learned in the last section. Let's apply that method to this problem.

Winning Times for the Women's 100 meter Race by Year



We start by drawing the line of best fit and finding its equation. We use the points $(1982, 10.9)$ and $(1958, 11.3)$.

The equation is $y = -0.017x + 43.9$

In year 2010, $y = -0.017(2010) + 43.9 = 9.73$ seconds

This shows a much slower decrease in winning times than linear extrapolation. This method (fitting a line to all of the data) is always more accurate for linear data and approximate linear data. However, the line of best fit in this case will not be useful in the future. For example, the equation predicts that around the year 2582 the time will be about zero seconds, and in years that follow the time will be negative!

Lesson Summary

- A **survey** is a method of collecting information about a population.
- **Experimental measurements** are data sets that are collected during experiments.
- **Linear interpolation** is used to estimate a data value between two experimental measurements. To do so, compute the line through the two adjacent measurements, then use that line to estimate the intermediate value.
- **Linear extrapolation** is used to estimate a data value either above or below the experimental measurements. Again, find the line defined by the two closest points and use that line to estimate the value.
- The **most accurate method** of estimating data values from a linear data set is to perform linear regression and estimate the value from the best-fit line.

Review Questions

1. Use the data from Example one (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example one (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example one (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.

4. Use the data from Example two (*Pregnant women and smoking*) to estimate the percent of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example two (*Pregnant women and smoking*) to estimate the percent of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example five (*Winning times*) to estimate the winning time for the female 100 meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15^{th} day of each month in the year 2006 in San Diego, California. Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.

Table 5.11:

Hours of daylight	High temperature (F)
10.25	60
11.0	62
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

8.
9. Using the table above to estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the prediction accurate? Find the answer using line of best fit.

Review Answers

1. About 21 years
2. 22.8 years
3. 26.5 years
4. 13.25 percent
5. 9.8 percent
6. 13.1 seconds
7. 70.5 F
8. 65 F. Prediction is not very good since we expect cooler temperatures for less daylight hours. The best fit line method of linear regression predicts 58.5 F.

5.7 Problem Solving Strategies: Use a Linear Model

Learning Objectives

- Read and understand given problem situations.
- Develop and apply the strategy: use a linear model.
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we have been estimating values using straight lines. When we use linear interpolation, linear extrapolation or predicting results using a line of best fit, it is called **linear modeling.** In this section, we will look at a few examples where data sets occurring in real-world problems can be modeled using linear relationships. From previous sections remember our problem solving plan:.

Step 1

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4

Look – Check and Interpret

Check to see if you used all your information and that the answer makes sense.

Example 1

Dana heard something very interesting at school. Her teacher told her that if you divide the circumference of a circle by its diameter you always get the same number. She tested this statement by measuring the circumference and diameter of several circular objects. The following table shows her results.

From this data, estimate the circumference of a circle whose diameter is 12 inches . What about 25 inches ? 60 inches ?

Table 5.12: **Diameter and Circumference of Various Objects**

Object	Diameter (inches)	Circumference (inches)
Table	53	170
Soda can	2.25	7.1
Cocoa tin	4.2	12.6

Table 5.12: (continued)

Object	Diameter (inches)	Circumference (inches)
Plate	8	25.5
Straw	.25	1.2
Propane tank	13.3	39.6
Hula Hoop	34.25	115

Solution

Let's use the problem solving plan.

Step 1

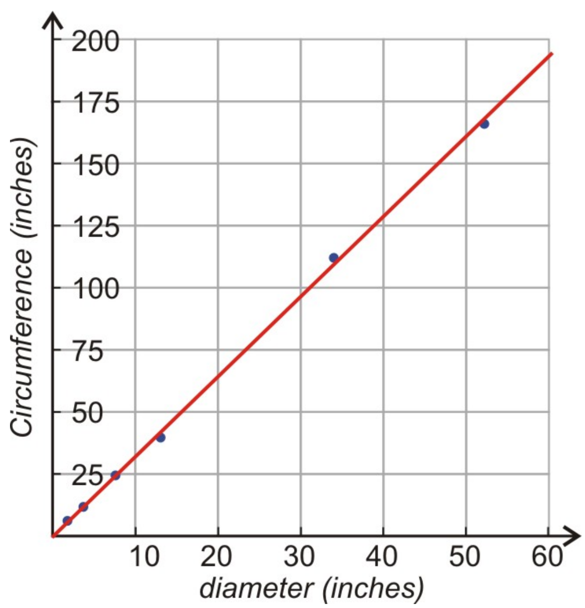
We define our variables.

$x =$ diameter of the circle in inches

$y =$ circumference of the circle in inches

We want to know the circumference when the diameter is 12, 25 or 60 inches .

Step 2 We can find the answers either by using the line of best fit or by using linear interpolation or extrapolation. We start by drawing the scatter plot of the data.



Step 3 Line of best fit

Estimate a line of best fit on the scatter plot.

Find the equation using points $(.25, 1.2)$ and $(8, 25.5)$.

$$\text{Slope} \qquad m = \frac{25.5 - 1.2}{8 - .25} = \frac{24.3}{7.75} = 3.14$$

$$y = 3.14x + b$$

$$1.2 = 3.14(.25) + b \Rightarrow b = 0.42$$

$$\text{Equation} \qquad y = 3.14x + 0.42$$

$$\text{Diameter} = 12 \text{ inches} \Rightarrow y = 3.14(12) + 0.42 = \underline{38.1 \text{ inches}}$$
$$\text{Diameter} = 25 \text{ inches} \Rightarrow y = 3.14(25) + 0.42 = \underline{78.92 \text{ inches}}$$
$$\text{Diameter} = 60 \text{ inches} \Rightarrow y = 3.14(60) + 0.42 = \underline{188.82 \text{ inches}}$$

In this problem the slope = 3.14. This number should be very familiar to you – it is the number rounded to the hundredths place. Theoretically, the circumference of a circle divided by its diameter is always the same and it equals 3.14 or π.

You are probably more familiar with the formula $C = \pi \cdot d$.

Note: The calculator gives the line of best fit as $y = 3.25x - 0.57$, so we can conclude that we luckily picked two values that gave the correct slope of 3.14. Our line of best fit shows that there was more measurement error in other points.

Step 4 Check and Interpret

The circumference of a circle is πd and the diameter is simply d. If we divide the circumference by the diameter we will get π. The slope of the line is 3.14, which is very close to the exact value of π. There is some error in the estimation because we expect the y-intercept to be zero and it is not.

The reason the line of best fit method works the best is that the data is very linear. All the points are close to the straight line but there is some slight measurement error. The line of best fit averages the error and gives a good estimate of the general trend.

Note: The linear interpolation and extrapolation methods give estimates that aren't as accurate because they use only two points in the data set. If there are measurement errors in the points that are being used, then the estimates will lose accuracy. Normally, it is better to compute the line of best fit with a calculator or computer.

Example 2

A cylinder is filled with water to a height of 73 *centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at two second intervals. The table below shows the height of the water level in the cylinder at different times.*

a) *Find the water level at* 15 seconds.

b) *Find the water level at* 27 seconds .

Water Level in Cylinder at Various Times

Table 5.13: **Water Level in Cylinder at Various Times**

Time (seconds)	Water level (cm)
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1

Table 5.13: (continued)

Time (seconds)	Water level (cm)
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1

Solution

Let's use the problem solving plan.

Step 1

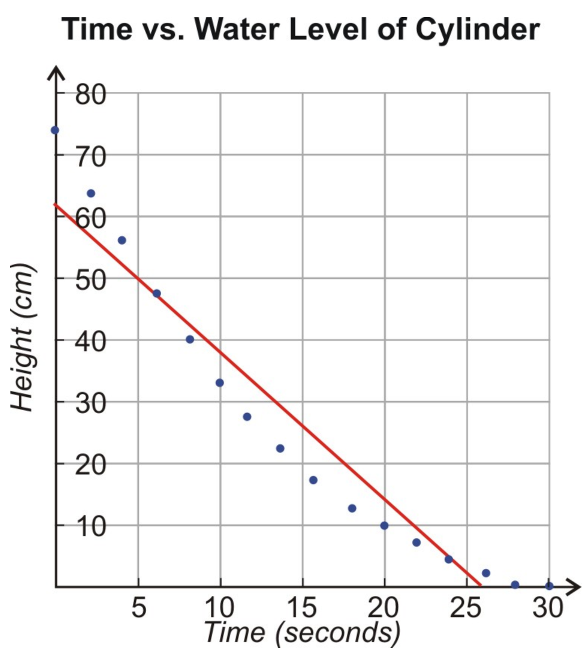
Define our variables

$x =$ time in seconds

$y =$ water level in centimeters

We want to know the water level at time 15, 27 and -5 seconds.

Step 2 We can find the answers either by using the line of best fit or by using linear interpolation or extrapolation. We start by drawing the scatter plot of the data.



Step 3 *Method 1* Line of best fit

Draw an estimate of the line of best fit on the scatter plot. Find the equation using points $(6, 47.2)$ and $(24, 3.9)$.

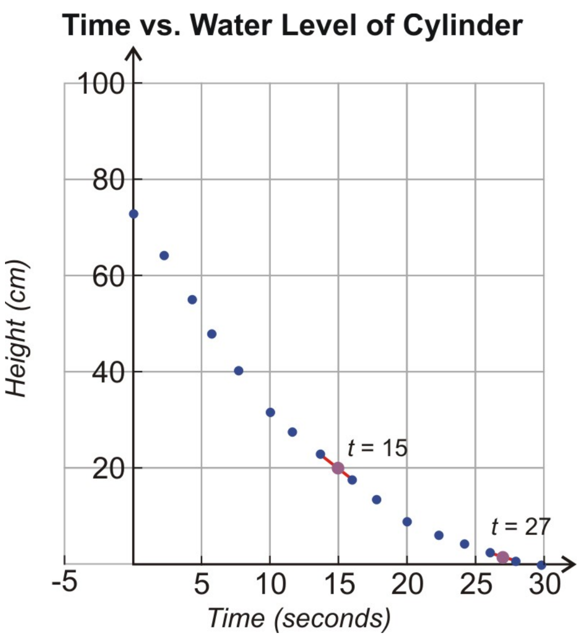
Slope	$m = \frac{3.9 - 47.2}{24 - 6} = \frac{-43.3}{18} = -2.4$
	$y = -2.4x + b$
	$47.2 = -2.4(6) + b \Rightarrow b = 61.6$
Equation	$y = -2.4x + 61.6$

$$\text{Time} = 15 \text{ seconds} \Rightarrow y = -2.4(15) + 61.6 = \underline{25.6\text{cm}}$$
$$\text{Time} = 27 \text{ seconds} \Rightarrow y = -2.4(27) + 61.6 = \underline{-3.2 \text{ cm}}$$

The line of best fit does not show us accurate estimates for the height. The data points do not appear to fit a linear trend so the line of best fit is close to very few data points.

Method 2: Linear interpolation or linear extrapolation.

We use linear interpolation to find the water level for the times 15 and 27 seconds, because these points are between the points we know.



Time = 15 seconds

Connect points (14, 21.9) and (16, 17.1) and find the equation of the straight line.

$$m = \frac{17.1 - 21.9}{16 - 14} = \frac{-4.8}{2} = -2.4y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$$

Equation $y = -2.4x + 55.5$

Plug in $x = 15$ and obtain $y = -2.4(15) + 55.5 = 19.5$ cm

Time = 27 seconds

Connect points (26, 2) and (28, 0.7) and find the equation of the straight line.

$$m = \frac{0.7 - 2}{28 - 26} = \frac{-1.3}{2} = -.65$$
$$y = -.65x + b \Rightarrow 2 = -.65(26) + b \Rightarrow b = 18.9$$

Equation $y = -.65x = 18.9$

Plug in $x = 27$ and obtain $y = -.65(27) + 18.9 = 1.35$ cm

We use linear extrapolation to find the water level for time -5 seconds because this point is smaller than the points in our data set.

Step 4 Check and Interpret

In this example, the linear interpolation and extrapolation method gives better estimates of the values that we need to solve the problem. Since the data is not linear, the line of best fit is not close to many of the points in our data set. The linear interpolation and extrapolation methods give better estimates because we do not expect the data to change greatly between the points that are known.

Lesson Summary

- Using linear interpolation, linear extrapolation or prediction using a line of best fit is called **linear modeling.**
- The four steps of the **problem solving plan** are:

1. Understand the problem
2. Devise a plan – Translate
3. Carry out the plan – Solve
4. Look – Check and Interpret

Review Questions

The table below lists the predicted life expectancy based on year of birth (US Census Bureau).

Use this table to answer the following questions.

1. Make a scatter plot of the data
2. Use a line of best fit to estimate the life expectancy of a person born in 1955.
3. Use linear interpolation to estimate the life expectancy of a person born in 1955.
4. Use a line of best fit to estimate the life expectancy of a person born in 1976.
5. Use linear interpolation to estimate the life expectancy of a person born in 1976.
6. Use a line of best fit to estimate the life expectancy of a person born in 2012.
7. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
8. Which method gives better estimates for this data set? Why?

Table 5.14:

Birth Year	Life expectancy in years
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

9.
10. The table below lists the high temperature for the first day of the month for year
11. 2006
12. in San Diego, California (Weather Underground). Use this table to answer the following questions.
13. Draw a scatter plot of the data
14. Use a line of best fit to estimate the temperature in the middle of the 4^{th} month (month 4.5).
15. Use linear interpolation to estimate the temperature in the middle of the4^{th} month (month 4.5).
16. Use a line of best fit to estimate the temperature for month 13 (January 2007).
17. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
18. Which method gives better estimates for this data set? Why?

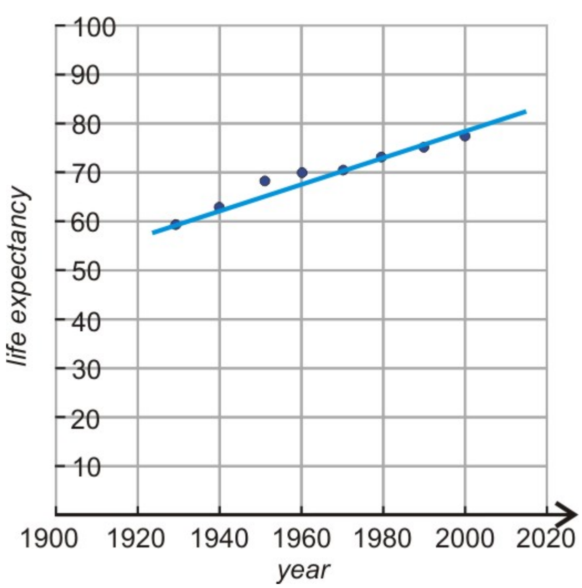
Table 5.15:

Month number	Temperature (F)
1	63
2	66
3	61
4	64
5	71
6	78
7	88
8	78
9	81
10	75
11	68
12	69

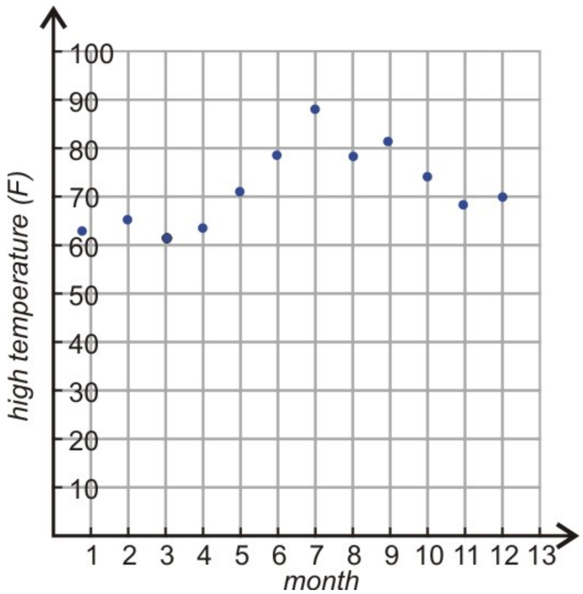
19.

Review Answers

1. Equation of line of best fit using points $(1940, 62.9)$ and $(1990, 75.4)$ $y = .25x - 422.1$.



2. 66.7 years
3. $y = .15x - 224.3$, 69.0 years
4. 71.9 years
5. $y = .29x - 500.5$, 72.5 years
6. 80.9 years
7. $y = .16x - 243$, 78.9 years
8. A line of best fit gives better estimates because data is linear.
9. Equation of line of best fit using points $(2,66)$ and $(10,75)$ $y = 1.125x + 63.75$



10. 68.8 F
11. $y = 7x + 36$, 67.5 F
12. 78.4 F
13. $y = x + 57$, 70 F
14. Linear interpolation and extrapolation give better estimates because data is not linear.

Chapter 6

Graphing Linear Inequalities; Introduction to Probability

6.1 Inequalities Using Addition and Subtraction

Learning Objectives

- Write and graph inequalities in one variable on a number line.
- Solve an inequality using addition.
- Solve an inequality using subtraction.

Introduction

Inequalities are similar to equations in that they show a relationship between two expressions. We solve and graph inequalities in a similar way to equations. However, there are some differences that we will talk about in this chapter. The main difference is that for linear inequalities the answer is an interval of values whereas for a linear equation the answer is most often just one value.

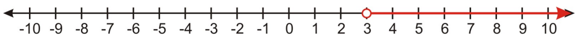
When writing inequalities we use the following symbols

$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

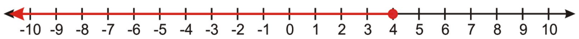
Let's start with the simple inequality $x > 3$

We read this inequality as "x is greater than 3". The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality by a number line graph.



Consider another simple inequality $x \leq 4$

We read this inequality as "x is less than or equal to 4". The solution is the set of all real numbers that equal four or less than four. We graph this solution set on the number line.



In a graph, we use an empty circle for the endpoint of a strict inequality ($x > 3$) and a filled circle if the equal sign is included ($x \leq 4$).

Example 1

Graph the following inequalities on the number line.

a) $x < -3$

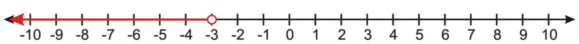
b) $x \geq 6$

c) $x > 0$

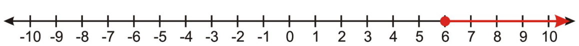
d) $x \leq 8$

Solution

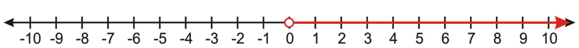
a) The inequality $x < -3$ represents all real numbers that are less than -3. The number -3 is not included in the solution and that is represented by an open circle on the graph.



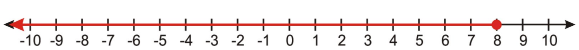
b) The inequality $x \geq 6$ represents all real numbers that are greater than or equal to six. The number six is included in the solution and that is represented by a closed circle on the graph.



c) The inequality $x > 0$ represents all real numbers that are greater than zero. The number zero is not included in the solution and that is represented by an open circle on the graph.

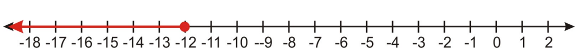


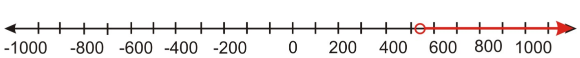
d) The inequality $x \leq 8$ represents all real numbers that are less than or equal to eight. The number eight is included in the solution and that is represented by a closed circle on the graph.

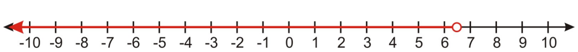


Example 2

Write the inequality that is represented by each graph.

a)


b)


c)


d)

Solution:

a) $x \leq -12$

b) $x > 540$
c) $x < 65$
d) $x \geq$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

Example 3

Write each statement as an inequality and graph it on the number line.

a) You must maintain a balance of at least $2500 in your checking account to get free checking.

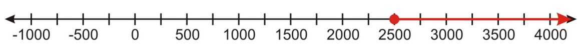
b) You must be at least 48 inches tall to ride the "Thunderbolt" Rollercoaster.

c) You must be younger than 3 years old to get free admission at the San Diego Zoo.

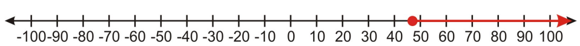
d) The speed limit on the interstate is 65 miles per hour.

Solution:

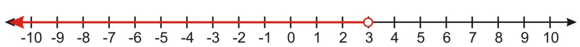
a) The inequality is written as $x \geq 2500$. The words "at least" imply that the value of $2500 is included in the solution set.



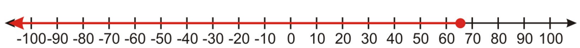
b) The inequality is written as $x \geq 48$. The words "at least" imply that the value of 48 inches is included in the solution set.



c) The inequality is written as $x < 3$.



d) Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.



Solve an Inequality Using Addition

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations. For inequalities of this type:

$x - a < b$ or $x - a > b$

We isolate the x by adding the constant a to both sides of the inequality.

Example 4

Solve each inequality and graph the solution set.

a) $x - 3 < 10$

b) $x - 1 > -10$

c) $x - 1 \leq -5$

d) $x - 20 \geq 14$

Solution:

a)

To solve the inequality $x - 3 < 10$

Add 3 to both sides of the inequality. $x - 3 + 3 < 10 + 3$

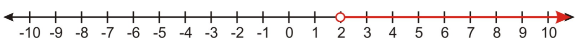
Simplify $x < 13$

b)

To solve the inequality $x - 1 > -10$

Add 12 to both sides of the inequality $x - 12 + 12 > -10 + 12$

Simplify $x > 2$

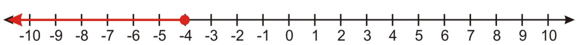


c)

To solve the inequality $x - 1 \leq -5$

Add 1 to both sides of the inequality $x - 1 + 1 \leq -5 + 1$

Simplify to obtain $x \leq -4$

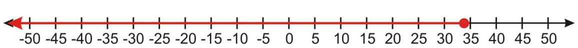


d)

To solve the inequality $x - 20 \leq 14$

Add 20 to both sides of the inequality : $x - 20 + 20 \leq 14 + 20$

Simplify $x \leq 34$



Solve an Inequality Using Subtraction

For inequalities of this type:

$x + 1 < b$ or $x + 1 > b$

We isolate the x by subtracting the constant a on both sides of the inequality.

Example 5

Solve each inequality and graph the solution set.

a) $x + 2 < 7$

b) $x + 8 \leq -7$

c) $x + 4 > 13$

d) $x + 5 \geq \frac{3}{4}$

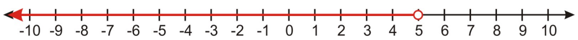
Solution:

a)

To solve the inequality $x + 2 < 7$

Subtract 2 on both sides of the inequality $x + 2 - 2 < 7 - 2$

Simplify to obtain $x < 5$

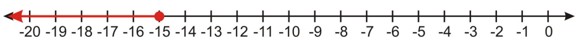


b)

To solve the inequality $x + 8 \leq -7$

Subtract 8 on both sides of the inequality $x + 8 - 8 \leq -7 - 8$

Simplify to obtain : $x < -15$

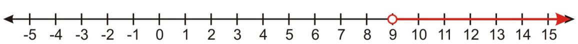


c)

To solve the inequality $x + 4 > 13$

Subtract 4 on both sides of the inequality $x + 4 - 4 > 13 - 4$

Simplify $x > 9$



d)

To solve the inequality $x + 5 \geq \frac{3}{4}$

Subtract 5 on both sides of the inequality $x + 5 - 5 \geq -\frac{3}{4} - 5$

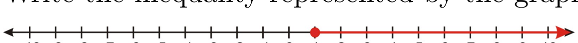
Simplify to obtain : $x \geq -5\frac{3}{4}$

Lesson Summary

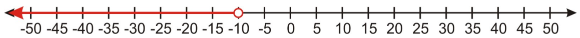
- The answer to an **inequality** is often an **interval of values**. Common **inequalities** are:
- $>$ is greater than
- $\geq$ is greater than or equal to
- $>$ is less than
- $\leq$ is less than or equal to
- Solving inequalities with **addition** and **subtraction** works just like solving an equation. To solve, we isolate the variable on one side of the equation.

Review Questions

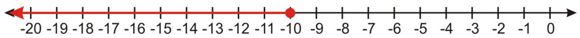
1. Write the inequality represented by the graph.



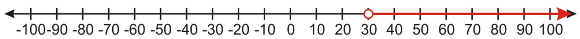
2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.



4. Write the inequality represented by the graph.



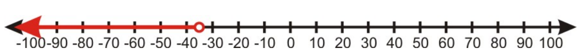
Graph each inequality on the number line.

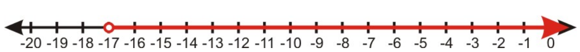
5. $x < -35$
6. $x > -17$
7. $x \geq 20$
8. $x \leq 3$

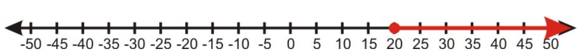
Solve each inequality and graph the solution on the number line.

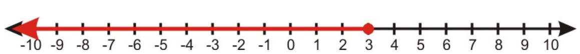
9. $x - 5 < 35$
10. $x + 15 \geq -60$
11. $x - 2 \leq 1$
12. $x - 8 > -20$
13. $x + 11 > 13$
14. $x + 65 < 100$
15. $x - 32 \leq 0$
16. $x + 68 \geq 75$

Review Answers

1. $x \geq 1$
2. $x < -10$
3. $x \leq -10$
4. $x > 30$
5. 

6. 

7. 

8. 

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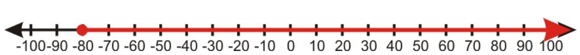
10. 

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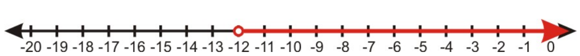
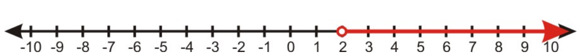
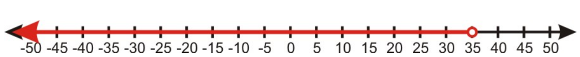
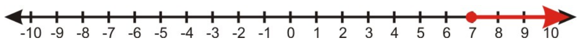
12. 

13. 

14. 

15. 

16. 

6.2 Inequalities Using Multiplication and Division

Learning Objectives

- Solve an inequality using multiplication.
- Solve an inequality using division.
- Multiply or divide an inequality by a negative number.

Introduction

In this section, we consider problems where we find the solution of an inequality by multiplying or dividing both sides of the inequality by a number.

Solve an Inequality Using Multiplication

Consider the problem $\frac{x}{5} \le 3$

To find the solution we multiply both sides by 5. $5 \cdot \frac{x}{5} \le 3.5$

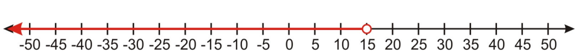
We obtain $x \le 15$

The answer of an inequality can be expressed in four different ways:

1. **Inequality notation** The answer is simply expressed as $x < 15$.
2. **Set notation** The answer is $\{x|x < 15\}$. You read this as "the set of all values of x, such that x is a real number less than 15".
3. **Interval notation** uses brackets to indicate the range of values in the solution.

Square or **closed brackets** "[" and "]" indicate that the number next to the bracket is included in the solution set.
Round or **open brackets** "(" and ")" indicate that the number next to the bracket is not included in the solution set.
Interval notation also uses the concept of **infinity** ∞ and **negative infinity** $-\infty$.
The interval notation solution for our problem is $(-\infty, 15)$.

1. **Solution graph** shows the solution on the real number line. A closed circle on a number indicates that the number is included in the solution set. While an open circle indicates that the number is not included in the set. For our example, the solution graph is drawn here.



Example 1

a) $[-4, 6]$ says that the solutions is all numbers between -4 and 6 **including** -4 and 6.

b) $(8, 24)$ says that the solution is all numbers between 8 and 24 but **does not include** the numbers 8 and 24.

c) $[3, 12)$ says that the solution is all numbers between 3 and 12, **including** 3 but **not including** 12.

d) $(-10,)$ says that the solution is all numbers greater that -10, **not including** -10.

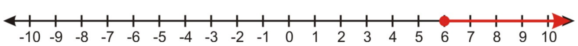
e) $(,)$ says that the solution is all real numbers.

Solving an Inequality Using Division

Consider the problem.	$2x \geq 15$
To find the solution we multiply both sides by 2.	$\frac{2x}{2} \geq \frac{12}{2}$
We obtain.	$x \geq 6$

Let's write the solution in the four different notations you just learned:

Inequality notation	$x \geq 6$
Set notation	$x \mid x \geq 6$
Interval notation	$[6, \infty)$

Solution graph 

Multiplying and Dividing an Inequality by a Negative Number

We solve an inequality in a similar way to solving a regular equation. We can add or subtract numbers on both sides of the inequality. We can also multiply or divide **positive** numbers on both sides of an inequality without changing the solution.

Something different happens if we multiply or divide by **negative** numbers. **In this case, the inequality sign changes direction**.

For example, to solve $-3x < 9$

We divide both sides by –3. The inequality sign changes from < to > because we divide by a negative number. $\frac{-3x}{-3} > \frac{9}{-3}$

$$x > -3$$

We can explain why this happens with a simple example. We know that two is less than three, so we can write the inequality.

$$2 < 3$$

If we multiply both numbers by -1 we get -2 and -3, but we know that -2 is greater than -3.

$$-2 > -3$$

You see that multiplying both sides of the inequality by a negative number caused the inequality sign to change direction. This also occurs if we divide by a negative number.

Example 2

Solve each inequality. Give the solution in inequality notation and interval notation.

a) $4x < 24$

b) $-9x \geq -\frac{3}{5}$

c) $-5x \leq 21$

d) $12 > -30$

Solution:

a)

Original problem.	$4x < 24$
Divide both sides by 4.	$\frac{4x}{4} < \frac{24}{4}$
Simplify	$x < 6$ or $(\infty, 6)$ Answer

b)

Original problem :	$-9x\frac{-3}{5}$
Divide both sides by -9.	$\frac{-9}{-9} \leq \frac{\cancel{-3}}{5} \cdot \frac{1}{\cancel{-9}_3}$ Direction of the inequality is changed
Simplify.	$x \geq \frac{1}{15}$ or $\left[\frac{1}{15}, \infty\right)$ Answer

Original problem :	$-5x \leq 21$
Divide both sides by -5.	$\frac{-5x}{-5} \geq \frac{21}{-5}$ Direction of the inequality is changed
Simplify.	$x \geq -\frac{21}{5}$ or $\left[\frac{21}{5}, \infty\right)$ Answer

d)

Original problem	$12x > -30$
Divide both sides by 12.	$\frac{12x}{12} > \frac{-30}{12}$
Simplify.	$x > -\frac{5}{5}$ or $\left(\frac{-5}{2}, \infty\right)$ Answer

Example 3

Solve each inequality. Give the solution in inequality notation and solution graph.

a) $\frac{x}{2} > 40$

b) $\frac{x}{-3} \leq -12$

c) $\frac{x}{25} < \frac{3}{2}$

d) $\frac{x}{-7} \geq 9$

Solution

a)

Original problem $\frac{x}{2} > 40$

Multiply both sides by 2. $2 \cdot \frac{x}{2} > 40 \cdot 2$ Direction of inequality is NOT changed

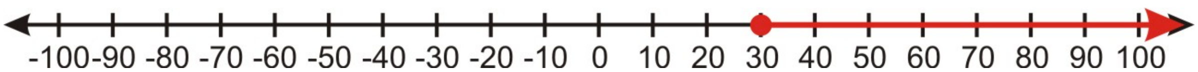
Simplify. $x > -80$ Answer

b)

Original problem $\frac{x}{-3} \le -12$

Multiply both sides by -3. $-3 \cdot \frac{x}{-3} \ge -12 \cdot (-3)$ Direction of inequality is changed

Simplify. $x \ge 36$ Answer



c)

Original problem $\frac{x}{25} < \frac{3}{2}$

Multiply both sides by 25. $25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25$ direction of inequality is NOT changed

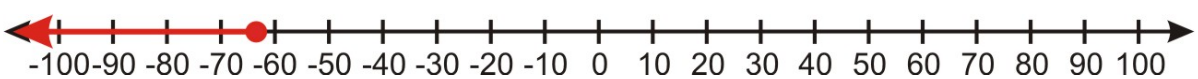
Simplify. $x < \frac{75}{2}$ or $x < 37.5$ Answer

d)

Original problem $\frac{x}{-7} \ge 9$

Multiply both sides by -7. $-7 \cdot \frac{x}{-7} \le 9 \cdot (-7)$ Direction of inequality is changed

Simplify. $x \le -63$ Answer



Lesson Summary

- There are four ways to represent an inequality:

1. ***Equation notation*** $x \ge 2$
2. ***Set notation*** $x \ge 2$
3. ***Interval notation*** $[2, \infty)$

Closed brackets "[" and "]" mean inclusive, parentheses "("and ")" mean exclusive.

4. ***Solution graph*** 

- When multiplying or dividing both sides of an inequality by a negative number, you need to ***reverse the inequality***.

Review Questions

Solve each inequality. Give the solution in inequality notation and solution graph.

1. $3x \leq 6$
2. $\frac{x}{5} > -\frac{3}{10}$
3. $-10x > 250$
4. $\frac{x}{-7} \geq -5$

Solve each inequality. Give the solution in inequality notation and interval notation.

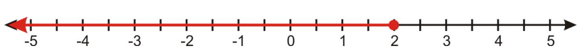
5. $9x > -\frac{3}{4} >$
6. $-\frac{x}{15} > \leq 5$
7. $620x > 2400$
8. $\frac{x}{20} \geq -\frac{7}{40}$

Solve each inequality. Give the solution in inequality notation and set notation.

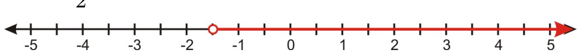
9. $-0.5x \leq 7.5$
10. $75x \geq 125$
11. $\frac{x}{-3} > -\frac{10}{9}$
12. $\frac{x}{-15} < 8$

Review Answers

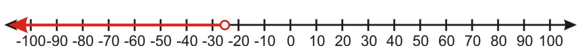
1. $x \leq 2$ or



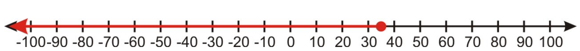
2. $x > -\frac{3}{2}$ or



3. $x < -25$ or



4. $x \leq 35$ or



5. $x > -\frac{1}{12}$ or $(-\frac{1}{12}, \infty)$
6. $x \geq -75$ or $[-75, \infty)$
7. $x < 3.9$ or $(-\infty, 3.9)$
8. $x \geq -\frac{7}{2}$ or $[-\frac{7}{2}, \infty)$
9. $x \geq -15$ or $\{x$ is a real number $\mid x \geq -15$
10. $x \geq \frac{5}{3}$ or $\{x$ is a real number $\mid x \geq \frac{5}{3}$
11. $x < -\frac{10}{3}$ or $\{x$ is a real number $\mid x < -\frac{10}{3}$
12. $x > -120$ or $\{x$ is a real number $\mid x > -120$

6.3 Multi-Step Inequalities

Learning Objectives

- Solve a two-step inequality.

- Solve a multi-step inequality.
- Identify the number of solutions of an inequality.
- Solve real-world problems using inequalities.

Introduction

In the last two sections, we considered very simple inequalities which required one-step to obtain the solution. However, most inequalities require several steps to arrive at the solution. As with solving equations, we must use the order of operations to find the correct solution. In addition **remember that when we multiply or divide the inequality by a negative number the direction of the inequality changes**.

The general procedure for solving multi-step inequalities is as follows.

1. Clear parenthesis on both sides of the inequality and collect like terms.

2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.

3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.

Solve a Two-Step Inequality

Example 1

Solve each of the following inequalities and graph the solution set.

a)$6x - 5 < 10$

b)$-9x < -5x - 15$

c) $-\frac{9x}{5} \le 24$

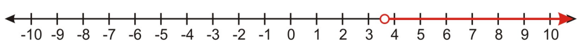
Solution

a)

Original problem :	$6x - 5 < 10$
Add 5 to both sides : .	$6x - 5 + 5 < 10 + 5$
Simplify.	$6x < 15$
Divide both sides by 6.	$\frac{6x}{6} < \frac{15}{6}$
Simplify.	$x < \frac{5}{2}$ Answer

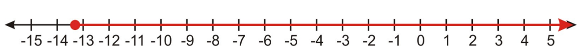
b)

Original problem.	$-9x \leq -5x - 15$
Add 5x to both sides.	$-9x + 5x \leq -5x + 5x - 15$
Simplify.	$-4x < -15$
Divide both sides by -4.	$\frac{-4x}{-4} > \frac{-15}{-4}$ Inequality sign was flipped
Simplify.	$x > \frac{15}{4}$ Answer



c)

Original problem.	$-9x \leq 24$
Multiply both sides by 5.	$\frac{-9x}{5}.5 \leq 24.5$
Simplify.	$-9x \leq 120$
Divide both sides by -9.	$\frac{-9x}{-9} > \frac{120}{-9}$ Inequality sign was flipped
Simplify.	$x \geq -\frac{40}{3}$ Answer



Solve a Multi-Step Inequality

Example 2

Each of the following inequalities and graph the solution set.

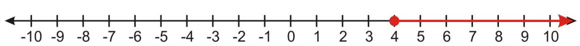
a) $\frac{9x}{5} - 7 \geq -3x + 12$

b) $-25x + 12 \leq -10x - 12$

Solution

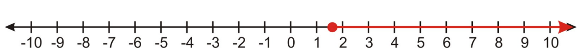
a)

Original problem	$\frac{9x}{5} - 7 \geq -3x + 12$
Add $3x$ to both sides.	$\frac{9x}{5} + 3x - 7 \geq -3x + 3x + 12$
Simplify.	$\frac{24x}{5} - 7 \geq 12$
Add 7 to both sides.	$\frac{24x}{5} - 7 + 7 \geq 12 + 7$
Simplify.	$\frac{24x}{5} - 7 \geq 19$
Multiply 5 to both sides.	$5 \cdot \frac{24x}{5} \geq 5.19$
Simplify.	$24x \geq 95$
Divide both sides by 24.	$\frac{24x}{24} \geq \frac{95}{24}$
Simplify.	$x \geq \frac{95}{24}$ Answer



b)

Original problem	$-25x + 12 \leq -10x - 12$
Add 5 to both sides.	$-25x + 10x + 12 \leq -10x + 10x - 12$
Simplify.	$-15x + 12 \leq -12$
Subtract 12 from both sides.	$-15x + 12 - 12 \leq -12 - 12$
Simplify.	$-15x \leq -24$
Divide both sides by -15.	$\frac{-15x}{-15} \geq \frac{-24}{-15}$ Inequality sign was flipped
Simplify.	$x \geq \frac{8}{5}$ Answer



Example 3

Solve the following inequalities.

a) $4x - 2(3x - 9) \leq -4(2x - 9)$

b) $\frac{5x-1}{4} > -2(x + 5)$

Solution

a)

Original problem	$4x - 2(3x - 9) \leq -4(2x - 9)$
Simplify parentheses.	$4x - 6x + 18 \leq -8x + 36$
Collect like terms.	$- - 2x + 18 \leq -8x + 36$
Add $8x$ to both sides.	$-2x + 8x + 18 \leq -8x + 8x + 36$
Simplify.	$-6x + 18 \leq 36$
Subtract 18 from both sides.	$-6x + 18 - 18 \leq 36 - 18$
Simplify.	$6x \leq 18$
Divide both sides by 6.	$\frac{6x}{6} \leq \frac{18}{6}$
Simplify.	$x \leq 3$ Answer

b)

Original problem	$\frac{5x-1}{4} > -2(x + 5)$
Simplify parenthesis.	$\frac{5x-1}{4} > -2x - 10$
Multiply both sides by 4.	$4.\frac{5x-1}{4} > 4(-2x - 10)$
Simplify.	$5x - 1 > -8x - 40$
Add $8x$ to both sides.	$5x + 8x - 1 > -8x + 8x - 40$
Simplify.	$13x - 1 > -40$
Add 1 to both sides.	$13 - 1 + 1 > -40 + 1$
Simplify.	$13x > -39$
Divide both sides by 13.	$\frac{13x}{13} > -\frac{39}{13}$
Simplify.	$x > -3$ Answer

Identify the Number of Solutions of an Inequality

Inequalities can have:

- A set that has an infinite number of solutions.
- No solutions
- A set that has a discrete number of solutions.

Infinite Number of Solutions

The inequalities we have solved so far all have an infinite number of solutions. In the last example, we saw that the inequality

$\frac{5x-1}{4} > -2(x + 5)$ has the solution $x > -3$

This solution says that all real numbers greater than -3 make this inequality true. You can see that the solution to this problem is an infinite set of numbers.

No solutions

$$\begin{aligned} \text{Consider the inequality} \quad & x - 5(> x + 6 \\ \text{This simplifies to} \quad & -5 > 6 \end{aligned}$$

This statements is not true for any value of x. We say that this inequality has no solution.

Discrete solutions

So far we have assumed that the variables in our inequalities are real numbers. However, in many real life situations we are trying to solve for variables that represent integer quantities, such as number of people, number of cars or number of ties.

Example 4

Raul is buying ties and he wants to spend \$200 *or less on his purchase. The ties he likes the best cost* \$50. *How many ties could he purchase?*

Solution

Let $x =$ the number of ties Raul purchases.

We can write an inequality that describes the purchase amount using the formula.

(number of ties) $\times$ (price of a tie) $\leq \$200$ or $50x \leq 200$

We simplify our answer. $x \leq 4$

This solution says that Raul bought four or less ties. Since ties are discrete objects, the solution set consists of five numbers $\{0, 1, 2, 3, 4\}$.

Solve Real-World Problems Using Inequalities

Sometimes solving a word problem involves using an inequality.

Example 5

In order to get a bonus this month, Leon must sell at least 120 *newspaper subscriptions. He sold* 85 *subscriptions in the first three weeks of the month. How many subscriptions must Leon sell in the last week of the month?*

Solution

Step 1

We know that Leon sold 85 subscriptions and he must sell at least 120 subscriptions.

We want to know the least amount of subscriptions he must sell to get his bonus.

Let $x =$ the number of subscriptions Leon sells in the last week of the month.

Step 2

The number of subscriptions per month must be greater than 120.

We write

$$85 + x \geq 120$$

Step 3

We solve the inequality by subtracting 85 from both sides $x \geq 35$

Answer Leon must sell 35 or more subscriptions in the last week to get his bonus.

Step 4:

To check the answer, we see that $85 + 35 = 120$. If he sells 35 or more subscriptions the number of subscriptions sold that month will be 120 or more.

Example 6

Virena's Scout Troup is trying to raise at least \$650 *this spring. How many boxes of cookies must they sell at* \$4.50 *per box in order to reach their goal?*

Solution

Step 1

Virena is trying to raise at least $650

Each box of cookies sells for $4.50

Let $x =$ number of boxes sold

The inequality describing this problem is:

$$450x \geq 650.$$

Step 3

We solve the inequality by dividing both sides by 4.50

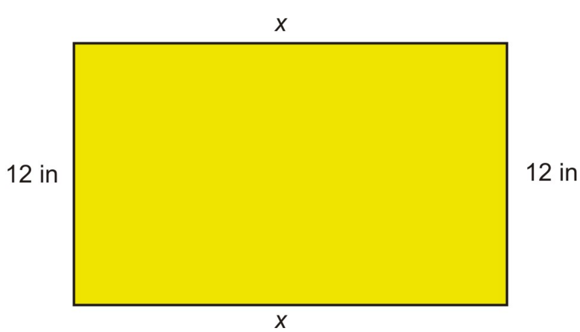
$$x \geq 1444.44$$

Answer We round up the answer to 145 since only whole boxes can be sold.

Step 4

If we multiply 145 by $4.50 we obtain $652.50. If Virena's Troop sells more than 145 boxes, they raise more that $650.

The answer checks out.



Example 7

The width of a rectangle is 20 inches . *What must the length be if the perimeter is at least* 180 inches?

Solution

Step 1

width = 20 inches

Perimeter is at least 180 inches

What is the smallest length that gives that perimeter?

Let $x =$ length of the rectangle

Step 2

Formula for perimeter is Perimeter $= 2 \times$ length $+ 2 \times$ width

Since the perimeter must be at least 180 inches , we have the following equation.

$$2x + 2(20) \geq 180$$

Step 3

We solve the inequality.

Simplify.

$$2x + 40 \geq 180$$

Subtract 40 from both sides.

$$2x \geq 140$$

Divide both sides by 2.

$$x \geq 70$$

Answer The length must be at least 70 inches .

Step 4

If the length is at least 70 inches and the width is 20 inches, then the perimeter can be found by using this equation.

$$2(70) + 2(20) = 180 \text{ inches}$$

The answer check out.

Lesson Summary

- The **general procedure** for solving multi-step inequalities is as follows.

1. Clear parentheses on both sides of the inequality and collect like terms.
2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.
3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.

- Inequalities can have **multiple solutions, no solutions**, or **discrete solutions**.

Review Questions

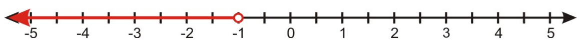
Solve the following inequalities and give the solution in set notation and show the solution graph.

1. $4x + 3 < -1$
2. $2x < 7x - 36$
3. $5x > 8x + 27$
4. $5 - x < 9 + x$
5. $4 - 6x \leq 2(2x + 3)$
6. $5(4x + 3) \geq 9(x - 2) - x$
7. $2(2x - 1) + 3 < 5(x + 3) - 2x$
8. $8x - 5(4x + 1) \geq -1 + 2(4x - 3)$
9. $2(7x - 2) - 3(x + 2) < 4x - (3x + 4)$

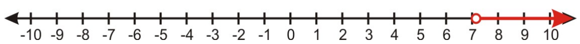
10. $\frac{2}{3}x - \frac{1}{2}(4x - 1) \geq x + 2(x - 3)$
11. At the San Diego Zoo, you can either pay \$22.75 for the entrance fee or \$71 for the yearly pass which entitles you to unlimited admission. At most how many times can you enter the zoo for the \$22.75 entrance fee before spending more than the cost of a yearly membership?
12. Proteek's scores for four tests were $82, 95, 86$ and 88. What will he have to score on his last test to average at least 90 for the term?

Review Answers

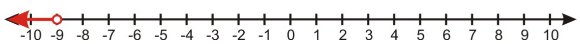
1. $\{ x \mid x \text{ is a real number}, x < -1 \}$



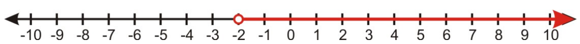
2. $\{ x \mid x \text{ is a real number}, x > \frac{36}{5} \}$



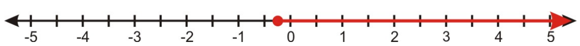
3. $\{ x \mid x \text{ is a real number}, x < -9 \}$



4. $\{ x \mid x \text{ is a real number}, x > -2 \}$

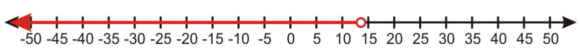


5. $\{ x \mid x \text{ is a real number}, x \geq -\frac{1}{5} \}$

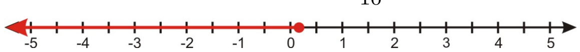


6. $\{ x \mid x \text{ is a real number}, x \geq -\frac{33}{12} \}$

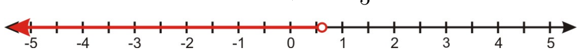
7. $\{ x \mid x \text{ is a real number}, x < 14 \}$



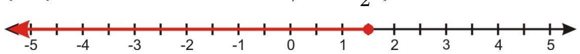
8. $\{ x \mid x \text{ is a real number}, x \leq \frac{1}{10} \}$



9. $\{ x \mid x \text{ is a real number}, x < \frac{3}{5} \}$



10. $\{ x \mid x \text{ is a real number}, x \leq \frac{3}{2} \}$



11. At most 3 times.
12. At least 99.

6.4 Compound Inequalities

Learning Objectives

- Write and graph compound inequalities on a number line.
- Solve a compound inequality with *"and"*.
- Solve a compound inequality with *"or"*.
- Solve compound inequalities using a graphing calculator (TI family).
- Solve real-world problems using compound inequalities

Introduction

In this section, we will solve compound inequalities. In previous sections, we obtained solutions that gave the variable either as greater than or as less than a number. In this section we are looking for solutions where the variable can be in two or more intervals on the number line.

There are two types of compound inequalities:

1. Inequalities joined by the word *"and"*.

The solution is a set of values greater than a number *and* less than another number.

$$a < x < b$$

In this case we want values of the variable for which *both* inequalities are true.

2. Inequalities joined by the word *"or"*.

The solution is a set of values greater than a number or less than another number.

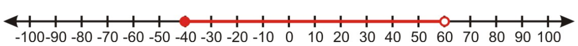
$x < a$ or $x > b$

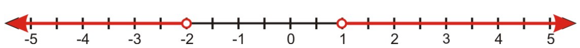
In this case, we want values for the variable in which *at least one* of the inequalities is true.

Write and Graph Compound Inequalities on a Number Line

Example 1

Write the inequalities represented by the following number line graphs.

a) 

b) 

c) 

d) 

Solution

a) The solution graph shows that the solution is any value between -40 and 60, including -40 but not 60. Any value in the solution set satisfies both inequalities.

$x \geq -40$ and $x < 60$

This is usually written as the following compound inequality.

$$-40 \leq x < 60$$

b) The solution graph shows that the solution is any value greater than 1 (not including 1) or any value less than -2 (not including -2). You can see that there can be no values that can satisfy both these conditions at the same time. We write:

$x > 1$ or $x < -2$

c) The solution graph shows that the solution is any value greater than 4 (including 4) or any value less than -1 (including -1). We write:

$x \geq 4$ or $x \leq -1$

d) The solution graph shows that the solution is any value less than 25 (not including 25) and any value greater than -25 (not including -25). Any value in the solution set satisfies both conditions.

$x > -25$ and $x < 25$

This is usually written as $-25 < x < 25$.

Example 2

Graph the following compound inequalities on the number line.

a) $-4 \leq x \leq 6$

b) $x < 0$ or $x > 2$

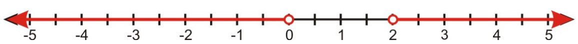
c) $x \geq -8$ or $x \leq -20$

d) $-15 < x \leq 85$

Solution

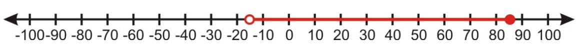
a) The solution is all numbers between -4 and 6 including both -4 and 6.

b) The solution is either numbers less than 0 or numbers greater than 2 not including 0 or 2.



c) The solution is either numbers greater than or equal to -8 or less than or equal to -20.

d) The solution is numbers between -15 and 85, not including -15 but including 85.



Solve a compound Inequality With *"and"*

When we solve compound inequalities, we separate the inequalities and solve each of them separately. Then, we combine the solutions at the end.

Example 3

Solve the following compound inequalities and graph the solution set.

a) $-2 < 4x - 5 \leq 11$

b) $3x - 5 < x + 9 \leq 5x + 13$

Solution

a) First, we rewrite the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$-2 < 4x - 5$		$4x - 5 \leq 11$
$3 < 4x$	and	$4x \leq 16$
$\frac{3}{4} < x$		$x \leq 4$

Answer $\frac{3}{4} < x$ and $x \leq 4$. This can be written as $\frac{3}{4}x \leq 4$.

b) Rewrite the compound inequality as two separate inequalities by using *and.* Then solve each inequality separately.

$$
\begin{aligned}
3x - 5 &< x + 9 \\
2x &< 14 \\
x &< 7
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
x + 9 &\le 5x + 13 \\
-4 &\le 4x \\
-1 &\le x \text{ or } x \ge -1
\end{aligned}
$$

Answer $x < 7 >$ and $x \ge -1$. This can be written as $-1 \le x < 7$.

Solve a Compound Inequality With "or"

Consider the following example.

Example 4

Solve the following compound inequalities and graph the solution set.

a) $9 - 2x \le 3$ or $3x + 10 \le 6 - x$

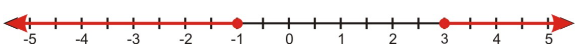
b) $\frac{x-2}{6} \le 2x - 4$ or $\frac{x-2}{6} > x + 5$

Solution

a) Solve each inequality separately.

$$
\begin{aligned}
9 - 2x &\le 3 \\
-2x &\le -6 \\
x &\ge 3
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
3x + 10 &\le 6 - x \\
4x &\le -4 \\
x &\le -1
\end{aligned}
$$

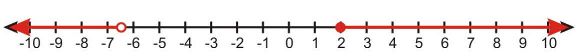
Answer $x \ge 3$ or $x \le -1$



b) Solve each inequality separately.

$$
\begin{aligned}
\frac{x-2}{6} &\le 2x - 4 \\
x - 2 &\le 6(2x - 4) \\
x - 2 &\le 12x - 24 \\
22 &\le 11x \\
2 &\le x
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
\frac{x-2}{6} &> x + 5 \\
x - 2 &> 6(x + 5) \\
x - 2 &> 6x + 30 \\
-32 &> 5x \\
-6.4 &> x
\end{aligned}
$$

Answer $x \ge 2$ or $x < -64$



Solve Compound Inequalities Using a Graphing Calculator (TI-83/84 family)

This section explains how to solve simple and compound inequalities with a graphing calculator.

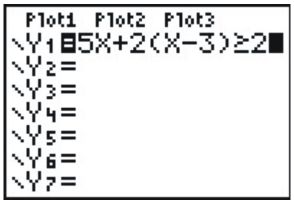
Example 5

Solve the following inequalities using the graphing calculator.

a) $5x + 2(x - 3) \geq 2$

b) $7x - 2 < 10x + 1 < 9x + 5$

c) $3x + 2 \leq 10$ or $3x + 2 \geq 15$



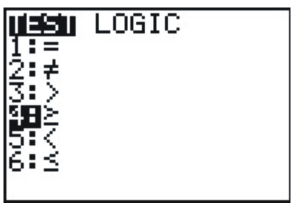
Solution

a) $5x +_{2} (x - 3) \geq 2$

Step 1 **Enter the inequality.**

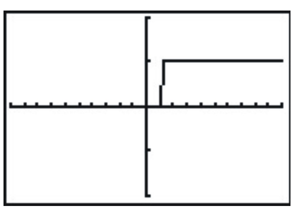
Press the [**Y=**] button.

Enter the inequality on the first line of the screen.



$$Y_1 = 5x + 2(x - 3) \geq 2$$

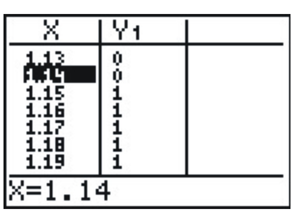
The $\geq$ symbol is entered by pressing [**TEST**] [**2nd**] [**MATH**] and choose option 4.



Step 2 **Read the solution.**

Press the [**GRAPH**] button.

Because the calculator translates a true statement with the number 1 and a false statement with the number 0, you will see a step function with the y–value jumping from 0 to 1. The solution set is the values of x for which the graph shows $y = 1$.



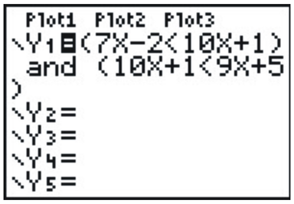
Note: You need to press the [**WINDOW**] key or the [**ZOOM**] key to adjust window to see full graph.

The solution is $x \geq \frac{8}{7} = 1.42857\ldots$, which is why you can see the y value changing from 0 to 1 at 1.14.

b) $7x - 2 < 10x + 1 < 9x + 5$

This is a compound inequality $7x - 2 < 10x + 1$ and $10x + 1 < 9x + 5$.

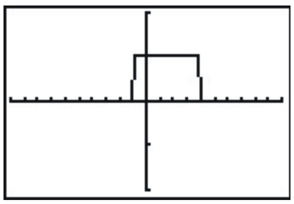
To enter a compound inequality:



Press the [**Y=**] button.

Enter the inequality as $Y_1 = (7x - 2 < 10x + 1)$ *AND* $(10x + 1 < 9x + 5$

To enter the [**AND**] symbol press [**TEST**], choose [**LOGIC**] on the top row and choose option 1.



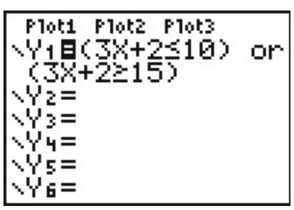
The resulting graph looks as shown at the right.

The solution are the values of x for which $y = 1$.

In this case $-1 < x < 4$.

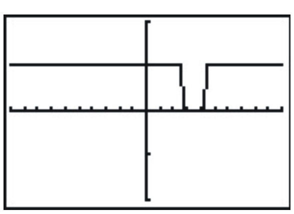
c) $3x + 2 \leq 10$ or $3x + 2 \geq 15$

This is a compound inequality $3x + 2 \leq 10$ or $3x + 2 \geq 15$



Press the [**Y=**] button.

Enter the inequality as $Y_1 = (3x + 2 \leq 10)$ *OR* $(3x + 2 \geq 15)$ To enter the [**OR**] symbol press [**TEST**], choose [**LOGIC**] on the top row and choose option 2.



The resulting graph looks as shown at the right. The solution are the values of x for which $y = 1$. In this case, $x \leq 2.7$ or $x \geq 4.3$.

Solve Real-World Problems Using Compound Inequalities

Many application problems require the use of compound inequalities to find the solution.

Example 6

The speed of a golf ball in the air is given by the formula $v = -32t + 80$, *where* t is the time since the ball was hit. When is the ball traveling between 20 ft/sec and 30 ft/sec*?*

Solution

Step 1

We want to find the times when the ball is traveling between 20 ft/sec and 30 ft/sec.

Step 2

Set up the inequality $20 \le v \le 30$

Step 3

Replace the velocity with the formula $v = -32t + 80$.

$$20 \le -32t + 80 \le 30$$

Separate the compound inequality and solve each separate inequality.

$$\begin{aligned} 20 &\le -32t + 80 \\ 32t &\le 60 \\ t &\le 1.875 \end{aligned} \qquad \text{and} \qquad \begin{aligned} -32t + 80 &\le 30 \\ 50 &\le 32t \\ 1.56 &\le t \end{aligned}$$

Answer $1.56 \le t \le 1.875$

Step 4 To check plug in the minimum and maximum values of t into the formula for the speed.

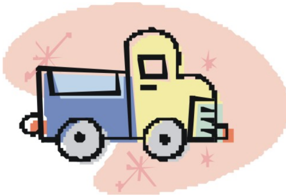
For $t = 1.56$, $v = -32t + 80 = -32(1.56) + 80 = 30$ ft/sec

For $t = 1.875$, $v = -32t + 80 = -32(1.875) + 80 = 20$ ft/sec

So the speed is between 20 and 30 ft/sec . The answer checks out.

Example 7

William's pick-up truck gets between 18 *to* 22 *miles per gallon of gasoline. His gas tank can hold* 15*gallons of gasoline. If he drives at an average speed of* 40 miles *per hour how much driving time does he get on a full tank of gas?*



Solution

Step 1 We know

The truck gets between 18 and 22 miles/gallon

There are 15 gallons in the truck's gas tank

William drives at an average of 40 miles/hour

Let $t =$ driving time

Step 2 We use dimensional analysis to get from time per tank to miles per gallon.

$$\frac{t\ \cancel{\text{hours}}}{1\ \cancel{\text{tank}}} \times \frac{1\ \cancel{\text{tank}}}{15\ \text{gallons}} \times \frac{40\ \text{miles}}{1\ \text{hours}} = \frac{40t}{45}\ \frac{\text{miles}}{\text{gallon}}$$

Step 3 Since the truck gets between 18 to 22 miles/gallon , we set up the compound inequality.

$$18 \leq \frac{40t}{15} \leq 22$$

Separate the compound inequality and solve each inequality separately.

$$\begin{aligned} 18 &\leq \frac{40t}{15} & & & \frac{40t}{15} &\leq 22 \\ 270 &\leq 40t & &\text{and} & 40t &\leq 330 \\ 6.75 &\leq t & & & t &\leq 8.25 \end{aligned}$$

Answer $6.75 \leq t \leq 8.25$. Andrew can drive between 6.75 and 8.25 hours on a full tank of gas.

Step 4

For $t = 6.75$, we get $\frac{40t}{15} \frac{40(6.75)}{15} = 18$ miles per gallon.

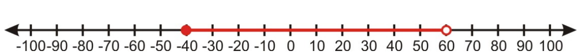
For $t = 8.25$, we get $\frac{40t}{15} \frac{40(8.25)}{15} = 18$ miles per gallon.

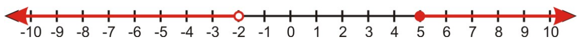
Lesson Summary

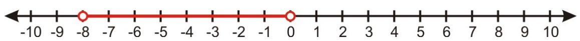
- **Compound inequalities** combine two or more inequalities with **"and"** or **"or"**.
- **"And"** combinations mean the only solutions for both inequalities will be solutions to the compound inequality.
- **"Or"** combinations mean solutions to either inequality will be solutions to the compound inequality.

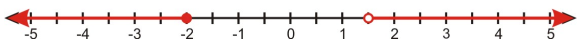
Review Questions

Write the compound inequalities represented by the following graphs.

1. 

2. 

3. 

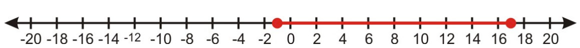
4. 

Solve the following compound inequalities and graph the solution on a number line.

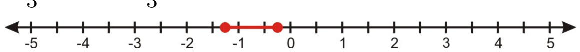
5. $-5 \leq x - 4 \leq 13$
6. $1 \leq 3x + 4 \leq 4$
7. $-12 \leq 2 - 5x \leq 7$
8. $\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
9. $-2\frac{2x-1}{3} < -1$
10. $4x - 1 \geq 7$ or $\frac{9x}{2} < 3$
11. $3 - x < -4$ or $3 - x > 10$
12. $\frac{2x+3}{4} < 2$ or $-\frac{x}{5} + 3\frac{2}{5}$
13. $2x - 7 \leq -3$ or $2x - 3 > 11$
14. $4x + 3 \leq 9$ or $-5x + 4 \leq -12$
15. To get a grade of B in her Algebra class, Stacey must have an average grade greater than or equal to 80 and less than 90. She received the grades of $92, 78, 85$ on her first three tests. Between which scores must her grade fall if she is to receive a grade of B for the class?

Review Answers

1. $-40 \leq x \leq 70$
2. $x < -2$ or $x \geq 5$
3. $-8 < x < 0$
4. $x \leq -2$ or $x > 1.5$
5. $-1 \leq x \leq 17$

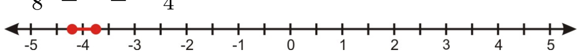


6. $-\frac{4}{3} \leq x \leq -\frac{1}{3}$

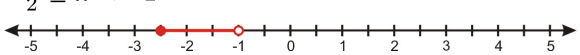


7. $-1 \leq x \leq \frac{14}{5}$

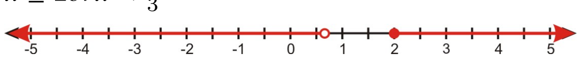
8. $-\frac{33}{8} \leq x \leq -\frac{15}{4}$



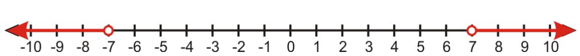
9. $-\frac{5}{2} \leq x < -1$



10. $x \geq 2 or x < \frac{2}{3}$

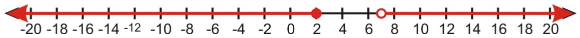


11. $x > 7$ or $x < -7$

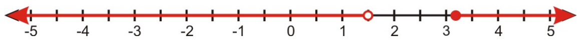


12. $x < \frac{5}{2}$ or $x > 13$

13. $x \leq 2$ or $x > 7$



14. $x < \frac{3}{2}$ or $x \geq \frac{16}{5}$



15. $65 \leq x < 105$

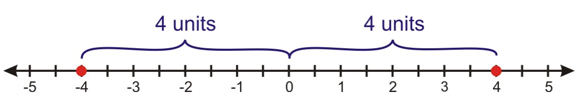
6.5 Absolute Value Equations

Learning Objectives

- Solve an absolute value equation.
- Analyze solutions to absolute value equations.
- Graph absolute value functions.
- Solve real-world problems using absolute value equations.

Introduction

The **absolute value** of a number is its distance from zero on a number line. There are always two numbers on the number line that are the same distance from zero. For instance, the numbers 4 and -4 are both a distance of 4 units away from zero.



$|\ 4\ |$ represents the distance from 4 to zero which equals 4.

$|\ -4\ |$ represents the distance from -4 to zero which also equals 4.

In fact, for any real number x,

$|\ x\ |= x$ if x is not negative (that is, including $x = 0$.)

$|\ x\ |= -x$ if x is negative.

Absolute value has no effect on a positive number but changes a negative number into its positive inverse.

Example 1

Evaluate the following absolute values.

a) $|\ 25\ |$

b) $|\ -120\ |$

c) $|\ -3\ |$

d) $|\ 55\ |$

e) $|\ -\frac{5}{4}\ |$

Solution:

a) $|\ 25\ |= 25$ Since 25 is a positive number the absolute value does not change it.

b) $|\ -120\ |= 120$ Since -120 is a negative number the absolute value makes it positive.

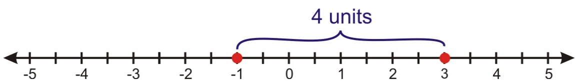
c) $|\ -3\ |= 3$ Since -3 is a negative number the absolute value makes it positive.

d) $| 55 |= 55$ Since 55 is a positive number the absolute value does not change it.

e) $| -\frac{5}{4} |= \frac{5}{4}$ Since is a negative number the absolute value makes it positive.

Absolute value is very useful in finding the distance between two points on the number line. The distance between any two points a and b on the number line is $| a - b | \;\; or \;\; | b - a |$.

For example, the distance from 3 to -1 on the number line is $| 3 - (-1) |=| 4 |= 4$.



We could have also found the distance by subtracting in the reverse order, $| -1 - 3 |=| -4 |= 4$.

This makes sense because the distance is the same whether you are going from 3 to -1 or from -1 to 3.

Example 2

Find the distance between the following points on the number line.

a) 6 and 15

b) -5 and 8

c) -3 and -12

Solutions

Distance is the absolute value of the difference between the two points.

a) Distance $= | 6 - 15 |=| -9 |= 9$

b) Distance $= | -5 - 8 |=| -13 |= 13$

c) Distance $= | -3 - (-12) |=| 9 |= 9$

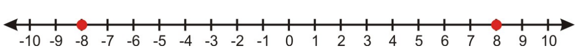
Remember: When we computed the change in x and the change in y as part of the slope computation, these values were positive or negative, depending on the direction of movement. In this discussion, "distance" means a positive distance only.

Solve an Absolute Value Equation

We now want to solve equations involving absolute values. Consider the following equation.

$$| x |= 8$$

This means that the distance from the number x to zero is 8. There are two possible numbers that satisfy this condition 8 and -8.



When we solve absolute value equations we always consider4 two possibilities.

1. The expression inside the absolute value sign is not negative.

2. The expression inside the absolute value sign is negative.

Then we solve each equation separately.

Example 3

Solve the following absolute value equations.

a) $| 3 |= 3$

b) $|10| = 10$

Solution

a) There are two possibilities $x = 3$ and $x = -3$.

b) There are two possibilities $x = 10$ and $x = -10$.

Analyze Solutions to Absolute Value Equations

Example 4

Solve the equation and interpret the answers.

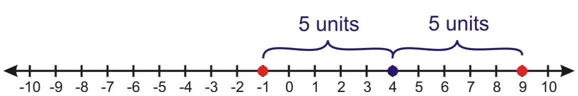
Solution

We consider two possibilities. The expression inside the absolute value sign is not negative or is negative. Then we solve each equation separately.

$$\begin{aligned} x - 4 &= 5 \\ x &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - 4 &= -5 \\ x &= -1 \end{aligned}$$

Answer $x = 9$ and $x = -1$.

Equation $|x - 4| = 5$ can be interpreted as "what numbers on the number line are 5 units away from the number 4?" If we draw the number line we see that there are two possibilities 9 and -1.



Example 5

Solve the equation $|x + 3| = 2$ *and interpret the answers.*

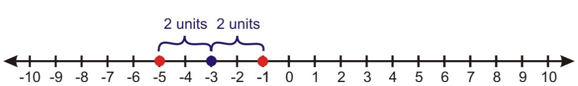
Solution

Solve the two equations.

$$\begin{aligned} x + 3 &= 2 \\ x &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 3 &= -2 \\ x &= 5 \end{aligned}$$

Answer $x = -5$ and $x = -1$.

Equation $| x + 3 = 2$ can be re-written as $|x - (-3)| = 2$. We can interpret this as "what numbers on the number line are 2 units away from -3?" There are two possibilities -5 and -1.



Example 6

Solve the equation $|2x - 7| = 6$ *and interpret the answers.*

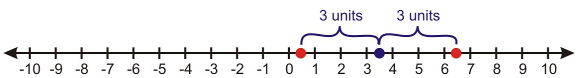
Solution

Solve the two equations.

$$2x-7=-6 \qquad\qquad 2x-7=6$$
$$2x=13 \qquad \text{and} \qquad 2x=1$$
$$x=\frac{13}{2} \qquad\qquad x=\frac{1}{2}$$

Answer $x=\frac{13}{2}$ and $x=\frac{1}{2}$

The interpretation of this problem is clearer if the equation $|2x-7|=6$ was divided by 2 on both sides. We obtain $|x-\frac{7}{2}|=3$. The question is "What numbers on the number line are 3 units away from $\frac{7}{2}$?" There are two possibilities $\frac{13}{2}$ and $\frac{1}{2}$.



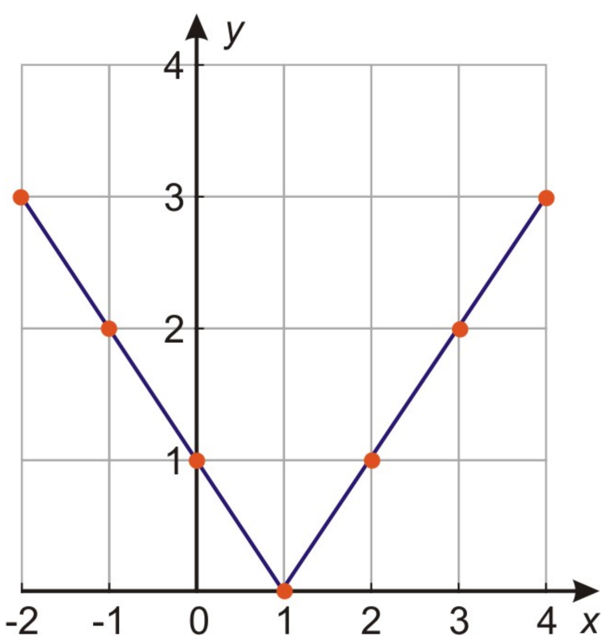
Graph Absolute Value Functions

You will now learn how to graph absolute value functions. Consider the function:

$$y=|x-1|$$

Let's graph this function by making a table of values.

x	$y=\|x-1\|$
-2	$y=\|-2-1\|=\|-3\|=3$
-1	$y=\|-1-1\|=\|-2\|=2$
0	$y=\|0-1\|=\|-1\|=1$
1	$y=\|1-1\|=\|0\|=0$
2	$y=\|2-1\|=\|1\|=1$
3	$y=\|3-1\|=\|2\|=2$
4	$y=\|4-1\|=\|3\|=3$



You can see that the graph of an absolute value function makes a big "V". It consists of two line rays (or line segments), one with positive slope and one with negative slope joined at the **vertex** or **cusp**.

We saw in previous sections that to solve an absolute value equation we need to consider two options.

1. The expression inside the absolute value is not negative.

2. The expression inside the absolute value is negative.

The graph of $y = |x - 1|$ is a combination of two graphs.

Option 1

$$y = x - 1$$

when $x - 1 \geq 0$

Option 2

$y = -(x - 1)$ or $y = -x + 1$

when $x - 1 < 0$

These are both graphs of straight lines.

The two straight lines meet at the vertex. We find the vertex by setting the expression inside the absolute value equal to zero.

$x - 1 = 0$ or $x = 1$

We can always graph an absolute value function using a table of values. However, we usually use a simpler procedure.

Step 1 Find the vertex of the graph by setting the expression inside the absolute value equal to zero and solve for x.

Step 2 Make a table of values that includes the vertex, a value smaller than the vertex and a value larger than the vertex. Calculate the values of y using the equation of the function.

Step 3 Plot the points and connect with two straight lines that meet at the vertex.

Example 7

Graph the absolute value function: $y = |x + 5|$.

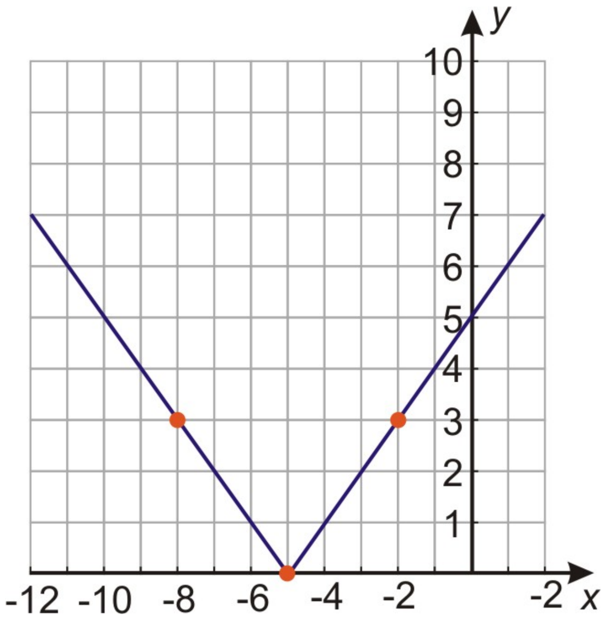
Solution

Step 1 Find the vertex $x + 5 = 0$ or $x = -5$ vertex.

Step 2 Make a table of values.

x	$y = \lvert x + 5 \rvert$
-8	$y = \lvert -8 + 5 \rvert = \lvert -3 \rvert = 3$
-5	$y = \lvert -5 + 5 \rvert = \lvert 0 \rvert = 0$
-2	$y = \lvert -2 + 5 \rvert = \lvert 3 \rvert = 3$

Step 3 Plot the points and draw two straight lines that meet at the vertex.



Example 8

Graph the absolute value function $y = |3x - 12|$.

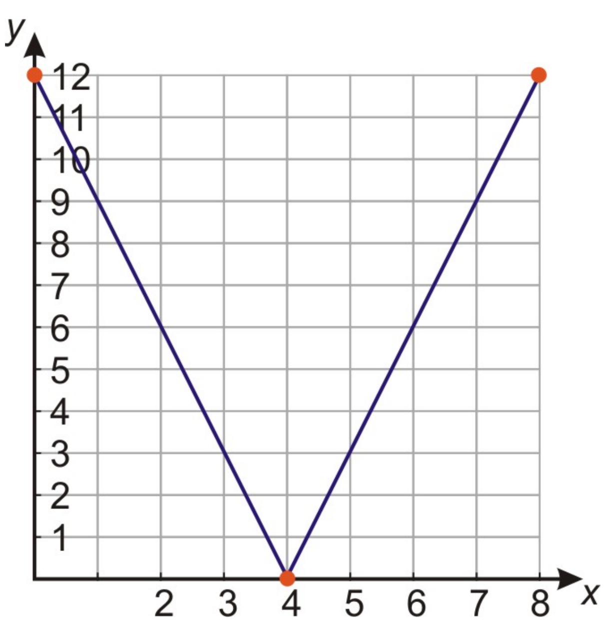
Solution

Step 1 Find the vertex $3x - 12 = 0$ so $x = 4$ is the vertex.

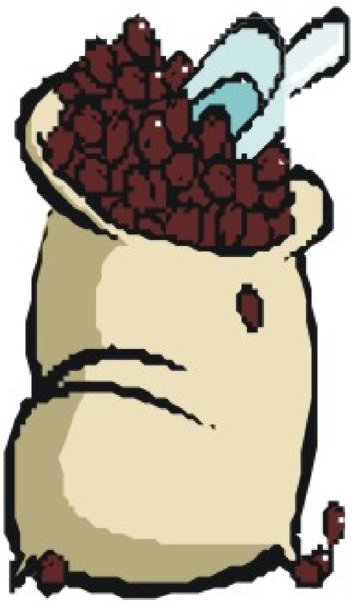
Step 2 Make a table of values:

x	$y = \mid 3x - 12 \mid$
0	$y = \mid 3(0) - 12 \mid = \mid -12 \mid = 12$
4	$y = \mid 3(4) - 12 \mid = \mid 0 \mid = 0$
8	$y = \mid 3(8) - 12 \mid = \mid 12 \mid = 12$

Step 3 Plot the points and draw two straight lines that meet at the vertex.



Solve Real-World Problems Using Absolute Value Equations



Example 9

A company packs coffee beans in airtight bags. Each bag should weigh 16 *ounces but it is hard to fill each bag to the exact weight. After being filled, each bag is weighed and if it is more than* 0.25 *ounces overweight or underweight it is emptied and repacked. What are the lightest and heaviest acceptable bags?*

Solution

Step 1

We know that each bag should weigh 16 ounces.

A bag can weigh 0.25 ounces more or less than 16 ounces.

We need to find the lightest and heaviest bags that are acceptable.

Let x = weight of the coffee bag in ounces.

Step 2

The equation that describes this problem is written as $|\, x - 16 \,|\ 0.25$.

Step 3

Consider the positive and negative options and solve each equation separately.

$$x - 16 = 0.25 \qquad \text{and} \qquad x - 16 = -0.25$$

$$x = 16.25 \qquad\qquad\qquad x = 15.75$$

Answer The lightest acceptable bag weighs 15.75 ounces and the heaviest weighs 16.25 ounces.

Step 4

We see that $16.25 - 16 = 0.25$ ounces and $16 - 15.75 = 0.25$ ounces. The answers are 0.25 ounces bigger and smaller than 16 ounces respectively.

The answer checks out.

Lesson Summary

- The absolute value of a number is its distance from zero on a number line.

$|x| = x$ if x is not negative.

$|x| = -x$ if x is negative.

- An equation with an absolute value in it **splits into two equations.**

1. The expression within the absolute value is **positive,** then the absolute value signs do nothing and can be omitted.

2. The expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.

Review Questions

Evaluate the absolute values.

1. $|250|$
2. $|-12|$
3. $|-\frac{2}{5}|$
4. $|\frac{1}{10}|$

Find the distance between the points.

5. 12 and -11
6. 5 and 22
7. -9 and -18
8. -2 and 3

Solve the absolute value equations and interpret the results by graphing the solutions on the number line.

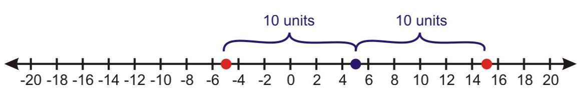
9. $|x-5| = 10$
10. $|x+2| = 6$
11. $|5x-2| = 3$
12. $|4x-1| = 19$

Graph the absolute value functions.

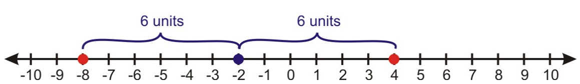
13. $y = |x+3|$
14. $y = |x-6|$
15. $y = |4x+2|$
16. $y = |\frac{x}{3} - 4|$
17. A company manufactures rulers. Their $12-$ inch rulers pass quality control if they within $\frac{1}{32}$ inches of the ideal length. What is the longest and shortest ruler that can leave the factory?

Review Answers

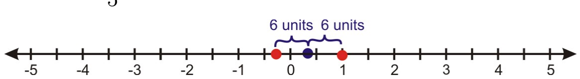
1. 250
2. 12
3. $\frac{2}{5}$
4. $\frac{1}{10}$
5. 23
6. 17
7. 9
8. 5
9. 15 and -5



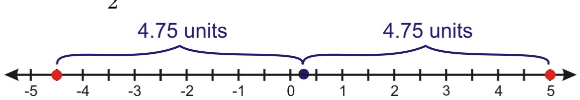
10. 4 and -8



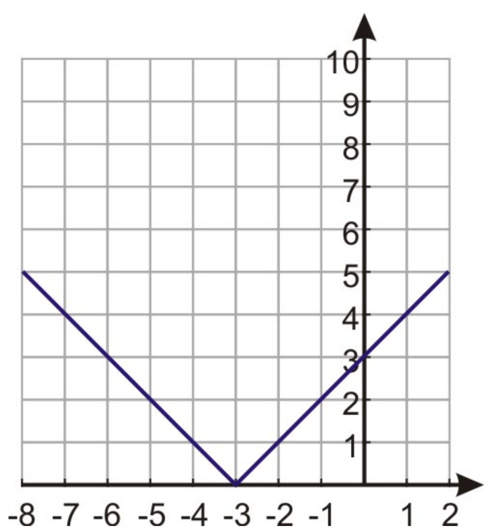
11. 1 and $-\frac{1}{5}$



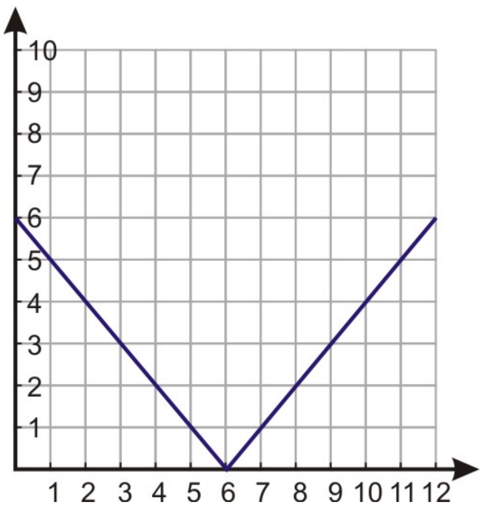
12. 5 and $-\frac{9}{2}$



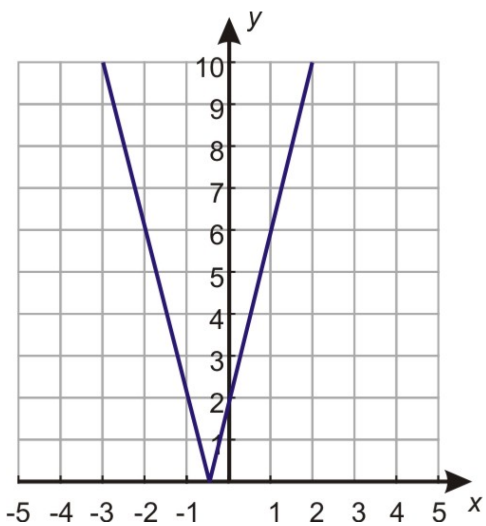
13.



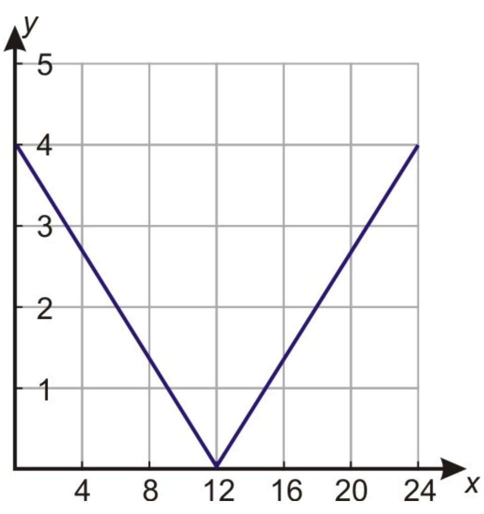
14.



15.



16.



17. $11\frac{31}{32}$ and $12\frac{1}{32}$

6.6 Absolute Value Inequalities

Learning Objectives

- Solve absolute value inequalities.
- Rewrite and solve absolute value inequalities as compound inequalities.
- Solve real-world problems using absolute value inequalities.

Introduction

Absolute value inequalities are solved in a similar way to absolute value equations. In both cases, you must consider the two options.

1. The expression inside the absolute value is not negative.

2. The expression inside the absolute value is negative.

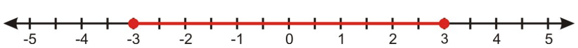
Then we solve each inequality separately.

Solve Absolute Value Inequalities

Consider the inequality

$$|x| \leq 3$$

Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero is less than or equal to 3. The following graph shows this solution:

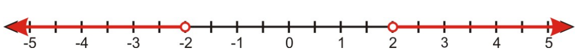


Notice that this is also the graph for the compound inequality $-3 \leq x \leq 3$.

Now consider the inequality

$$|x| > 2$$

Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero are more than 2. The following graph shows this solution.



Notice that this is also the graph for the compound inequality $x < -2$ or $x > 2$.

Example 1

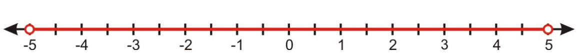
Solve the following inequalities and show the solution graph.

a) $|x| < 6$

b) $|x| \geq 2.5$

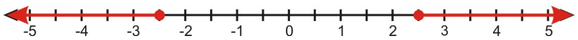
Solution

a) $|x| < 5$ represents all numbers whose distance from zero is less than 5.



Answer $-5 < x < 5$

b)$|\ x\ |\geq 2.5$ represents all numbers whose distance from zero is more than or equal to 2.5.



Answer $x \leq -2.5$ or $x \geq 2.5$

Rewrite and Solve Absolute Value Inequalities as Compound Inequalities

In the last section you saw that absolute value inequalities are compound inequalities.

Inequalities of the type $|\ x\ |< a$ can be rewritten as $-a < x < a$

Inequalities of the type $|\ x\ |< b$ can be rewritten as $x < -b$ or $x > b$

To solve an absolute value inequality, we separate the expression into two inequalities and solve each of them individually.

Example 2

Solve the inequality $|\ x - 3\ |< 7$ *and show the solution graph.*

Solution

Rewrite as a compound inequality.

Write as two separate inequalities.

$x - 3 < 7$ and $x - 3 < 7$

Solve each inequality

$x < 10$ and $x > -4$

The solution graph is 

Example 3

Solve the inequality $|\ 4x + 6\ |\leq 13$ *and show the solution graph.*

Solution

Rewrite as a compound inequality.

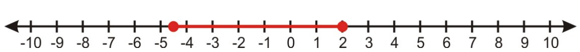
Write as two separate inequalities

$4x + 5 \leq 13$ and $4x + 5 \geq -13$

Solve each inequality:

$4x \leq 8$ and $4x \geq -18$

$x \leq 2$ and $x \geq -\frac{9}{2}$

The solution graph is 

Example 4

Solve the inequality $|\ x + 12\ |> 2$ *and show the solution graph.*

Solution

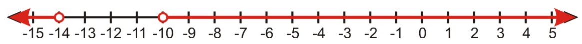
Rewrite as a compound inequality.

Write as two separate inequalities.

$x + 12 < -2$ or $x + 12 > 2$

Solve each inequality

$x < -14$ or $x > -10$

The solution graph is 

Example 5

Solve the inequality $| 8x - 15 | \geq 9$ *and show the solution graph.*

Rewrite as a compound inequality.

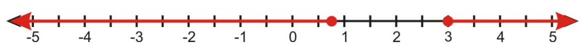
Write as two separate inequalities.

$8x - 15 \leq -9$ or $8x - 15 \geq 9$

Solve each inequality

$8x \leq 6$ or $8x \geq 24$

$x \leq \frac{3}{4}$ or $x \geq 3$

The solution graph is 

Solve Real-World Problems Using Absolute Value Inequalities

Absolute value inequalities are useful in problems where we are dealing with a range of values.

Example 6:

The velocity of an object is given by the formula $v = 25t - 80$ *where the time is expressed in seconds and the velocity is expressed in feet per seconds. Find the times when the magnitude of the velocity is greater than or equal to* 60 feet *per second.*

Solution

Step 1

We want to find the times when the velocity is greater than or equal to 60 feet per second

Step 2

We are given the formula for the velocity $v = 25t - 80$

Write the absolute value inequality $| 25t - 80 | \geq 60$

Step 3

Solve the inequality

$25t - 80 \geq 60$ or $25t - 80 \leq -60$

$25t \geq 140$ or $25t \leq 20$

$t \geq 5.6$ or $t \leq 0.8$

Answer: The magnitude of the velocity is greater than 60 ft/sec for times less than 0.8 seconds and for times greater than 5.6 seconds.

Step 4 When $t = 0.8$ seconds , $v = 25(0.8) - 80 = -60$ ft/sec. The magnitude of the velocity is 60 ft/sec. The negative sign in the answer means that the object is moving backwards.

When $t = 5.6$ seconds , $v = 25(0.8) - 80 = -60$ ft/sec.

To find where the magnitude of the velocity is greater than 60 ft/sec, check values in each of the following time intervals: $t \leq 0.8, 0.8 \leq t \leq 5.6$ and $t \geq 5.6$.

Check $t = 0.5$: $v = 25(0.5) - 80 = -67.5$ ft/sec

Check $t = 2$: $v = 25(2) - 80 = -30$ ft/sec

Check $t = 6$: $v = 25(6) - 80 = 70$ ft/sec

You can see that the magnitude of the velocity is greater than 60 ft/sec for $t \geq 5.6$ or $t \leq 0.8$.

The answer checks out.

Lesson Summary

- Like absolute value equations, inequalities with absolute value split into two inequalities. One where the expression within the absolute value is negative and one where it is positive.
- Inequalities of the type $|\ x\ | < a$ can be rewritten as $-a < x < a$.
- Inequalities of the type $|\ x\ | > b$ can be rewritten as $-x < -b$ or $x > b$.

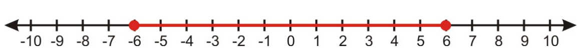
Review Questions

Solve the following inequalities and show the solution graph.

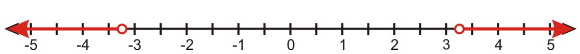
1. $|x| \leq 6$
2. $|x| > 3.5$
3. $|x| < 12$
4. $|\frac{x}{5}| \leq 6$
5. $|7x| \geq 21$
6. $|x - 5| > 8$
7. $|x + 7| < 3$
8. $|x - \frac{3}{4}| \leq \frac{1}{2}$
9. $|2x - 5| \geq 13$
10. $|5x + 3| < 7$
11. $|\frac{x}{3} - 4| \leq 2$
12. $|\frac{2x}{7} + 9| > \frac{5}{7}$
13. A three month old baby boy weighs an average of 13 p*ounds* . He is considered healthy if he is 2.5 l*bs* more or less than the average weight. Find the weight range that is considered healthy for three month old boys.

Review Answers

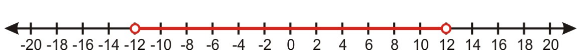
1. $-6 \leq x \leq 6$



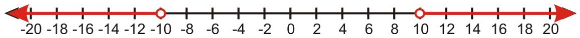
2. $x < -3.5$ or $x > 3.5$



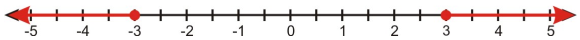
3. $-12 < x < 12$



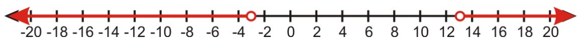
4. $x < -10$ or $x > 10$



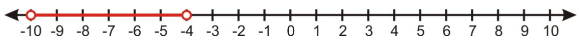
5. $x \leq -3$ or $x \geq 3$



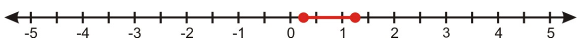
6. $x < -3$ or $x > 13$



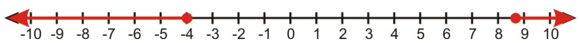
7. $-10 < x < -4$



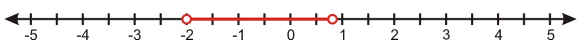
8. $\frac{1}{4} \leq x \leq \frac{5}{4}$



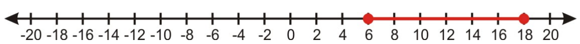
9. $x \leq -4$ or $x \geq 9$



10. $-2 < x < \frac{4}{5}$



11. $6 \leq x \leq 18$



12. $x < -34$ or $x > -29$

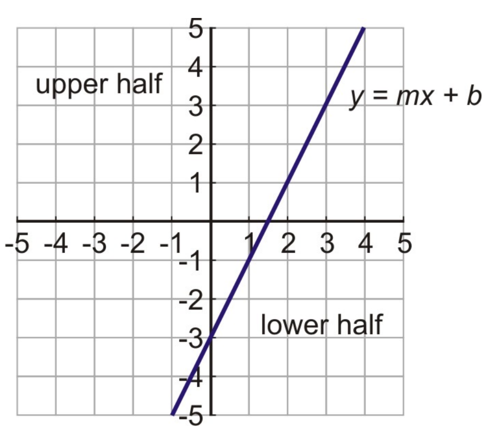
13. A healthy weight is $10.5 \text{ lb} \leq x \leq 15.5 \text{ lb}$.

6.7 Linear Inequalities in Two Variables

Learning Objectives

- Graph linear inequalities in one variable on the coordinate plane.
- Graph linear inequalities in two variables.
- Solve real-world problems using linear inequalities

Introduction



A linear inequality in two variables takes the form

$y > mx + b$ or $y < mx + b$

Linear inequalities are closely related to graphs of straight lines. A straight line has the equation $y = mx+b$. When we graph a line in the coordinate plane, we can see that it divides the plane in two halves.

The solution to a linear inequality includes all the points in one of the plane halves. We can tell which half of the plane the solution is by looking at the inequality sign.

$>$ The solution is the half plane above the line.

$\geq$ The solution is the half plane above the line and also all the points on the line.

$<$ The solution is the half plane below the line.

$\leq$ The solution is the half plane below the line and also all the points on the line.

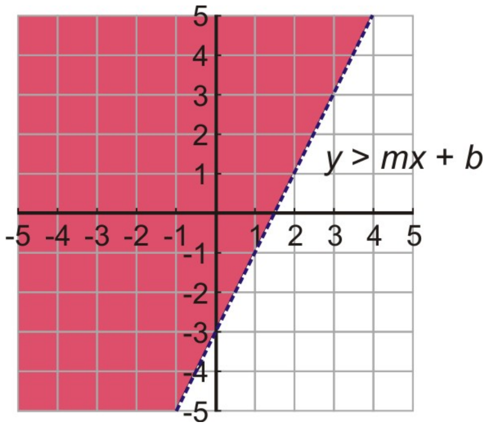
(Above the line means for a given $x-$coordinate, all points with $y-$values greater than the $y-$value are on the line)

For a strict inequality, we draw a **dashed line** to show that the points on the line are not part of the solution.

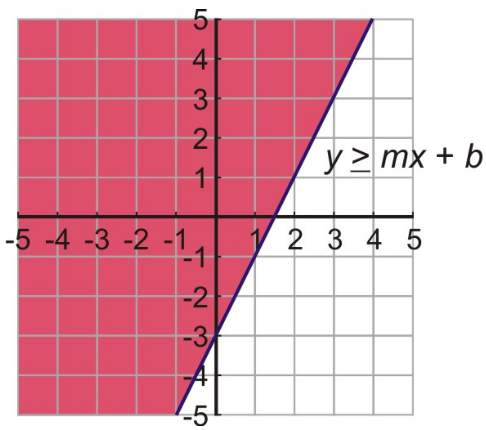
For an inequality that includes the equal sign, we draw a **solid line** to show that the points on the line are part of the solution.

Here is what you should expect linear inequality graphs to look like.

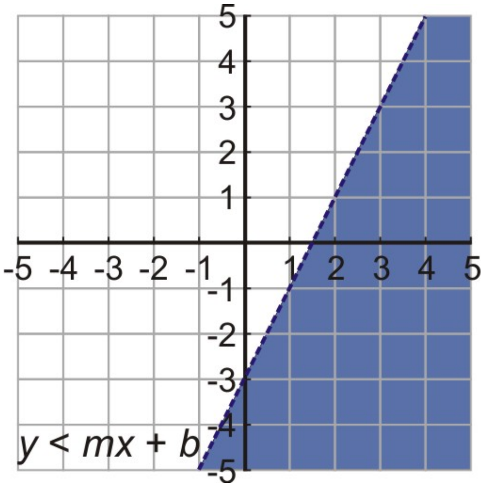
The solution of $y > mx + b$ is the half plane above the line. The dashed line shows that the points on the line are not part of the solution.



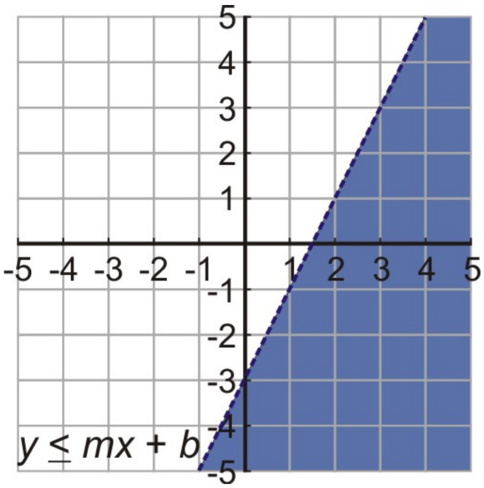
The solution of $y \geq mx + b$ is the half plane above the line and all the points on the line.



The solution of $y < mx + b$ is the half plane below the line.



The solution of $y \leq mx + b$ is the half plane below the line and all the points on the line.



Graph Linear Inequalities in One Variable in the Coordinate Plane

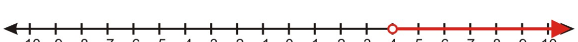
In the last few sections, we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line and when we graph an equation of the type $y = b$ we get a horizontal line.

Example 1

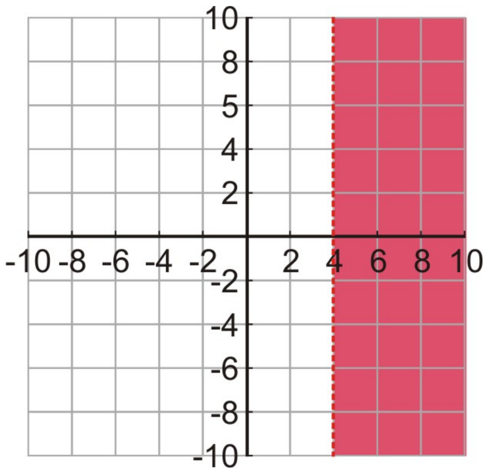
Graph the inequality $x > 4$ *on the coordinate plane.*

Solution

First, let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than four but not including four. The solution is represented by a line.



In two dimensions we are also concerned with values of y, and the solution to $x > 4$ consists of all coordinate points for which the value of x is bigger than four. The solution is represented by the half plane to the right of $x = 4$.

The line $x = 4$ is dashed because the equal sign is not included in the inequality and therefore points on the line are not included in the solution.

Example 2

Graph the inequality $y \leq 6$ *on the coordinate plane.*

Solution

The solution is all coordinate points for which the value of y is less than or equal than 6. This solution is represented by the half plane below the line $y = 6$.

The line $y = 6$ is solid because the equal sign is included in the inequality sign and the points on the line are included in the solution.

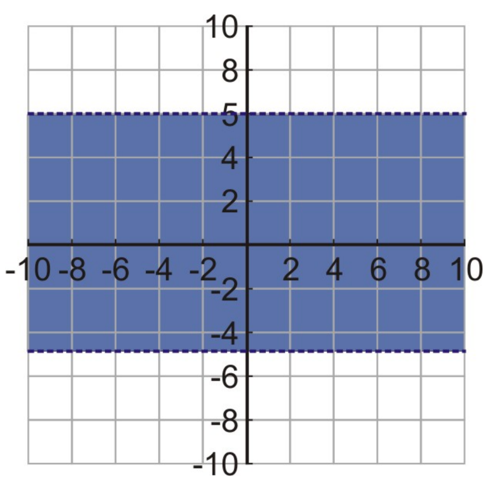
Example 3

Graph the inequality $| 6 |< 5$

Solution

The absolute value inequality $| 6 |< 5$ can be re-written as $-5 < y < 5$. This is a compound inequality which means

$y > -5$ and $y < 5$



In other words, the solution is all the coordinate points for which the value of y is larger than -5 **and** smaller than 5. The solution is represented by the plane between the horizontal lines $y = -5$ and $y = 5$.

Both horizontal lines are dashed because points on the line are not included in the solution.

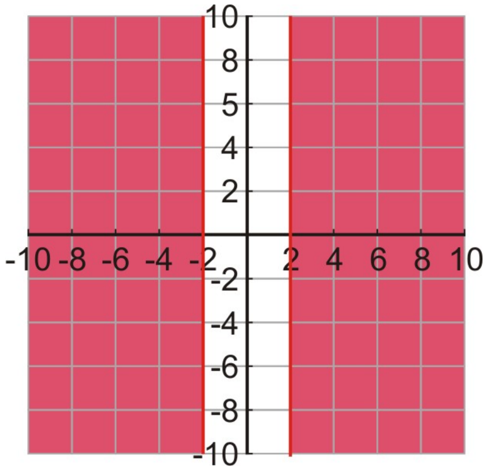
Example 4

Graph the inequality $|x| \geq 2$.

Solution

The absolute value inequality $|x| \geq 2$ can be re-written as a compound inequality:

$x \leq -2$ or $x \geq 2$



In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 and greater than or equal to 2. The solution is represented by the plane to the left of the vertical line $x = -2$ and the plane to the right of line $x = 2$.

Both vertical lines are solid because points on the line are included in the solution.

Graph Linear Inequalities in Two Variables

The general procedure for graphing inequalities in two variables is as follows.

Step 1: Re-write the inequality in slope-intercept form $y = mx + b$. Writing the inequality in this form lets you know the direction of the inequality

Step 2 Graph the line of equation $y = mx + b$ using your favorite method. (For example, plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, etc.). Draw a dashed line if the equal sign in not included and a solid line if the equal sign is included.

Step 3 Shade the half plane above the line if the inequality is greater than. Shade the half plane under the line if the inequality is less than.

Example 5

Graph the inequality $y \geq 2x - 3$.

Solution

Step 1

The inequality is already written in slope-intercept form $y \geq 2x - 3$.

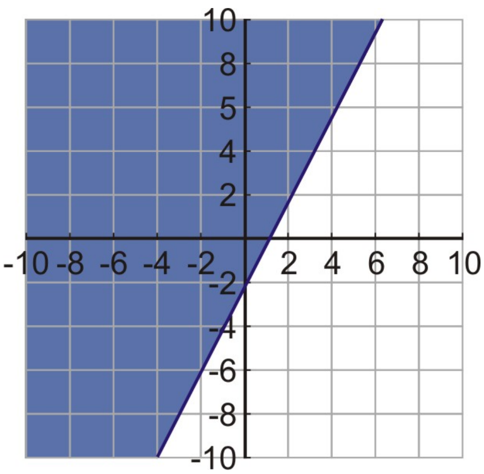
x	y
-1	$2(-1)-3=-5$
0	$2(0)-3=-3$
1	$2(1)-3=-1$

Step 2

Graph the equation $y = 2x - 3$ by making a table of values.

Step 3

Graph the inequality. We shade the plane above the line because y is greater than. The value $2x-3$ defines the line. The line is solid because the equal sign is included.



Example 6

Graph the inequality $5x - 2y > 4$.

Solution

Step 1

Rewrite the inequality in slope-intercept form.

$$-2y > -5x + 4$$
$$y > \frac{5}{2}x - 2$$

Notice that the inequality sign changed direction due to division of negative sign.

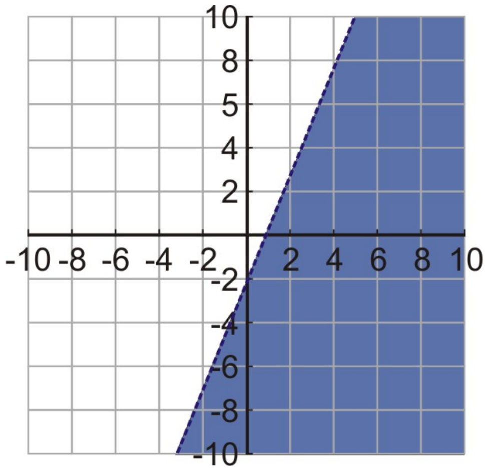
Step 2

Graph the equation $y > \frac{5}{2}x - 2$ by making a table of values.

x	y
-2	$\frac{5}{2}(-2)-2=-7$
0	$\frac{5}{2}(0)-2=-2$
2	$\frac{5}{2}(2)-2=3$

Step 3

Graph the inequality. We shade the plane **below** the line because the inequality in slope-intercept form is less than. The line is dashed because the equal sign in not included.



Example 7

Graph the inequality $y + 4 \leq -\frac{x}{3} + 5$.

Solution

Step 1

Rewrite the inequality in slope-intercept form $y \leq -\frac{x}{3} + 1$

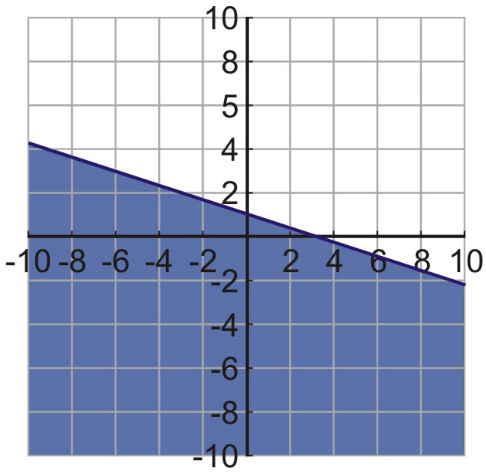
Step 2

Graph the equation $y = -\frac{x}{3} + 1$ by making a table of values.

x	y
-3	$-\frac{(-3)}{3} + 1 = 2$
0	$-\frac{0}{3}(0) + 1 = 1$
3	$-\frac{3}{3} + 1 = 0$

Step 3

Graph the inequality. We shade the plane below the line. The line is solid because the equal sign in included.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example 8

A pound of coffee blend is made by mixing two types of coffee beans. One type costs \$9 *per pound and another type costs* \$7 *per pound. Find all the possible mixtures of weights of the two different coffee beans for which the blend costs* \$8.50 *per pound or less.*

Solution

Let's apply our problem solving plan to solve this problem.

Step 1:

Let $x =$ weight of \$9 per pound coffee beans in pounds

Let $y =$ weight of \$7 per pound coffee beans in pounds

Step 2

The cost of a pound of coffee blend is given by $9x + 7y$.

We are looking for the mixtures that cost \$8.50 or less.

We write the inequality $9x + 7y \le 8.50$.

Step 3

To find the solution set, graph the inequality $9x + 7y \le 8.50$.

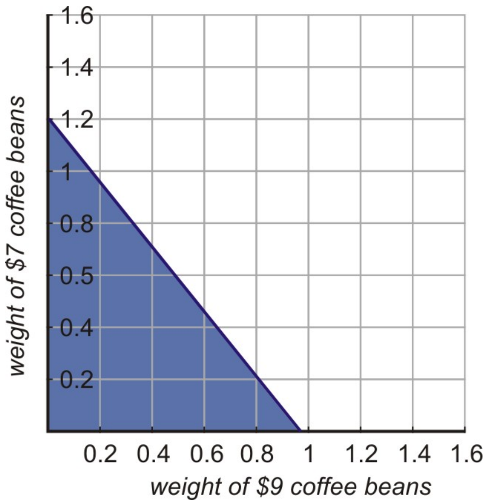
Rewrite in slope-intercept $y \le -1.29x + 1.21$.

Graph $y = -1.29x + 1.21$ by making a table of values.

x	y
0	1.21
1	-0.08
2	-1.37

Step 4

Graph the inequality. The line will be solid. We shade below the line.



Notice that we show only the first quadrant of the coordinate plane because the weight values should be positive.

The blue-shaded region tells you all the possibilities of the two bean mixtures that will give a total less than or equal to \$8.50.

Example 9

Julian has a job as an appliance salesman. He earns a commission of \$60 *for each washing machine he sells and* \$130 *for each refrigerator he sells. How many washing machines and refrigerators must Julian sell in order to make* \$1000 *or more in commission?*

Solution Let's apply our problem solving plan to solve this problem.

Step 1

Let x= number of washing machines Julian sells

Let y = number of refrigerators Julian sells

Step 2

The total commission is given by the expression $60x + 130y$.

We are looking for total commission of \$1000 or more. We write the inequality. $60x + 130y \geq 1000$.

Step 3

To find the solution set, graph the inequality $60x + 130y \geq 1000$.

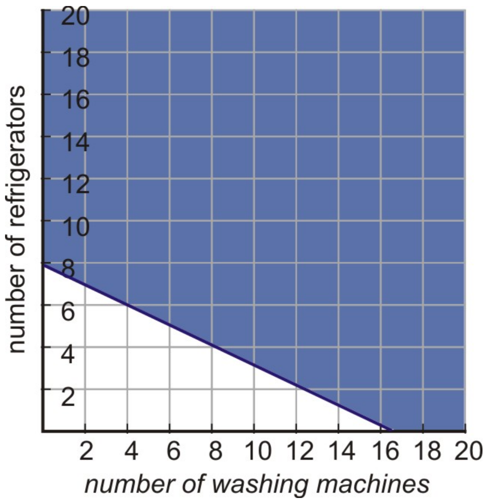
Rewrite it in slope-intercept $y \geq -.46x + 7.7$.

Graph $y = -.46x + 7.7$ by making a table of values.

x	y
0	7.7
2	6.78
4	5.86

Step 4

Graph the inequality. The line will be solid. We shade above the line.



Notice that we show only the first quadrant of the coordinate plane because dollar amounts should be positive. Also, only the points with integer coordinates are possible solutions.

Lesson Summary

- The general procedure for graphing inequalities in two variables is as follows:

Step 1

Rewrite the inequality in slope-intercept form $y = mx + b$.

Step 2

Graph the line of equation $y = mx + b$ by building a table of values.

Draw a dashed line if the equal sign in not included and a solid line if the it is included.

Step 3

Shade the half plane above the line if the inequality is greater than.

Shade the half plane under the line if the inequality is less than.

Review Questions

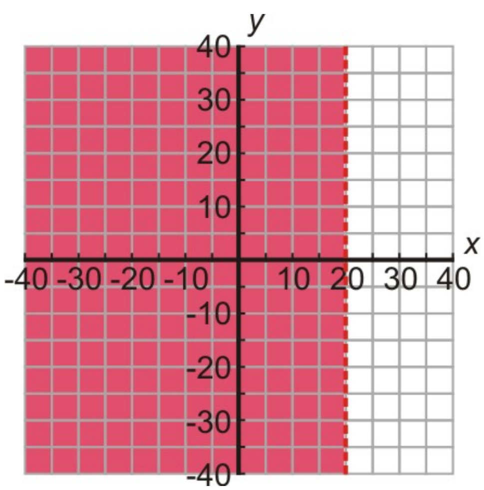
Graph the following inequalities on the coordinate plane.

1. $x < 20$
2. $y \geq -5$
3. $| x |> 10$
4. $| y |\leq 7$
5. $y \leq 4x + 3$
6. $y > -\frac{x}{2} - 6$
7. $3x - 4y \geq 12$
8. $x + 7y < 5$
9. $6x + 5y > 1$
10. $y + 5 \leq -4x + 10$
11. $x - \frac{1}{2}y \geq 5$

12. $30x + 5y < 100$
13. An ounce of gold costs \$670 and an ounce of silver costs \$13. Find all possible weights of silver and gold that makes an alloy that costs less than \$600 per ounce.
14. A phone company charges 50 cents cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and night time minutes would you have to use to pay more than \$20 over a 24 hour period?

Review Answers

1.



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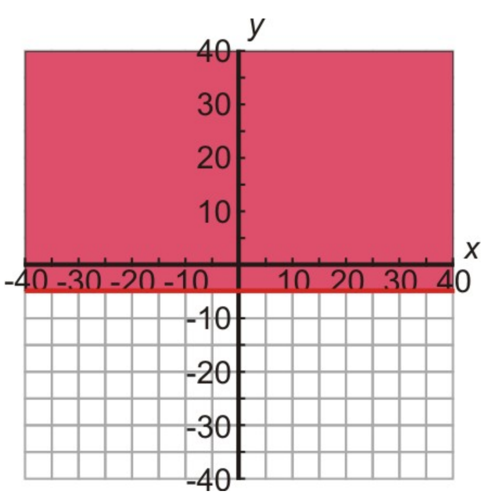

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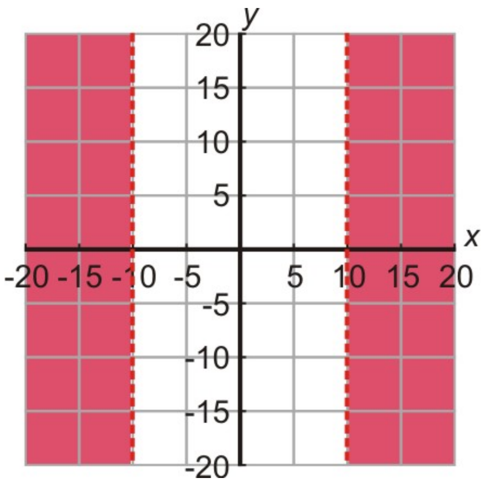
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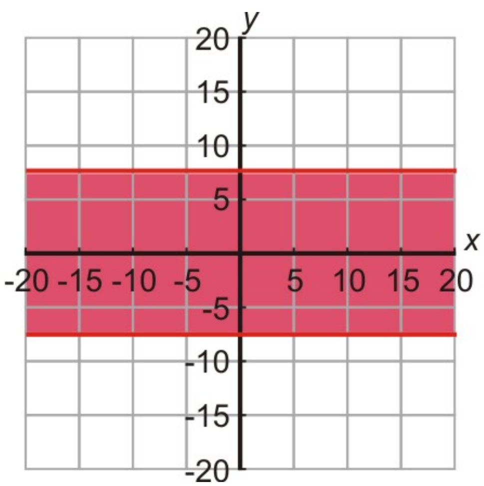
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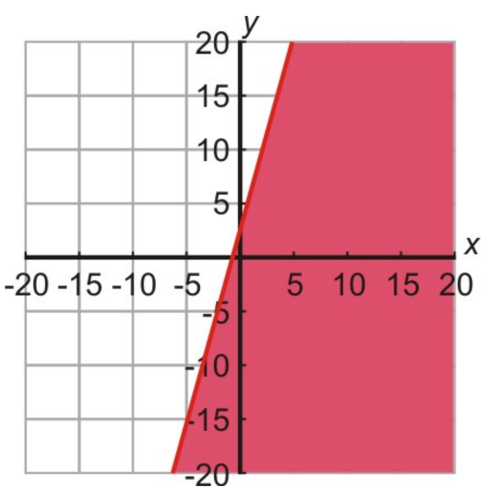
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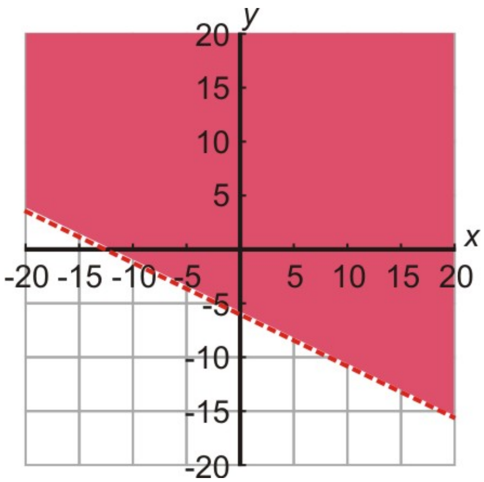
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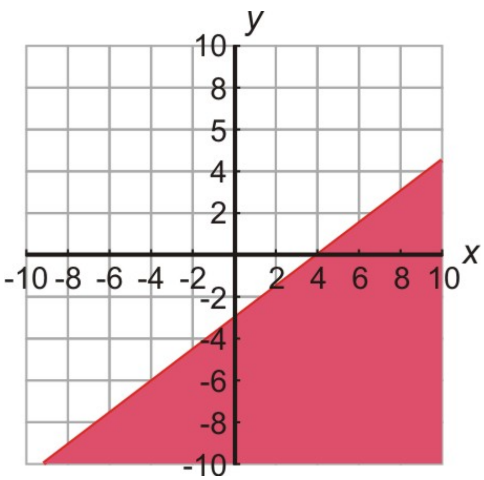
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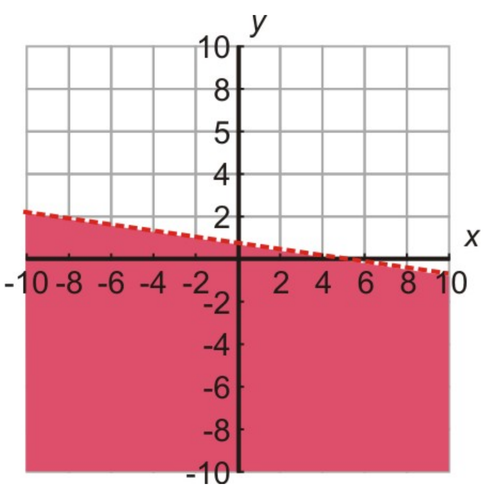
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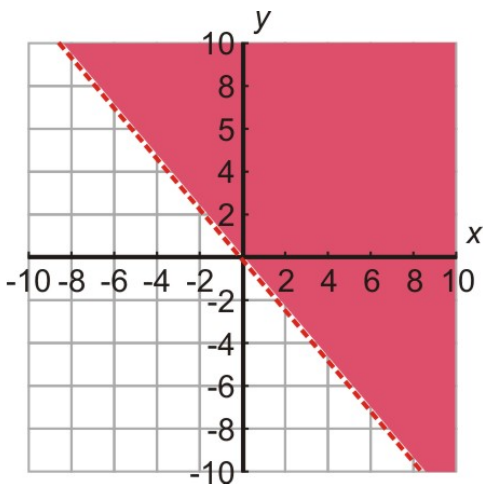

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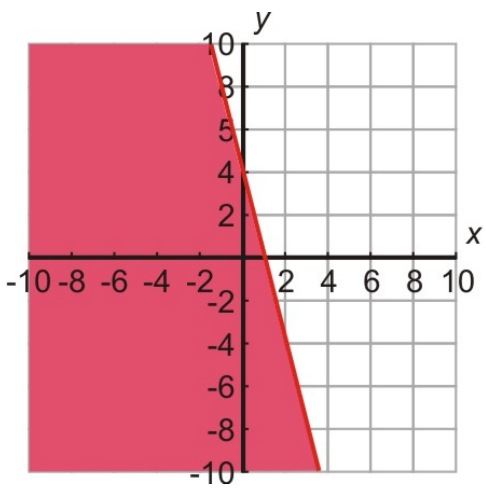

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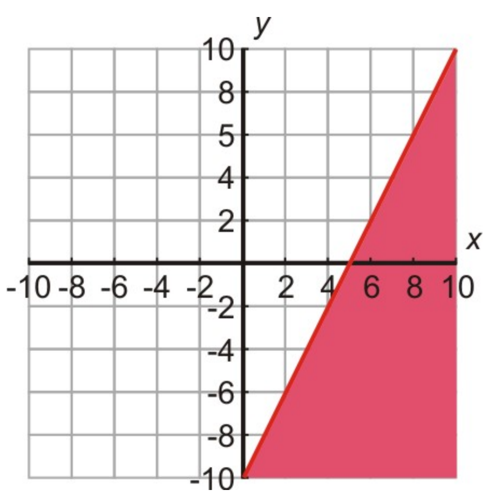

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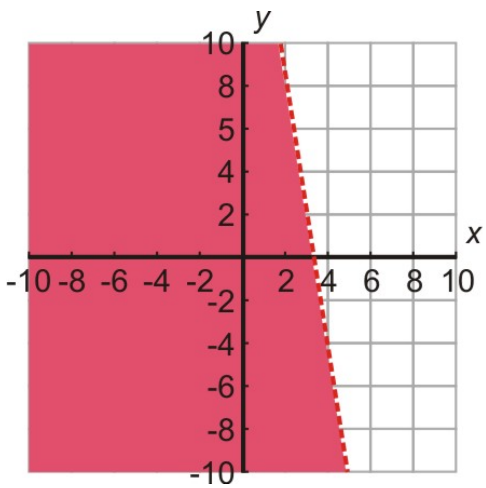
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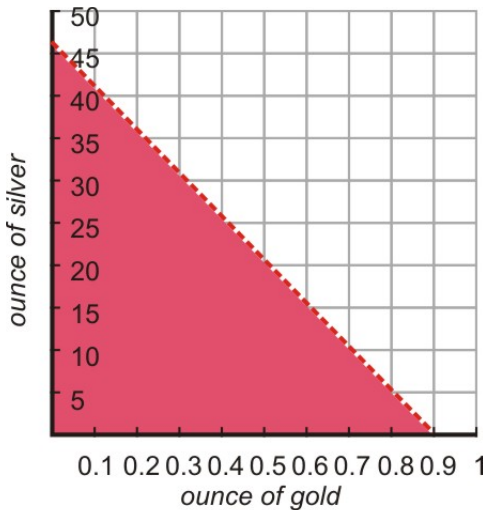
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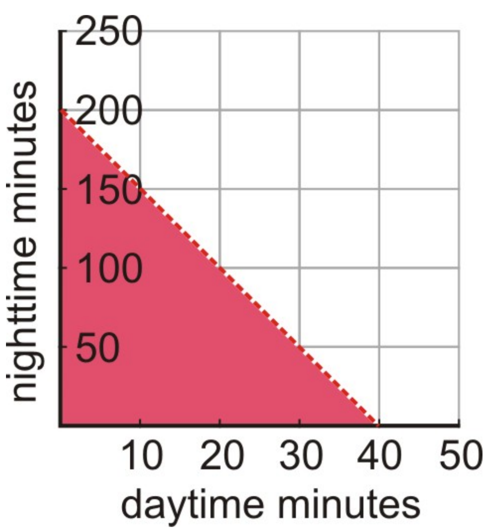
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Chapter 7

Solving Systems of Equations and Inequalities

7.1 Linear Systems by Graphing

Learning Objectives

- Determine whether an ordered pair is a solution to a system of equations.
- Solve a system of equations graphically.
- Solve a system of equations graphically with a graphing calculator.
- Solve word problems using systems of equations.

Introduction

In this lesson, we will discover methods to determine if an ordered pair is a solution to a system of two equations. We will then learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we will look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations consists of a set of equations that must be solved together.

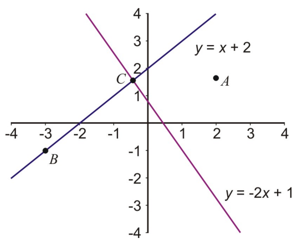
Consider the following system of equations.

$$\begin{aligned} y &= x + 2 \\ y &= -2x + 1 \end{aligned}$$

Since the two lines are in a system we deal with them together by graphing them on the same coordinate axes. The lines can be graphed using your favorite method. Let's graph by making a table of values for each line.

Line 1 $y = x + 2$

x	y
0	2
1	3



Line 2 $y = -2x + 1$

x	y
0	1
1	-1

A solution for a single equation is any point that lies on the line for that equation. A solution for a system of equations is any point that lies on both lines in the system.

For Example

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In particular, this point marks the intersection of the two lines. It solves both equations, so it solves the system. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered paid that solves both equations.

You can confirm the solution by plugging it into the system of equations, and confirming that the solution works in each equation.

Example 1

Determine which of the points $(1, 3)$, $(0, 2)$ or $(2, 7)$ *is a solution to the following system of equations.*

$$\begin{aligned} y &= 4x - 1 \\ y &= 2x + 3 \end{aligned}$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point $(1, 3)$

$$\begin{aligned} y &= 4x - 1 \\ 3^? &=^? 4(1) - 1 \\ 3 &= 3 \end{aligned}$$

The solution checks.

$$\begin{aligned} y &= 2x + 3 \\ 3 &\stackrel{?}{=} 2(1) + 3 \\ 3 &\neq 5 \end{aligned}$$

The solution does not check.

Point $(1,3)$ is on line $y = 4x - 1$ but it is not on line $y = 2x + 3$ so it is not a solution to the system.

Point $(0,2)$

$$\begin{aligned} y &= 4x - 1 \\ 2 &\stackrel{?}{=} 4(0) - 1 \\ 2 &\neq -1 \end{aligned}$$

The solution does not check.

Point $(0,2)$ is not on line $y = 4x - 1$ so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point $(2,7)$

$$\begin{aligned} y &= 4x - 1 \\ 7 &\stackrel{?}{=} 4(2) - 1 \\ 7 &= 7 \end{aligned}$$

The solution checks.

$$\begin{aligned} y &= 2x + 3 \\ 7 &\stackrel{?}{=} 2(2) + 3 \\ 7 &= 7 \end{aligned}$$

The solution checks.

Point $(2,7)$ is a solution to the system since it lies on both lines.

Answer The solution to the system is point $(2,7)$.

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point which lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions. It is exact only when the x and y values of the solution are integers. However, this method it is a great at offering a visual representation of the system of equations and demonstrates that the solution to a system of equations is the intersection of the two lines in the system.

Example 2

(The equations are in slope-intercept form)

Solve the following system of equations by graphing.

$$y = 3x - 5$$
$$y = -2x + 5$$

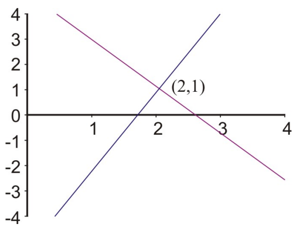
Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, let's make a table of values for each line.

Line 1 $y = 3x - 5$

x	y
1	-2
2	1



Line 2 $y = -2x + 5$

x	y
1	3
2	1

Answer The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point $(2, 1)$. So the solution is $x = 2$, $y = 1$ or $(2, 1)$.

Example 3

(The equations are in standard form)

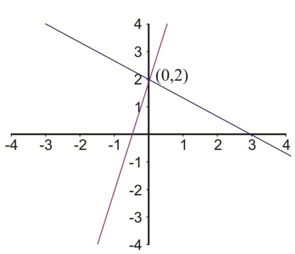
Solve the following system of equations by graphing

$$2x + 3y = 6$$
$$4x - y = -2$$

Solution

Graph both lines on the same coordinate axis using your method of choice.

Here we will graph the lines by finding the $x-$ and $y-$intercepts of each of the lines.



Line 1 $2x+3y=6$

$x-$intercept set $y=0 \Rightarrow 2x=6 \Rightarrow x=3$ which results in point $(3,0)$.

$y-$intercept set $x=0 \Rightarrow 3y=6 \Rightarrow y=2$ which results in point $(0,2)$.

Line 2 $-4x+y=2$

$x-$intercept: set $y=0 \Rightarrow -4x=2 \Rightarrow x=-\frac{1}{2}$ which results in point $(-\frac{1}{2},0)$.

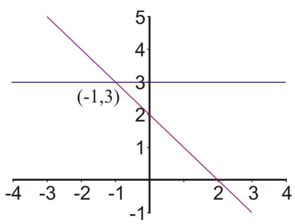
$y-$intercept: set $x=0 \Rightarrow y=2$ which results in point $(0,2)$

Answer The graph shows that the lines intersect at point $(0,2)$. Therefore, the solution to the system of equations is $x=0,\ y=2$.

Example 4:

Solve the following system by graphing.

$$\begin{aligned} y &= 3 \\ x+y &= 2 \end{aligned}$$



Line 1 $y=3$ is a horizontal line passing through point $(0,3)$.

Line 2 $x+y=2$

$x-$intercept: $(2,0)$

$y-$intercept: $(0,2)$

Answer The graph shows that the solution to this system is $(-1,3)$ $x=-1,\ y=3$.

These examples are great at demonstrating that the solution to a system of linear equations means the point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can see, however, that determining a solution from a graph would require very careful graphing of the lines, and is really only practical when you are certain that the solution gives integer values for x and y. In most cases, this method can only offer approximate solutions to systems of equations. For exact solutions other methods are necessary.

Solving a System of Equations Using the Graphing Calculator

A graphing calculator can be used to find or check solutions to a system of equations. In this section, you learned that to solve a system graphically, you must graph the two lines on the same coordinate axes and find the point of intersection. You can use a graphing calculator to graph the lines as an alternative to graphing the equations by hand.

Example 6

Solve the following system of equations using a graphing calculator.

$$x - 3y = 4$$
$$2x + 5y = 8$$

In order to input the equations into the calculator, they must be written in slope-intercept form (i.e., $y = mx + b$ form), or at least you must isolate y.

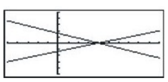
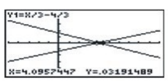
$$\begin{aligned} x - 3y &= 4 & & & y &= \frac{1}{3}x - \frac{4}{3} \\ & & \Rightarrow & & & \\ 2x + 5y &= 8 & & & y &= \frac{-2}{5}x - \frac{8}{5} \end{aligned}$$

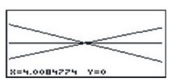
Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$Y_1 = \frac{x}{3} - \frac{4}{3}$$
$$T_2 = -\frac{2x}{3} - \frac{8}{5}$$

Now press [**GRAPH**]. The window below is set to $-5 \leq x \leq 10$ and $-5 \leq x \leq 5$.

The first screen below shows the screen of a TI-83 family graphing calculator with these lines graphed.



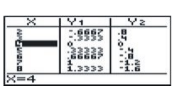
There are a few different ways to find the intersection point.

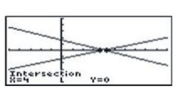
Option 1 Use [**TRACE**] and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be on the bottom of the screen. The second screen above shows the values to be $X = 4.0957447$ and $Y = .03191489$.

Use the [**ZOOM**] function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X = 4$ and $Y = 0$.

Option 2 Look at the table of values by pressing [**2nd**] [**GRAPH**]. The first screen below shows a table of values for this system of equations. Scroll down until the values for and are the same. In this case this occurs at $X = 4$ and $Y = 0$.

Use the [**TBLSET**] function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by taking smaller values of Table 1.



Option 3 Using the [**2nd**] [**TRACE**] function gives the screen in the second screen above.

Scroll down and select intersect.

The calculator will display the graph with the question [**FIRSTCURVE**]? Move the cursor along the first curve until it is close to the intersection and press [**ENTER**].

The calculator now shows [**SECONDCURVE**]?

Move the cursor to the second line (if necessary) and press [**ENTER**].

The calculator displays [**GUESS**]?

Press [**ENTER**] and the calculator displays the solution at the bottom of the screen (see the third screen above).

The point of intersection is $X = 4$ and $Y = 0$.

Notes:

- When you use the "intersect" function, the calculator asks you to select [**FIRSTCURVE**]? and [**SECONDCURVE**]? in case you have more than two graphs on the screen. Likewise, the [**GUESS**]? is requested in case the curves have more than one intersection. With lines you only get one point of intersection, but later in your mathematics studies you will work with curves that have multiple points of intersection.

- Option 3 is the only option on the graphing calculator that gives an exact solution. Using trace and table give you approximate solutions.

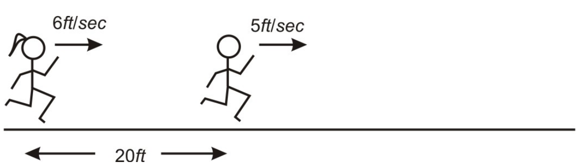
Solve Real-World Problems Using Graphs of Linear Systems

Consider the following problem

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet *per second and Nadia can run at a speed of* 6 feet *per second. To be a good sport Nadia likes to give Peter a head start of* 20 feet*. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?*

Draw a sketch

At time, $t = 0$:



Formulas

Let's define two variables in this problem.

$t =$ the time from when Nadia starts running

$d =$ the distance of the runners from the starting point.

Since we have two runners we need to write equations for each of them. This will be the **system of equations** for this problem.

Here we use the formula distance = speed × time

Nadia's equation $d = 6t$

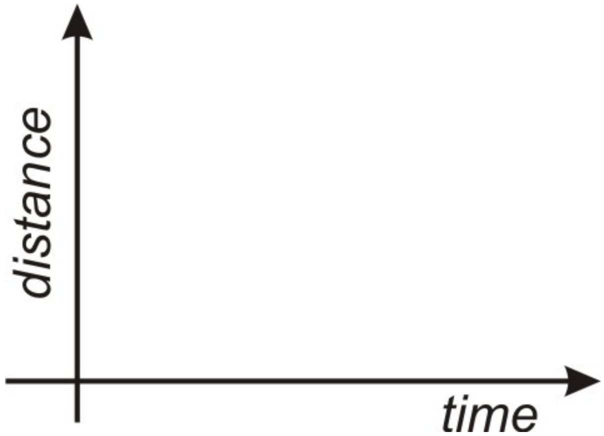
Peter's equation $d = 5t + 20$

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate graph.

Time should be on the horizontal axis since it is the independent variable.

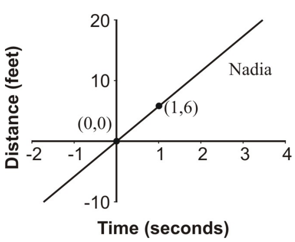
Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines. In this case, we will use the slope-intercept method since it makes more sense physically.

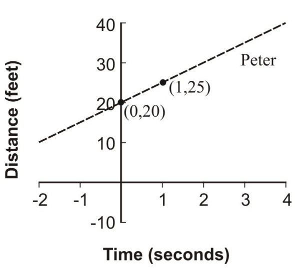
To graph the line that describes Nadia's run, start by graphing the $y-$intercept $(0,0)$. If you do not see that this is the $y-$intercept, try plugging in the test-value of $x = 0$.

The slope tells us that Nadia runs 6 feet every one second so another point on the line is $(1,6)$. Connecting these points gives us Nadia's line.

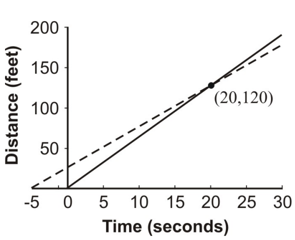


To graph the line that describes Peter's run, again start with the $y-$intercept. In this case, this is the point $(0,20)$.

The slope tells us that Peter runs 5 feet every one second so another point on the line is $(1,25)$. Connecting these points gives us Peter's line.



In order to find when and where Nadia and Peter meet, we will graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet 20 seconds after Nadia starts running and 120 feet from the starting point.

Review Questions

Determine which ordered pair satisfies the system of linear equations.

1. $y = 3x - 2$
   $y = -x$
   (a) $(1, 4)$
   (b) $(2, 9)$
   (c) $(\frac{1}{2}, -\frac{1}{2})$
2. $y = 2x - 3$
   $y = x + 5$
   (a) $(8, 13)$
   (b) $(-7, 6)$
   (c) $(0, 4)$
3. $2x + y = 8$
   $5x + 2y = 10$
   (a) $(-9, 1)$
   (b) $(-6, 20)$
   (c) $(14, 2)$
4. $3x + 2y = 6$
   $y = \frac{x}{2} - 3$
   (a) $(3, -\frac{3}{2})$
   (b) $(-4, 3)$
   (c) $(\frac{1}{2}, 4)$

Solve the following systems using the graphing method.

5. $y = x + 3$
   $y = -x + 3$
6. $y = 3x - 6$
   $y = -x + 6$
7. $2x = 4$
   $y = -3$
8. $y = -x + 5$
   $-x + y = 1$
9. $x + 2y = 8$
   $5x + 2y = 0$
10. $3x + 2y = 12$
    $4x - y = 5$
11. $5x + 2y = -4$
    $x - y = 2$
12. $2x + 4 = 3y$
    $x - 2y + 4 = 0$

13. $y = \frac{x}{2} - 3$
    $2x - 5y = 5$
14. $y = 4$
    $x = 8 - 3y$
15. Solve the following problems by using the graphing method.
16. Mary's car is 10 years old and has a problem. The repair man indicates that it will cost her $1200 to repair her car. She can purchase a different, more efficient car for $4500. Her present car averages about $2000 per year for gas while the new car would average about $1500 per year. Find the number of years for when the total cost of repair would equal the total cost of replacement.
17. Juan is considering two cell phone plans. The first company charges $120 for the phone and $30 per month for the calling plan that Juan wants. The second company charges $40 for the same phone, but charges $45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
18. A tortoise and hare decide to race 30 feet. The hare, being much faster, decided to give the tortoise a head start of 20 feet. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long will it be until the hare catches the tortoise?

Review Answers

1. (c)
2. (a)
3. (b)
4. (a)
5. $(0, 3)$
6. $(3, 3)$
7. $(2, -3)$
8. $(2, 3)$
9. $(-2, 5)$
10. $(2, 3)$
11. $(0, -2)$
12. $(4, 4)$
13. $(20, 7)$
14. $(-4, 4)$
15. 6.6 years
16. 5.33 months
17. 4.0 seconds

7.2 Solving Linear Systems by Substitution

Learning Objectives

- Solve systems of equations with two variables by substituting for either variable.
- Manipulate **standard form** equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Introduction

In this lesson, we will learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let's look again at the problem involving Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet *per second and Nadia can run at a speed of* 6 feet *per second. To be a good sport Nadia likes to give Peter a head start of* 20 feet*. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?*

In that example, we came up with two equations.

Nadia's equation	$d = 6t$
Peter's equation	$d = 5t + 20$

We have seen that each relationship produces its own line on a graph, but that to solve the system we find the point at which the lines intersect (Lesson 1). At that point the values for d and t satisfy **both** relationships.

In this simple example, this means that the d in Nadia's equation is the same as the d in Peter's. We can set the two equations equal to each other to solve for t.

$6t = 5t + 20$	Subtract $5t$ from both sides.
$t = 20$	Substitute this value for t into Nadia's equation.
$d = 6 \cdot 20 = 120$	

Even if the equations are not so obvious, we can use simple algebraic manipulation to find an expression for one variable in terms of the other. We can rearrange Peter's equation to isolate t.

$d = 5t + 20$	Subtract 20 from both sides.
$d - 20 = 5t$	Divide by 5.
$\frac{d - 20}{5} = t$	

We can now ***substitute*** this expression for t into Nadia's equation ($d = 6t$) to solve it.

$d = 6\left(\frac{d - 20}{5}\right)$	Multiply both sides by 5.
$5d = 6(d - 20)$	Distribute the 6.
$5d = 6d - 120$	Subtract $6d$ from both sides.
$-d = -120$	Divide by -1.
$d = 120$	Substitute value for d into our expression for t.
$t = \frac{120 - 20}{5} = \frac{100}{5} = 20$	

We find that Nadia and Peter meet 20 seconds **after they start racing, at a distance of** 120 yards **away**.

The method we just used is called the **Substitution Method**. In this lesson, you will learn several techniques for isolating variables in a system of equations, and for using the expression you get for solving systems of equations that describe situations like this one.

Example 1

Let us look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned} 2x + 3y &= 6 \\ -4x + y &= 2 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. It makes sense to use this equation to solve for y.

Solve the second equation for the y variable:

$$\begin{aligned} -4x + y &= 2 && \text{Add } 4x \text{ to both sides.} \\ y &= 2 + 4x \end{aligned}$$

Substitute this expression into the second equation.

$$\begin{aligned} 2x + 3(2 + 4x) &= 6 && \text{Distribute the 3.} \\ 2x + 6 + 12x &= 6 && \text{Collect like terms.} \\ 14x + 6 &= 6 && \text{Subtract 6 from both sides.} \\ 14x &= 0 \\ x &= 0 \end{aligned}$$

Substitute back into our expression for y.

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution ($x = 0$, $y = 2$) that we found when we graphed these functions (Lesson 7.1). As long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, consider a more complicated example. In the following example the solution gives fractional answers for both x and y, and so would be very difficult to solve by graphing alone!

Example 2

Solve the system

$$\begin{aligned} 2x + 3y &= 3 \\ 2x - 3y &= -1 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. Right now it doesn't matter which equation we use or which variable we solve for.

Solve the first equation for x

$$\begin{aligned} 2x + 3y &= 3 && \text{Subtract } 3y \text{ from both sides.} \\ 2x &= 3 - 3y && \text{Divide both sides by 2.} \\ x &= \frac{3 - 3y}{2} \end{aligned}$$

Substitute this expression into the second equation.

$$\begin{aligned}
\cancel{2}\cdot\frac{1}{\cancel{2}}(3-3y)-3y &= -1 && \text{Cancel the fraction and rewrite terms.}\\
3-3y-3y &= -1 && \text{Collect like terms.}\\
3-6y &= -1 && \text{Subtract 3 from both sides.}\\
-6y &= -4 && \text{Divide by } -6.\\
y &= \frac{2}{3}
\end{aligned}$$

Substitute into the expression and solve for x.

$$\begin{aligned}
x &= \frac{1}{2}\left(3-\cancel{3}\frac{2}{\cancel{3}}\right)\\
x &= \frac{1}{2}
\end{aligned}$$

So our solution is, $x = \frac{1}{2}$, $y = \frac{2}{3}$. You can see why the graphical solution $\left(\frac{1}{2}, \frac{2}{3}\right)$ might be difficult to read accurately.

Solving Real-World Problems Using Linear Systems

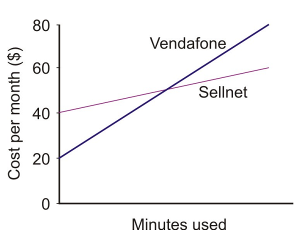
There are many situations where we can use simultaneous equations to help solve real-world problems. We may be considering a purchase. For example, trying to decide whether it is cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but do not know if you would really save any money by buying a new CD every month in that way. One example with which we are all familiar is considering phone contracts. Let's look at an example of that now.

Example 3

Anne is trying to choose between two phone plans. The first plan, with Vendafone costs \$20 *per month, with calls costing an additional* 25 cents *per minute. The second company, Sellnet, charges* \$40 *per month, but calls cost only* 8 cents *per minute. Which should she choose?*

Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the number of minutes is the independent variable, it will be our x. Cost is dependent on minutes. The *cost per month* is the *dependent* variable and will be assigned y.

For Vendafone	$y = 0.25x + 20$
For Sellnet	$y = 0.08x + 40$



By writing the equations in slope-intercept form ($y = mx + b$) you can visualize the situation in a simple sketched graph, shown right. The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line). In order to help Anne decide which to choose, we will determine where the two lines cross, by solving the two equations as a system. Since equation one gives us an expression for y $(0.25x + 20)$, we can substitute this expression directly into equation two.

$$\begin{aligned}
0.25x + 20 &= 0.08x + 40 && \text{Subtract 20 from both sides.}\\
0.25x &= 0.08x + 20 && \text{Subtract } 0.08x \text{ from both sides.}\\
0.17x &= 20 && \text{Divide both sides by 0.17.}\\
x &= 117.65 \text{ minutes} && \text{Rounded to two decimal places.}
\end{aligned}$$

We can now use our sketch, plus this information to provide an answer:

If Anne will use 117 *minutes or less every month, she should choose Vendafone. If she plans on using* 118 *or more minutes, she should choose Sellnet.*

Mixture Problems

Systems of equations crop up frequently when considering chemicals in solutions, and can even be seen in things like mixing nuts and raisins or examining the change in your pocket! Let's look at some examples of these.

Example 4

Nadia empties her purse and finds that it contains only nickels (worth 5 cents *each) and dimes (worth* 10 cents *each). If she has a total of* 7 *coins and they have a combined value of* 55 cents*, how many of each coin does she have?*

Since we have two types of coins, let's call the number of nickels x and the number of dimes will be our y. We are given two key pieces of information to make our equations, the number of coins and their value.

Number of coins equation	$x + y = 7$	(number of nickels) + (number of dimes)
The value equation	$5x + 10y = 55$	Since nickels are worth five cents and dimes ten cents

We can quickly rearrange the first equation to isolate x.

Image courtesy of Kevin@flickr.com/creativecommons

$x = 7 - y$	Now substitute into equation two.
$5(7 - y) + 10y = 55$	Distribute the 5.
$35 - 5y + 10y = 55$	Collect like terms.
$35 + 5y = 55$	Subtract 35 from both sides.
$5y = 20$	Divide by 5.
$y = 4$	Substitute back into equation one.
$+4 = 7$	Subtract 4 from both sides.
$x = 3$	

Solution

Nadia has 3 nickels and 4 dimes.

Sometimes the question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product**. Three examples are below.

Table 7.1:

Type of Mixture	First Equation	Second Equation
Coins (items with $ value)	Total number of items (n_1 n_2)	Total value (item value x no. of items)
Chemical solutions	Total solution volume ($V_1 + V_2$)	Amount of solute (vol x concentration)
Density of two substances	Total amount or volume of mix	Total mass (volume x density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of solute in the individual parts and in the final mixture. A solute is simply the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine. Even if the chemical is more exotic, we are still interested in the **total amount** of that chemical in each part. To find this, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 500 ml *of copper-sulfate solution with a* 15% *concentration. In order to do this, he wishes to use a high concentration solution* (60%) *and dilute it with a low concentration solution* (5%). *How much of each solution should he use?*

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals (0.6, 0.15 and 0.05). The two pieces of critical information we need is the final volume (500 ml) and the final amount of solute (15% of 500 ml = 75 ml). Our equations will look like this.

Volume equation	$x + y = 500$
Solute equation	$0.6x + 0.05y = 75$

You should see that to isolate a variable for substitution it would be easier to start with equation one.

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$$x + y = 500$$ Subtract y from both sides.

$$x = 500 - y$$ Now substitute into equation two.

$$0.6(500 - y) + 0.05y = 75$$ Distribute the 6.

$$300 - 0.6y + 0.05y = 75$$ Collect like terms.

$$300 - 0.55y = 75$$ Subtract 300 from both sides.

$$-0.55y = -225$$ Divide both sides by -0.55.

$$y = 409 \text{ ml}$$ Substitute back into equation for x.

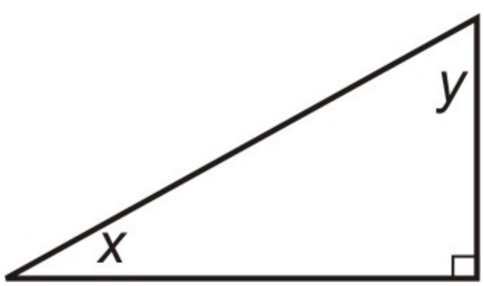
$$x = 500 - 409 = 91 \text{ ml}$$

Solution

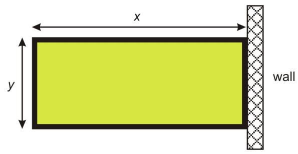
The chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

Review Questions

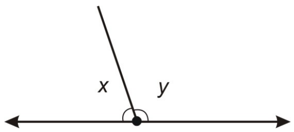
1. Solve the system: $x + 2y = 9$
   $3x + 5y = 20$
2. solve the system. $x - 3y = 10$
   $2x + y = 13$
3. Of the two non-right angles in a right angled triangle, one measures twice that of the other. What are the angles?



4. The sum of two numbers is 70. They differ by 11. What are the numbers?
5. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



6. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



7. I have \$15 and wish to buy five pounds of mixed nuts for a party. Peanuts cost \$2.20 per pound. Cashews cost \$4.70 per pound. How many pounds of each should I buy?
8. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and in 35%. How many liters of each should be mixed to give the acid needed for the experiment?
9. Bachelle wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is 19.3 g/cc and the density of silver is 10.5 g/cc . The jeweler told her that the volume of silver used was 10 cc and the volume of gold used was 20 cc . Find the combined density of her bracelet.

Review Answers

1. $x = -5$, $y = 7$
2. $x = 7$, $y = -1$
3. $x = 30^\circ$, $y = 60^\circ$
4. 29.5 and 40.5
5. $x = 120$ yards, $y = 80$ yards
6. $x = 81^\circ$, $y = 99^\circ$
7. 3.4 pounds of peanuts, 1.6 pounds of cashews
8. 0.8 liters of 10%, 0.2 liters of 35%
9. 16.4 g/cc

7.3 Solving Linear Systems by Elimination through Addition or Subtraction

Learning Objectives

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve real-world problems using linear systems by elimination.

Introduction

In this lesson, we will look at using simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y) this method is often referred to as **solving by elimination**. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Example 1

If one apple plus one banana costs \$1.25 *and one apple plus two bananas costs* \$2.00, *how much does it cost for one banana? One apple?*

It shouldn't take too long to discover that each banana costs \$0.75. You can see this by looking at the difference between the two situations. Algebraically, using a and b as the cost for apples and bananas, we get the following equations.

$$a + b = 1.25$$
$$a + 2b = 2.00$$

If you look at the difference between the two equations you see that the difference in items purchased is one banana, and the difference in money paid is 75 cents. So one banana costs 75 cents.

$$(a + 2b) - (a + b) = 2.00 - 1.25 \Rightarrow b = 0.75$$

To find out how much one apple costs, we subtract \$0.75 from the cost of one apple and one banana. So an apple costs 50 cents.

$$a + 0.75 = 1.25 \Rightarrow a = 1.25 - 0.75 \Rightarrow a = 0.50$$

To solve systems using addition and subtraction, we will be using exactly this idea. By looking at the sum or difference of the two equations, we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method requires us to combine two equations in such a way that the resulting equation has only one variable. We can then use simple linear algebra methods of solving for that variable. If required, we can always substitute the value we get for that variable back in either one of the original equations to solve for the remaining unknown variable.

Example 2

Solve the system by addition:

$$3x + 2y = 11$$
$$5x - 2y = 13$$

We will add everything on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right.

$$(3x + 2y) + (5x - 2y) = 11 + 13 \Rightarrow 8x = 24 \Rightarrow x = 3$$

A simpler way to visualize this is to keep the equations as they appear above, and to add in columns. However, just like adding units tens and hundreds, you MUST keep x's and y's in their own columns. You may also wish to use terms like "0y" as a placeholder!

$$\begin{aligned} 3x + 2y &= 11 \\ +\ (3x - 2y) &= 13 \\ \hline 8x + 0y &= 24 \end{aligned}$$

Again we get $8x = 24$ or $x = 3$.

To find a value for y we simply substitute our value for x back in.

Substitute $x = 3$ into the second equation.

$$\begin{aligned} 5 \cdot 3 - 2y &= 13 && \text{Since } 5 \times 3 = 15, \text{ we subtract 15 from both sides.} \\ -2y &= -2 && \text{Divide by 2 to get the value for } y. \\ y &= 1 \end{aligned}$$

The first example has a solution at $x = 3$ and $y = 1$. You should see that the method of addition works when the coefficients of one of the variables are opposites. In this case it is the coefficients of y that are opposites, being $+2$ in the first equation and -2 in the second.

There are other, similar, methods we can use when the coefficients are not opposites, but for now let's look at another example that can be solved with the method of addition.

Example 3

Dagwood21/www.flickr.com/creativecommons

Andrew is paddling his canoe down a fast moving river. Paddling downstream he travels at 7 miles *per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at* 1.5 miles *per hour. If he paddles equally hard in both directions, calculate, in miles per hour, the speed of the river and the speed Andrew would travel in calm water.*

Step One First, we convert our problem into equations. We have two unknowns to solve for, so we will call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew's speed is boosted by the river current, so his total speed is the canoe speed plus the speed of the river $(x + y)$. Upstream, his speed is hindered by the speed of the river. His speed upstream is $(x - y)$.

$$\begin{aligned} &\text{Downstream Equation} && x + y = 7 \\ &\text{Upstream Equation} && x - y = 1.5 \end{aligned}$$

Step Two Next, we are going to eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is $+1$ in the first equation and -1 in the second. Clearly $(+1) + (-1) = 0$, so this is the variable we will eliminate. To do this we add equation 1 to equation 2. We must be careful to collect like terms, and that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x+y)+(x-y)=7+1.5 \Rightarrow 2x=8.5 \Rightarrow x=4.25$$

Or, using the column method we used in example one.

$$\begin{array}{r} x+y=7 \\ +\,(x-y)=1.5 \\ \hline 2x+0y=8.5 \end{array}$$

Again you see we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, we'll plug it into the first equation.

Substitute $x = 3$ into the second equation:

$$\begin{aligned} 4.25+y&=7 && \text{Subtract 4.25 from both sides.} \\ y&=2.75 \end{aligned}$$

Solution

Andrew paddles at 4.25 miles *per hour. The river moves at* 2.75 miles *per hour.*

Solving Linear Systems Using Subtraction of Equations

Another, very similar method for solving systems is subtraction. In this instance, you are looking to have identical coefficients for x or y (including the sign) and then subtract one equation from the other. If you look at Example one you can see that the coefficient for x in both equations is $+1$. You could have also used the method of subtraction.

$$(x+y)-(x-y)=200-80 \Rightarrow 2y=120 \Rightarrow y=60$$

or

$$\begin{array}{r} x+y=200 \\ +\,(x-y)=-80 \\ \hline 0x+2y=120 \end{array}$$

So again we get $y = 60$, from which we can determine x. The method of subtraction looks equally straightforward, and it is so long as you remember the following:

1. Always put the equation you are subtracting in parentheses, and distribute the negative.
2. Don't forget to subtract the numbers on the right hand side.
3. Always remember that subtracting a negative is the same as adding a positive.

Example 4

Peter examines the coins in the fountain at the mall. He counts 107 coins*, all of which are either pennies or nickels. The total value of the coins is* \$3.47*. How many of each coin did he see?*

We have two types of coins. Let's call the number of pennies x and the number of nickels y. The total value of pennies is just x, since they are worth one cent each. The total value of nickels is $5y$. We are given two key pieces of information to make our equations. The number of coins and their value.

Number of Coins Equation	$x+y=107$	(number of pennies) + (number of nickels)
The Value Equation:	$x+5y=347$	pennies are worth $1c$, nickels are worth $5c$.

We will jump straight to the subtraction of the two equations.

$$\begin{aligned} x+y &= 107 \\ +(x+5y) &= -347 \\ \hline 4y &= -240 \end{aligned}$$

Let's substitute this value back into the first equation.

$$\begin{aligned} x+60 &= 107 && \text{Subtract 60 from both sides.} \\ x &= 47 \end{aligned}$$

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth \$3.00) for a total of \$3.47.

We have now learned three techniques for solving systems of equations.

1. Graphing
2. Substitution
3. Elimination

You should be starting to gain an understanding of which method to use when given a particular problem. For example, **graphing** is a good technique for seeing what the equations are doing, and when one service is less expensive than another. Graphing alone may not be ideal when an exact numerical solution is needed.

Similarly, **substitution** is a good technique when one of the coefficients in your equation is $+1$ or -1.

Addition or **subtraction** is ideal when the coefficient of one of the variables matches the coefficient of the same variable in the other equation. In the next lesson, we will learn the last technique for solving systems of equations exactly, when none of the coefficients match and the coefficient is not one.

Multimedia Link The following video contains three examples of solving systems of equations using multiplication and addition and subtraction as well as multiplication (which is the next topic). Khan Academy Systems of Equations (9:57) . Note that the narrator is not always careful about showing his work, and you should try to be neater in your mathematical writing.

Review Questions

1. Solve the system: $3x+4y=2.5$
   $5x-4y=25.5$
2. Solve the system $5x+7y=-31$
   $5x-9y=17$

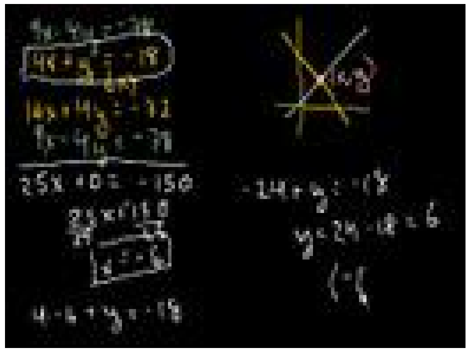


Figure 7.1: systems of equations (Watch on Youtube)

3. Solve the system $3y - 4x = -33$
   $5x - 3y = 40.5$
4. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of each candy bar and each fruit roll-up?
5. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
6. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12 miles journey costs $14.29 and a 17 miles journey costs $19.91, calculate:
   (a) the pick-up fee
   (b) the per-mile rate
   (c) the cost of a seven mile trip
7. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a seven minute call costs $4.25 and a 12 minute call costs $5.50, find each rate.
8. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $35 per hour, and the builder earns $28 per hour. Together they were paid $330.75, but the plumber earned $106.75 more than the builder. How many hours did each work?
9. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned $220. In his second week, he managed to sell 13 warranties and earned $280. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

Review Answers

1. $x = 3.5$, $y = -2$
2. $x = -2$, $y = -3$
3. $x = 7.5$, $y = -1$
4. Candy bars cost 48 cents each and fruit roll-ups cost 35 cents each.
5. The wind speed is 24 mph
6.

7. (a) \$.80
   (b) \$1.12
   (c) \$8.64
8. 75 cents per minute for the first 5 mins, 25 cents per minute additional
9. The plumber works 6.25 hours, the builder works 4 hours
10. Paul earns a base of \$7.00 per hour

7.4 Solving Systems of Equations by Multiplication

Learning objectives

- Solve a linear system by multiplying one equation.
- Solve a linear system of equations by multiplying both equations.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Introduction

We have now learned three techniques for solving systems of equations.

- Graphing, Substitution and Elimination (through addition and subtraction).

Each one of these methods has both strengths and weaknesses.

- **Graphing** is a good technique for seeing what the equations are doing, and when one service is less expensive than another, but graphing alone to find a solution can be imprecise and may not be good enough when an exact numerical solution is needed.

- **Substitution** is a good technique when one of the coefficients in an equation is $+1$ or -1, but can lead to more complicated formulas when there are no unity coefficients.

- **Addition** or **Subtraction** is ideal when the coefficients of either x or y match in both equations, but so far we have not been able to use it when coefficients do not match.

In this lesson, we will again look at the method of elimination that we learned in Lesson 7.3. However, the equations we will be working with will be more complicated and one can not simply add or subtract to eliminate one variable. Instead, we will first have to multiply equations to ensure that the coefficients of one of the variables are matched in the two equations.

Quick Review: Multiplying Equations

Consider the following questions

1. *If* 10 *apples cost* \$5*, how much would* 30 *apples cost?*
2. *If 3 bananas plus* 2 *carrots cost* \$4*, how much would* 6 *bananas plus* 4 *carrots cost?*

You can look at the first equation, and it should be obvious that each apple costs \$0.50. 30 apples should cost \$15.00.

Looking at the second equation, it is not clear what the individual price is for either bananas or carrots. Yet we know that the answer to question two is \$8.00. How?

If we look again at question one, we see that we can write the equation $10a = 5$ (a being the cost of one apple).

To find the cost of 30, we can either solve for a them multiply by 30, or we can multiply both sides of the equation by three.

$$30a = 15$$
$$a = \frac{1}{2} \text{ or } 0.5$$

Now look at the second question. We could write the equation $3b + 2c = 4$.

We see that we need to solve for $(6b + 4c)$ which is simply two times the quantity $(3b + 2c)$!

Algebraically, we are simply multiplying the entire equation by two.

$$2(3b + 2c) = 2 \cdot 4 \qquad \text{Distribute and multiply.}$$
$$6b + 4c = 8$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

We can multiply every term in an equation by a fixed number (a **scalar**), it is clear that we could use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match. In the simplest case, the coefficient as a variable in one equation will be a multiple of the coefficient in the other equation.

Example 1

Solve the system.

$$7x + 4y = 17$$
$$5x - 2y = 11$$

It is quite simple to see that by multiplying the second equation by two the coefficients of y will be $+4$ and -4, allowing us to complete the solution by addition.

Take two times equation two and add it to equation one. Then divide both sides by 17 to find x.

$$10x - 4y = 22$$
$$+ (7x + 4y) = 17$$
$$\overline{\qquad\qquad\qquad}$$
$$17y = 34$$
$$x = 2$$

Now simply substitute this value for x back into equation one.

$$
\begin{aligned}
7 \cdot 2 + 4y &= 17 && \text{Since } 7 \times 2 = 14, \text{ subtract 14 from both sides.}\\
4y &= 3 && \text{Divide by 4.}\\
y &= 0.75
\end{aligned}
$$

Example 2

Anne is rowing her boat along a river. Rowing downstream, it takes her two minutes to cover 400 yards. Rowing upstream, it takes her eight minutes to travel the same 400 yards. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Step One First we convert our problem into equations. We need to know that *distance traveled* is equal to *speed* x time. We have two unknowns, so we will call the speed of the river x, and the speed that Anne rows at y. When traveling downstream her total speed is the boat speed plus the speed of the river $(x+y)$. Upstream her speed is hindered by the speed of the river. Her speed upstream is $(x-y)$.

$$
\begin{aligned}
\text{Downstream Equation} &\qquad 2(x+y) = 400\\
\text{Upstream Equation} &\qquad 8(x-y) = 400
\end{aligned}
$$

Distributing gives us the following system.

$$
\begin{aligned}
2x + 2y &= 400\\
8x - 8y &= 400
\end{aligned}
$$

Right now, we cannot use the method of elimination as none of the coefficients match. But, if we were to multiply the top equation by four, then the coefficients of y would be $+8$ and -8. Let's do that.

$$
\begin{aligned}
8x - 8y &= 1,600\\
+ (8x - 8y) &= 400\\
\hline
16x &= 2,000
\end{aligned}
$$

Now we divide by 16 to obtain $x = 125$.

Substitute this value back into the first equation.

$$
\begin{aligned}
2(125 + y) &= 400 && \text{Divide both sides by 2.}\\
125 + y &= 200 && \text{Subtract 125 from both sides.}\\
y &= 75
\end{aligned}
$$

Solution

Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

Solve a Linear System by Multiplying Both Equations

It is a straightforward jump to see what would happen if we have no matching coefficients and no coefficients that are simple multiples of others. Just think about the following fraction sum.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

This is an example of finding a **lowest common denominator**. In a similar way, we can always find a lowest common multiple of two numbers (the **lowest common multiple** of 2 and 3 is 6). This way we can always find a way to multiply equations such that two coefficients match.

Example 3

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2,060 miles *in five days. Anne travels* 880 miles *from Norfolk, Virginia, and it takes her three days. If Anne pays* \$840 *and Andrew pays* \$1845*, what does I-Haul charge*

a) per day?

b) per mile traveled?

First, we will set-up our equations. Again we have two unknowns, the **daily rate** (we will call this x), and the **rate per mile** (let's call this y).

Anne's equation	$3x+880y=840$
Andrew's Equation	$5x+2060y=1845$

We cannot simply multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y) we see that they both have a common multiple of 15 (in fact 15 is the **lowest common multiple**). So this time we need to multiply both equations:

Multiply Anne's equation by five:

$$15x+4400y=4200$$

Multiply Andrew's equation by three:

$$15x+6180y=5535$$

Subtract:

$$\begin{aligned} 15x+4400y&=4200\\ -(15x+6180y)&=5535\\ \hline -1780y&=-1335 \end{aligned}$$

Divide both sides by -1780

$$y=0.75$$

Substitute this back into Anne's equation.

$$3x + 880(0.75) = 840 \quad \text{Since } 880 \times 0.75 = 660, \text{ subtract 660 from both sides.}$$
$$3x = 180 \quad \text{Divide both sides by 3.}$$
$$x = 60$$

Solution

I-Haul charges $60 per day plus $0.75 per mile.

Comparing Methods for Solving Linear Systems

Now that we have covered the major methods for solving linear equations, let's review. For simplicity, we will look at the four methods (we will consider **addition** and **subtraction** one method) in table form. This should help you decide which method would be better for a given situation.

Table 7.2:

Method:	Best used when you...	Advantages:	Comment:
Graphing	...don't need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don't have to do any computation.	Can lead to imprecise answers with non-integer solutions.
Substitution	...have an explicit equation for one variable (e.g. $y = 14x + 2$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, thus it can lead to more complicated formulas
Elimination by Addition or Subtraction	...have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	...do not have any variables defined explicitly or any matching coefficients.	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

The table above is only a guide. You may like to use the graphical method for every system in order to better understand what is happening, or you may prefer to use the multiplication method even when a substitution would work equally well.

Example 4

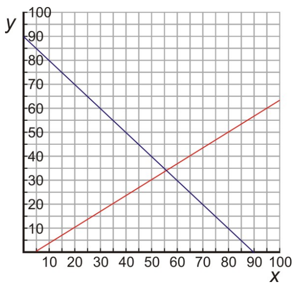
*Two angles are **complementary** when the sum of their angles is 90°. Angles A and B are complementary angles, and twice the measure of angle A is 9° more than three times the measure of angle B. Find the measure of each angle.*

First, we write out our two equations. We will use x to be the measure of Angle A and y to be the measure of Angle B. We get the following system

$$x + y = 90$$
$$2x = 3y + 9$$

The first method we will use to solve this system is the graphical method. For this we need to convert the two equations to $y = mx + b$ form

$$x + y = 90 \quad \Rightarrow \quad y = -x + 90$$
$$2x = 3y + 9 \quad \Rightarrow \quad y = \frac{2}{3}x - 3$$



The first line has a slope of -1 and a $y-$intercept of 90.

The second has a slope of $\frac{2}{3}$ and a $y-$intercept of -3.

In the graph, it appears that the lines cross at around $x = 55$, $y = 35$ but it is difficult to tell exactly! Graphing by hand is not the best method if you need to know the answer with more accuracy!

Next, we will try to solve by substitution. Let's look again at the system:

$$x + y = 90$$
$$2x = 3y + 9$$

We have already seen that we can solve for y with either equation in trying to solve the system graphically.

Solve the first equation for y.

$$y = 90 - x$$

Substitute into the second equation

$$2x = 3(90 - x) + 9 \qquad \text{Distribute the 3.}$$
$$2x = 270 - 3x + 9 \qquad \text{Add } 3x \text{ to both sides.}$$
$$5x = 270 + 9 = 279 \qquad \text{Divide by 5.}$$
$$x = 55.8^\circ$$

Substitute back into our expression for y.

$$y = 90 - 55.8 = 34.2^\circ$$

Solution

Angle A measures 55.8°. Angle B measures 34.2°

Finally, we will examine the method of elimination by multiplication.

Rearrange equation one to standard form

$$x + y = 90 \Rightarrow 2x + 2y = 180$$

Multiply equation two by two.

$$2x = 3y + 9 \Rightarrow 2x - 3y = 9$$

Subtract.

$$\begin{aligned} 2x + 2y &= 180 \\ -(2x - 2y) &= -9 \\ \hline 5y &= 171 \end{aligned}$$

Divide both sides by 5

$$y = 34.2$$

Substitute this value into the very first equation.

$$\begin{aligned} x + 34.2 &= 90 && \text{Subtract 34.2 from both sides.} \\ x &= 55.8^\circ \end{aligned}$$

Solution

Angle A measures 55.8°. *Angle B measures* 34.2°.

Even though this system looked ideal for substitution, the method of multiplication worked well, also. Once the algebraic manipulation was performed on the equations, it was a quick solution. You will need to decide yourself which method to use in each case you see from now on. Try to master all techniques, and recognize which technique will be most efficient for each system you are asked to solve.

Multimedia Link For even more practice, we have this video. One common type of problem involving systems of equations (especially on standardized tests) is "age problems." In the following video the narrator shows two examples of age problems, one involving a single person and one involving two people. Khan Academy Age Problems (7:13) .

Review Questions

1. Solve the following systems using multiplication.
   (a) $5x - 10y = 15$
   $3x - 2y = 3$
   (b) $5x - y = 10$
   $3x - 2y = -1$
   (c) $5x + 7y = 15$
   $7x - 3y = 5$

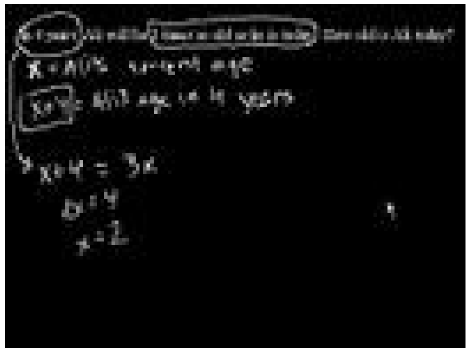


Figure 7.2: Age word problems (Watch on Youtube)

(d) $9x + 5y = 9$
$12x + 8y = 12.8$
(e) $4x - 3y = 1$
$3x - 4y = 4$
(f) $7x - 3y = -3$
$6x + 4y = 3$

2. Solve the following systems using any method.

(a) $x = 3y$
$x - 2y = -3$
(b) $y = 3x + 2$
$y = -2x + 7$
(c) $5x - 5y = 5$
$5x + 5y = 35$
(d) $y = -3x - 3$
$3x - 2y + 12 = 0$
(e) $3x - 4y = 3$
$4y + 5x = 10$
(f) $9x - 2y = -4$
$2x - 6y = 1$

3. Supplementary angles are two angles whose sum is 180°. Angles A and B are supplementary angles. The measure of Angle A is 18° less than twice the measure of Angle B. Find the measure of each angle.
4. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?
5. A 150 yards pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?
6. Mr. Stein invested a total of $100,000 in two companies for a year. Company A's stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?
7. A baker sells plain cakes for $7 or decorated cakes for $11. On a busy Saturday the baker started with 120 cakes, and sold all but three. His takings for the day were $991. How many plain cakes did he sell that day, and how many were decorated before they were sold?
8. Twice John's age plus five times Claire's age is 204. Nine times John's age minus three times Claire's age is also 204. How old are John and Claire?

Review Answers

1.
2. (a) $x = 0$, $y = -1.5$
   (b) $x = 3$, $y = 5$
   (c) $x = 1.25$, $y = 1.25$
   (d) $x = \frac{2}{3}$, $y = \frac{3}{5}$
   (e) $x = -\frac{8}{7}$, $y = -\frac{13}{7}$
   (f) $x = -\frac{3}{46}$, $y = \frac{39}{46}$
3.
4. (a) $x = -9$, $y = -3$
   (b) $x = 1$, $y = 5$
   (c) $x = 4$, $y = 3$
   (d) $x = -2$, $y = 3$
   (e) $x = \frac{13}{8}$, $y = \frac{15}{32}$
   (f) $x = -\frac{13}{25}$, $y = -\frac{17}{50}$
5. $A = 114°$, $B = 66°$
6. 30 liters of 5%, 70 liters of 15%
7. 51 yards and 99 yards
8. $68,750 in Company A, $31,250 in Company B
9. 74 plain, 43 decorated
10. John is 32, Claire is 28

7.5 Special Types of Linear Systems

Learning Objectives

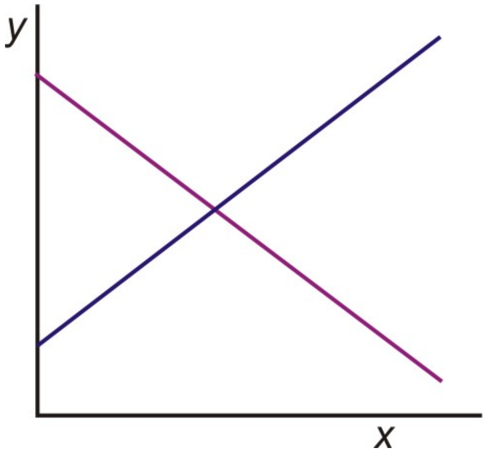
- Identify and understand what is meant by an **inconsistent linear system.**
- Identify and understand what is meant by a **consistent linear system.**
- Identify and understand what is meant by a **dependent linear system.**

Introduction

As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Determining the Type of System Graphically

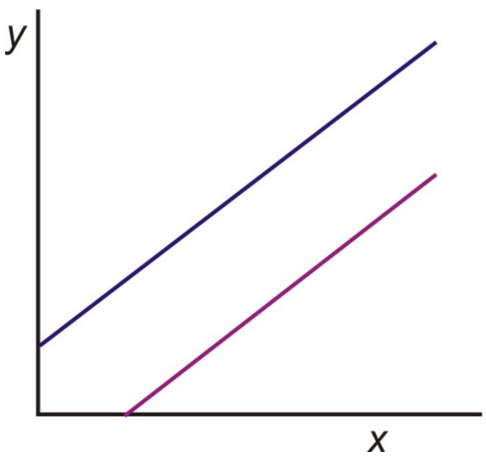
If we graph two lines on the same coordinate plane, three different situations may occur.



Case 1: The two lines intersect at a single point; hence the lines are not parallel.

If these lines were to represent a system of equations, the system would exactly one solution, where the lines cross.

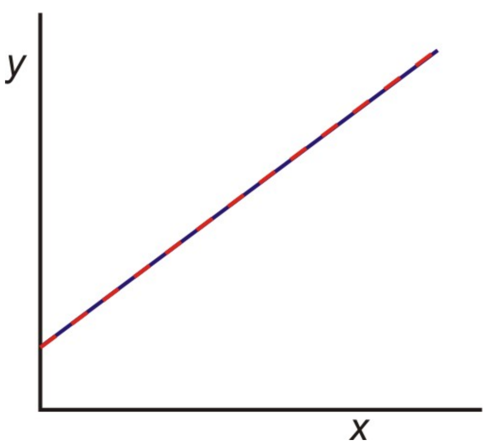
A system with exactly one solution is called a **consistent system.**



Case 2: The two lines do not intersect. The two lines are parallel.

If the lines represent a system of equations, then the system has no solutions.

A system with no solutions is called an **inconsistent system.**



Case 3: The two lines are identical. They intersect at all points on the line.

If this were a system of equations it would have an **infinite number** of solutions. Reason being, the two equations are really the same.

Such a system is called a **dependent system.**

To identify a system as **consistent**, **inconsistent**, or **dependent**, we can graph the two lines on the same graph and match the system with one of the three cases we discussed.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$intercepts of the two lines. To do this we must remember that:

- Lines that intersect have different slopes.
- Lines that are parallel have the same slope but different $y-$intercepts.
- Lines that have the same slope and the same $y-$intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$y = 3x + 2$$
$$y = -2x + 1$$

Solution

The equations are already in slope-intercept form. The slope of the first equation is 3 and the slope of the second equation is -2. Since the slopes are different, the lines must intersect at a single point. Therefore, the system has exactly one solution. This is a **consistent system**.

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$2x - 5y = 2$$
$$4x + y = 5$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{array}{ccccc} 2x - 5y = 2 & & -5y = -2x + 2 & & y = \frac{2}{5}x - \frac{2}{5} \\ & \Rightarrow & & \Rightarrow & \\ 4x + y = 5 & & y = -4x + 5 & & y = -4x + 5 \end{array}$$

The slopes of the two equations are different. Therefore, the lines must cross at a single point, and the system has exactly one solution. This is a **consistent system**.

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$3x = 5 - 4y$$
$$6x + 8y = 7$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{aligned} 3x &= 5 - 4y & & 4y = -3x + 5 & & y = \frac{-3}{4}x - \frac{5}{4} \\ & & \Rightarrow & & \Rightarrow & \\ 6x + 8y &= 7 & & 8y = -6x + 7 & & y = \frac{-3}{4}x + \frac{7}{8} \end{aligned}$$

The slopes of the two equations are the same but the $y-$intercepts are different, therefore the lines never cross and the system has no solutions. This is an **inconsistent system**.

Example 4

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x + y &= 3 \\ 3x + 3y &= 9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form.

$$\begin{aligned} x + y &= 3 & & y = -x + 3 & & y = -x + 3 \\ & & \Rightarrow & & \Rightarrow & \\ 3x + 3y &= 9 & & 3y = -3x + 9 & & y = -x + 3 \end{aligned}$$

The lines are identical. Therefore the system has an infinite number of solutions. It is a **dependent system.**

Determining the Type of System Algebraically

A third option for identifying systems as **consistent**, inconsistent or dependent is to solve the system algebraically using any method and use the result as a guide.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x - 3y &= 3 \\ 2x + y &= 9 \end{aligned}$$

Solution

Let's solve this system using the substitution method.

Solve the second equation for the y variable.

$$2x + y = 9 \Rightarrow y = -2x + 9$$

Substitute for y in the first equation.

$$\begin{aligned}10x - 3y &= 3\\10x - 3(-2x + 9) &= 3\\10x + 6x - 27 &= 3\\16x &= 30\\x &= \frac{15}{8}\end{aligned}$$

Substitute the value of x back into the second equation and solve for y.

$$2x + y = 9 \Rightarrow y = -2x + 9 \Rightarrow y = -2 \cdot \frac{15}{8} + 9 \Rightarrow y = \frac{21}{4}$$

Answer The solution to the system is $\left(\frac{15}{8}, \frac{21}{4}\right)$. The system is consistent since it has only one solution.

Another method to determine if the system of equations is an inconsistent, consistent or dependent system is to solve them algebraically using the elimination or substitution method.

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}3x - 2y &= 4\\9x - 6y &= 1\end{aligned}$$

Solution

Let's solve this system by the method of multiplication.

Multiply the first equation by 3.

$$\begin{aligned}3(3x - 2y = 4) & & & 9x - 6y = 12\\ & \Rightarrow & & \\9x - 6y = 1 & & & 9x - 6y = 1\end{aligned}$$

Add the two equations.

$$\begin{aligned}9x - 6y &= 12\\9x - 6y &= 1\\\hline 0 &= 13 \quad \text{This Statement is not true}\end{aligned}$$

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is not true, then the system is **inconsistent**.

Example 7

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}4x + y &= 3\\12x + 3y &= 9\end{aligned}$$

Solution

Let's solve this system by substitution.

Solve the first equation for y.

$$4x + y = 3 \Rightarrow y = -4x + 3$$

Substitute this expression for y in the second equation.

$$\begin{aligned} 12x + 3y &= 9 \\ 12x + 3(-4x + 3) &= 9 \\ 12x - 12x + 9 &= 9 \\ 9 &= 9 \end{aligned}$$

This is always a **true** statement.

Answer If, by trying to obtain a solution to a system, we arrive at a statement that is always true, then the system is dependent.

A second glance at the system in this example reveals that the second equation is three times the first equation so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Let's clarify this statement. An infinite number of solutions does not mean that any ordered pair (x, y) satisfies the system of equations. Only ordered pairs that solve the equation in the system are also solutions to the system.

For example, $(1, 2)$ is not a solution to the system because when we plug it into the equations it does not check out.

$$\begin{aligned} 4x + y &= 3 \\ 4(1) + 2 &\neq 3 \end{aligned}$$

To find which ordered pair satisfies this system, we can pick any value for x and find the corresponding value for y.

For $x = 1$, $4(1) + y = 3 \Rightarrow y = -1$

For $x = 2$, $4(2) + y = 3 \Rightarrow y = -3$

Let's summarize our finding for determining the type of system algebraically.

- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will always give a FALSE statement (for example $9 = 0$).
- A **dependent system** will always give a TRUE statement (such as $9 = 9$ or $0 = 0$).

Applications

In this section, we will look at a few application problems and see how consistent, inconsistent and dependent systems might arise in practice.

Example 8

A movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of \$45 *and then rent each movie for* \$2 *or they can choose not to pay the membership fee and rent each movie for* \$3.50*. How many movies would you have to rent before membership becomes the cheaper option?*

Solution

Let's translate this problem into algebra. Since there are two different options to consider, we will write two different equations and form a system.

The choices are membership option and no membership option.

Our variables are

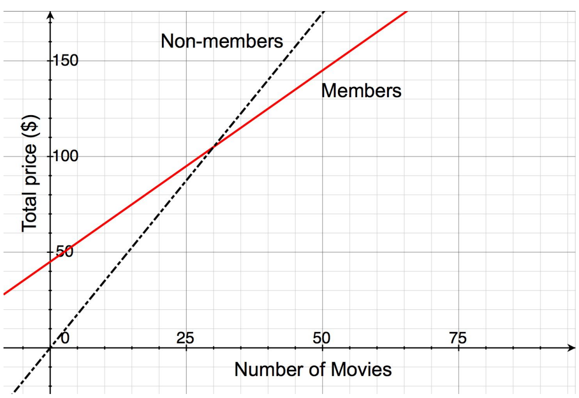
The number of movies you rent, let's call this x.

The total cost for renting movies, let's call this y.

Table 7.3:

	flat fee	rental fee	total
membership	\$45	$2x$	$y = 45 + 2x$
no membership	\$0	$3.50x$	$y = 3.5x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent movies. For the membership option, the rental fee is $2x$ since you would pay \$2 for each movie you rented. For the no membership option, the rental fee is $3.50x$ since you would pay \$3.50 for each movie you rented.



Our system of equations is

$$y = 45 + 2x$$
$$y = 3.50x$$

The graph of our system of equations is shown to the right.

This system can be solved easily with the method of substitution since each equation is already solved for y. Substitute the second equation into the first one

$$\begin{array}{l} y = 45 + 2x \\ \\ y = 3.50x \end{array} \Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies}$$

Answer You would have to rent 30 movies per year before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Let's examine a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (but the $y-$intercept is different). Let's change the previous problem so that this is the case.

Example 9

Two movie rental stores are in competition. Movie House charges an annual membership of \$30 *and charges* \$3 *per movie rental. Flicks for Cheap charges an annual membership of* \$15 *and charges* \$3 *per movie rental. After how many movie rentals would Movie House become the better option?*

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per move as Flicks for Cheap.

The lines that describe each option have different $y-$intercepts, namely 30 for Movie House and 15 for Flicks for Cheap. They have the same slope, three dollars per movie. This means that the lines are parallel and the system is inconsistent.

Let's see how this works algebraically:

Our variables are:

The number of movies you rent, let's call this x.

The total cost for renting movies, let's call this y.

Table 7.4:

	flat fee	rental fee	total
Movie House	\$30	$3x$	$y = 30 + 3x$
Flicks for Cheap	\$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is

$$y = 30 + 3x$$
$$y = 15 + 3x$$

Let's solve this system by substituting the second equation into the first equation.

$$\begin{matrix} y = 30 + 3x \\ \\ y = 15 + 3x \end{matrix} \quad \Rightarrow 15 + 3x = 30 + 3x \Rightarrow 15 = 30$$

This statement is always false.

Answer This means that the system is inconsistent.

Example 10

Peter buys two apples and three bananas for \$4. *Nadia buys four apples and six bananas for* \$8 *from the same store. How much does one banana and one apple costs?*

Solution

We must write two equations, one for Peter's purchase and one for Nadia's purchase.

Let's define our variables as

a is the cost of one apple

b is the cost of one banana

Table 7.5:

	Cost of Apples	Cost of Bananas	Total Cost
Peter	$2a$	$3b$	$2a + 3b = 4$
Nadia	$4a$	$6b$	$4a + 6b = 8$

The system of equations that describes this problem is:

$$2a + 3b = 4$$
$$4a + 6b = 8$$

Let's solve this system by multiplying the first equation by -2 and adding the two equations.

$$\begin{aligned} -2(2a + 3b = 4) \\ 4a + 6b = 8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -4a - 6b &= -8 \\ 4a + 6b &= 8 \\ \hline 0 + 0 &= 0 \end{aligned}$$

This statement is always true. This means that the system is **dependent**.

Looking at the problem again, we see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for $4 it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price ($8). Since the second equation does not give any new information, it is not possible to find out the price of each piece of fruit.

Answer The two equations describe the same line. This means that the system is dependent.

Review Questions

1. Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.
   (a) $3x - 4y = 13$
   $y = -3x - 7$
   (b) $\frac{3x}{5} + y = 3$
   $1.2x + 2y = 6$
   (c) $3x - 4y = 13$
   $y = -3x - 7$
   (d) $3x - 3y = 3$
   $x - y = 1$
   (e) $0.5x - y = 30$
   $0.5x - y = -30$
   (f) $4x - 2y = -2$
   $3x + 2y = -12$

2. Find the solution of each system of equations using the method of your choice. Please state whether the system is inconsistent or dependent.
   (a) $3x + 2y = 4$
   $-2x + 2y = 24$
   (b) $5x - 2y = 3$
   $2x - 3y = 10$
   (c) $3x - 4y = 13$
   $y = -3x - y$
   (d) $5x - 4y = 1$
   $-10x + 8y = -30$
   (e) $4x + 5y = 0$
   $3x = 6y + 4.5$
   (f) $-2y + 4x = 8$
   $y - 2x = -4$
   (g) $x - \frac{y}{2} = \frac{3}{2}$
   $3x + y = 6$
   (h) $0.05x + 0.25y = 6$
   $x + y = 24$
   (i) $x + \frac{2y}{3} = 6$
   $3x + 2y = 2$
3. A movie house charges \$4.50 for children and \$8.00 for adults. On a certain day, 1200 people enter the movie house and \$8,375 is collected. How many children and how many adults attended?
4. Andrew placed two orders with an internet clothing store. The first order was for thirteen ties and four pairs of suspenders, and totaled \$487. The second order was for six ties and two pairs of suspenders, and totaled \$232. The bill does not list the per-item price but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
5. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane's speed in still air and the jet-stream's speed?
6. Nadia told Peter that she went to the farmer's market and she bought two apples and one banana and that it cost her \$2.50. She thought that Peter might like some fruit so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so will only pay her for four apples. Nadia told him that the second time she paid \$6.00 for fruit. Please help Peter figure out how much to pay Nadia paid for four apples.

Review Answers

1.
2. (a) consistent
   (b) dependent
   (c) consistent
   (d) dependent
   (e) inconsistent
   (f) consistent
3.
4. (a) $x = -4$, $y = 8$
   (b) $x = -1$, $y = -4$
   (c) $x = -1$, $y = -4$
   (d) inconsistent
   (e) $x = -2.5$, $y = 2$

(f) dependent
(g) $x = \frac{9}{5}, y = \frac{3}{5}$
(h) $x = 0, y = 24$
(i) dependent

5. 350 children, 850 Adults
6. $\text{Ties} = \$23, \text{suspenders} = \$47$
7. $\text{Airplane speed} = 540 \text{ mph}, \text{ jet} - \text{streamspeed} = 60 \text{ mph}$
8. This represents an inconsistent system. Someone is trying to overcharge! It is not possible to determine the price of apples alone.

7.6 Systems of Linear Inequalities

Learning Objectives

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Introduction

In the last chapter, you learned how to graph a linear inequality in two variables. To do that you graphed the equation of the straight line on the coordinate plane. The line was solid for signs where the equal sign is included. The line was dashed for < or > where signs the equal sign is not included. Then you shaded above the line (if $y >$ or $y \geq$) or below the line (if $y <$ or $y \leq$).

In this section, we will learn how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes. Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example 1

Solve the following system.

$$\begin{aligned} 2x + 3y &\leq 18 \\ x - 4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection regions of the solution.

Let's rewrite each equation in slope-intercept form. This form is useful for graphing but also in deciding which region of the coordinate plane to shade. Our system becomes

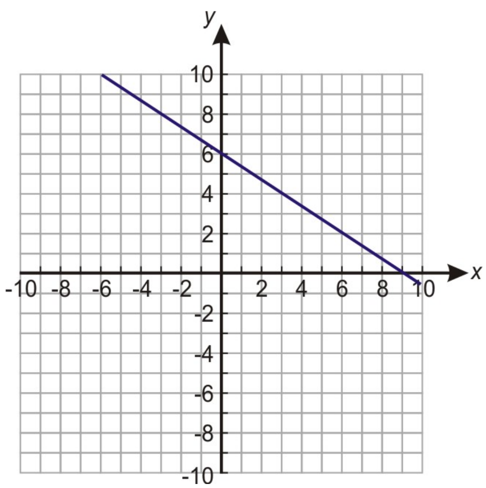
$$3y \leq -2x + 18 \qquad \Rightarrow \qquad y \leq \frac{-2}{3}x + 6$$

$$-4y \leq -x + 12 \qquad \qquad y \geq \frac{x}{4} - 3$$

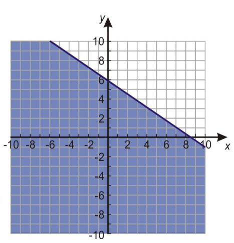
Notice that the inequality sign in the second equation changed because we divided by a negative number.

For this first example, we will graph each inequality separately and then combine the results.

We graph the equation of the line in the first inequality and draw the following graph.

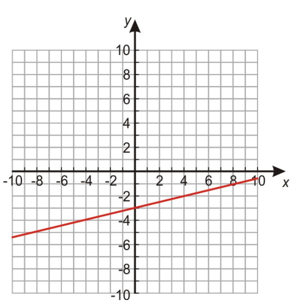


The line is solid because the equal sign is included in the inequality. Since the inequality is less than or equal to, we shade below the line.

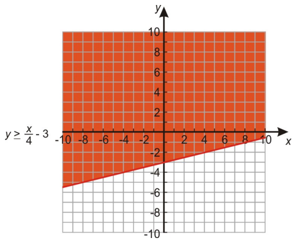


$y \leq -\frac{2}{3}x + 6$

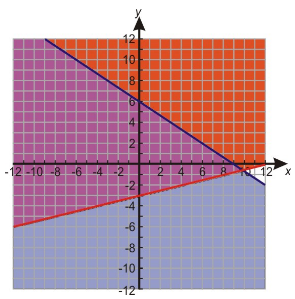
We graph the second equation in the inequality and obtain the following graph.



The line is solid again because the equal sign is included in the inequality. We now shade above because y is **greater** than or equal.



When we combine the graphs, we see that the blue and red shaded regions overlap. This overlap is where both inequalities work. Thus the purple region denotes the solution of the system.



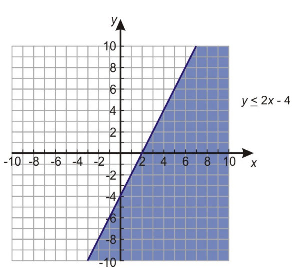
The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

Example 2

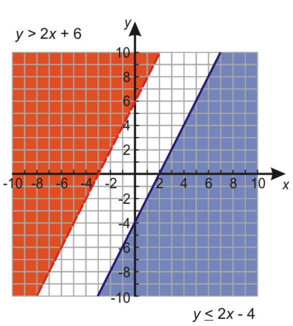
There are also situations where a system of inequalities has no solution. For example, let's solve this system.

$$y \leq 2x - 4$$
$$y > 2x + 6$$

Solution: We start by graphing the first line. The line will be solid because the equal sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equal sign is not included in the inequality. We must shade upward because y is greater than.

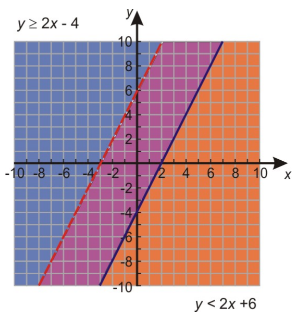


This graph shows no overlapping between the two shaded regions. We know that the lines will never intersect because they are parallel . The slope equals two for both lines. The regions will never overlap even if we extend the lines further.

This is an example of a system of inequalities with no solution.

For a system of inequalities, we can still obtain a solution even if the lines are parallel. Let's change the system of inequalities in Example 2 so the inequality signs for the two expressions are reversed.

$$y \geq 2x - 4$$
$$y < 2x + 6$$



The procedure for solving this system is almost identical to the previous one, except we shade upward for the first inequality and we shade downward for the second inequality. Here is the result.

In this case, the shaded regions do overlap and the system of inequalities has the solution denoted by the purple region.

Graph a System of More Than Two Linear Inequalities

In the previous section, we saw how to find the solution to a system of two linear inequalities. The solutions for these kinds of systems are always unbounded. In other words, the region where the shadings overlap continues infinitely in at least one direction. We can obtain **bounded** solutions by solving systems that contain more than two inequalities. In such cases the solution region will be bounded on four sides.

Let's examine such a solution by solving the following example.

Example 3

Find the solution to the following system of inequalities.

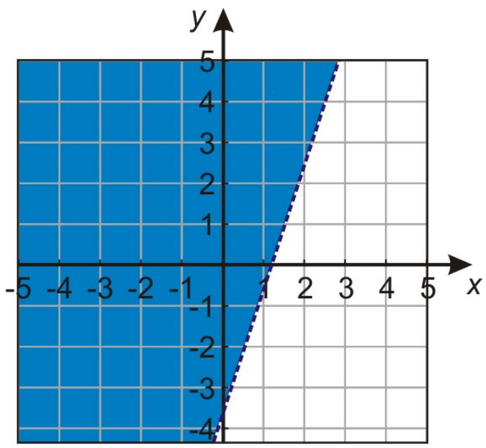
$$3x - y < 4$$
$$4y + 9x < 8$$
$$x \geq 0$$
$$y \geq 0$$

Solution

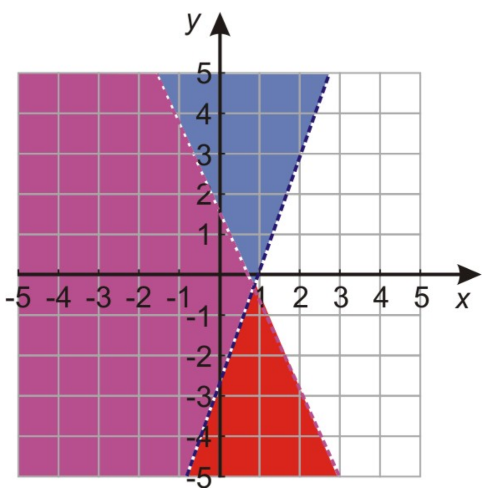
Let's start by writing our equation in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

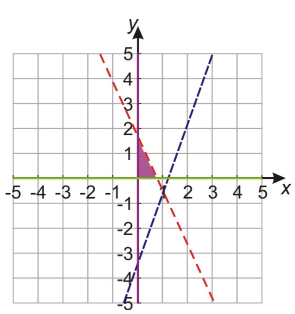
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$.



Next we graph $y < -\frac{9}{4}x + 2$



Finally we graph and $x \geq 0$ and $y \geq 0$, and the intersecting region is shown in the following figure.



The solution is **bounded** because there are lines on all sides of the solution region. In other words the solution region is a bounded geometric figure, in this case a triangle.

Write a System of Linear Inequalities

There are many interesting application problems that involve the use of system of linear inequalities. However, before we fully solve application problems, let's see how we can translate some simple word problems into algebraic equations.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend works in a certain region of the restaurant. The restaurant is also known for its great views but you have to sit in a certain area of the restaurant that offers these view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best views and be served by your friend.

Typically, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints.

Example 4

Write a system of linear inequalities that represents the following conditions.

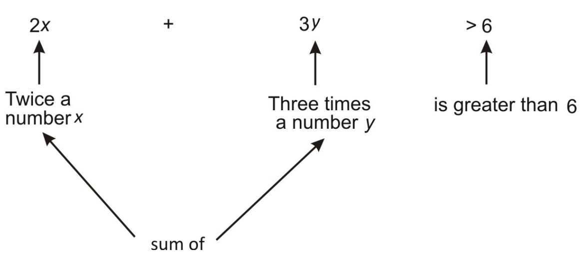
The sum of twice a number x and three times another number y is greater than 6*, and y is less than three times x.*

Solution

Let's take each statement in turn and write it algebraically:

1.The sum of twice a number x and three times another number y is greater than 6.

This can be written as



2. y is less than three times x.

This can be written as

$y <$ ↗ y is less than

$3x$ ↖ 3 times x

The system of inequalities arising from these statements is

$$\begin{aligned} 2x + 3y &> 6 \\ y &< 3x \end{aligned}$$

This system of inequalities can be solved using the method outlined earlier in this section. We will not solve this system because we want to concentrate on learning how to write a system of inequalities from a word problem.

Solve Real-World Problems Using Systems of Linear Inequalities

As we mentioned before, there are many interesting application problems that require the use of systems of linear inequalities. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the best possible value under those conditions. A typical example would be taking the limitations of materials and labor, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real life systems can have dozens or hundreds of variables, or more. In this section, we will only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the x,y−plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the systems of equations that give the solutions to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

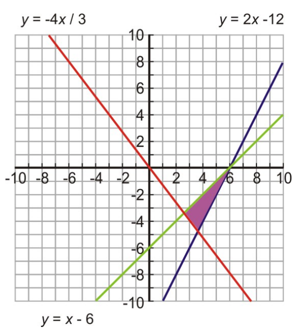
Example 5

Find the maximum and minimum value of $z = 2x + 5y$ *given the constraints.*

$$\begin{aligned} 2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\leq 6 \end{aligned}$$

Solution

Step 1: Find the solution to this system of linear inequalities by graphing and shading appropriately. To graph we must rewrite the equations in slope-intercept form.



$$\begin{aligned} y &\geq 2x - 12 \\ y &\geq -\frac{4}{3}x \\ y &\geq x - 6 \end{aligned}$$

These three linear inequalities are called the **constraints**.

The solution is the shaded region in the graph. This is called the **feasibility region**. That means all possible solutions occur in that region. However in order to find the optimal solution we must go to the next steps.

Step 2

Next, we want to find the corner points. In order to find them exactly, we must form three systems of linear equations and solve them algebraically.

System 1:

$$y = 2x - 12$$
$$y = -\frac{4}{3}x$$

Substitute the first equation into the second equation:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$

$$y = 2x - 12 \Rightarrow y - 2(3.6) - 12 \Rightarrow y = -4.8$$

The intersection point of lines is $(3.6, -4.8)$

System 2:

$$y = 2x - 12$$
$$y = x - 6$$

Substitute the first equation into the second equation.

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$

$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0$$

The intersection point of lines is $(6, 0)$.

System 3:

$$y = -\frac{4}{3}x$$
$$y = x - 6$$

Substitute the first equation into the second equation.

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$

$$y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43$$

The intersection point of lines is $(2.57, -3.43)$.

So the corner points are $(3.6, -4.8)$, $(6, 0)$ and $(2.57, -3.43)$.

Step 3

Somebody really smart proved that, for linear systems like this, the maximum and minimum values of the optimization equation will always be on the corners of the feasibility region. So, to find the solution to this exercise, we need to plug these three points into $z = 2x + 5y$.

$(3.6, -4.8)$	$z = 2(3.6) + 5(-4.8) = -16.8$
$(6, 0)$	$z = 2(6) + 5(0) = 12$
$(2.57, -3.43)$	$z = 2(2.57) + 5(-3.43) = -12.01$

The highest value of 12 occurs at point $(6, 0)$ and the lowest value of -16.8 occurs at $(3.6, -4.8)$.

In the previous example, we learned how to apply the method of linear programming out of context of an application problem. In the next example, we will look at a real-life application.

Example 6

You have \$10,000 *to invest, and three different funds from which to choose. The municipal bond fund has a* 5% *return, the local bank's CDs have a* 7% *return, and a high-risk account has an expected* 10% *return. To minimize risk, you decide not to invest any more than* \$1,000 *in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. Assuming the year-end yields are as expected, what are the optimal investment amounts?*

Solution

Let's define some *variables*.

x is the amount of money invested in the municipal bond at 5% return.

y is the amount of money invested in the bank's CD at 7% return.

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return.

z is the total interest returned from all the investments or $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z.

Now, let's write inequalities for the *constraints*.

You decide not to invest more than \$1000 in the high-risk account.

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs.

$$3y \leq x$$

Also we write expressions for the fact that we invest more than zero dollars in each account.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 10000 - x - y &\geq 0 \end{aligned}$$

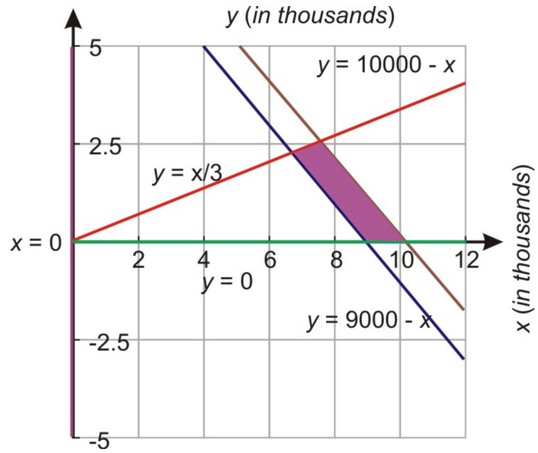
To summarize, we must maximize the expression $z = 1000 - .05x - .03y$.

Using the constraints,

$10000 - x - y \leq 1000$		$y \geq 9000 - x$
$3y \leq x$		$y \leq \frac{x}{3}$
$x \geq 0$	Let's rewrite each in slope-intercept form.	$x \geq 0$
$y \geq 0$		$y \geq 0$
$10000 - x - y \geq 0$		$y \leq 10000 - x$

Step 1 Find the solution region to the set of inequalities by graphing each line and shading appropriately.

The following figure shows the overlapping region.



The purple region is the feasibility region where all the possible solutions can occur.

Step 2 Next, we need to find the corner points of the shaded solution region. Notice that there are four intersection points. To find them we must pair up the relevant equations and solve the resulting system.

System 1:

$$y = -\frac{x}{3}x$$
$$y = 10000 - x$$

Substitute the first equation into the second equation.

$$\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow x = 7500$$
$$y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500$$

The intersection point is $(7500, 2500)$.

System 2:

$$y = -\frac{x}{3}x$$
$$y = 9000 - x$$

Substitute the first equation into the second equation.

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$

$$\frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is $(6750, 2250)$.

System 3:

$$y = 0$$
$$y = 10000 - x$$

The intersection point is $(10000, 0)$.

System 4:

$$y = 0$$
$$y = 9000 - x$$

The intersection point is $(9000, 0)$.

Step 3: In order to find the maximum value for z, we need to plug all intersection points into z and take the largest number.

$(7500, 2500)$	$z = 1000 - 0.05(7500) - 0.03(2500) = 550$
$(6750, 2250)$	$z = 1000 - 0.05(6750) - 0.03(2250) = 595$
$(10000, 0)$	$z = 1000 - 0.05(10000) - 0.03(0) = 500$
$(9000, 0)$	$z = 1000 - 0.05(9000) - 0.03(0) = 550$

Answer

The maximum return on the investment of \$595 occurs at point $(6750, 2250)$. This means that

\$6,750 is invested in the municipal bonds.

\$2,250 is invested in the bank CDs.

\$1,000 is invested in the high-risk account.

Review Questions

Find the solution region of the following systems of inequalities

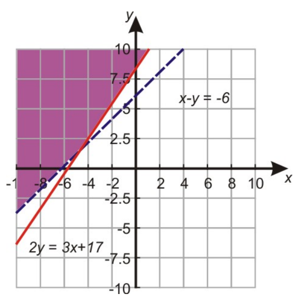
1. $x - y < -6$
   $2y \geq 3x + 17$
2. $4y - 5x < 8$
   $-5x \geq 16 - 8y$
3. $5x - y \geq 5$
   $2y - x \geq -10$
4. $5x + 2y \geq -25$
   $3x - 2y \leq 17$
   $x - 6y \geq 27$
5. $2x - 3y \leq 21$
   $x + 4y \leq 6$
   $3x + y \geq -4$
6. $12x - 7y < 120$
   $7x - 8y \geq 36$
   $5x + y \geq 12$

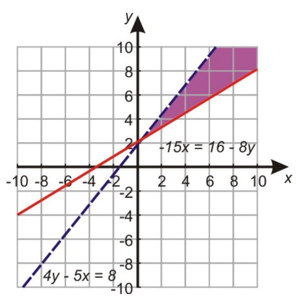
Solve the following linear programming problems:

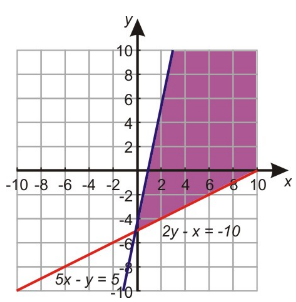
7. Given the following constraints find the maximum and minimum values of $z = -x + 5y$
   $x + 3y \leq 0$
   $x - y \geq 0$
   $3x - 7y \leq 16$

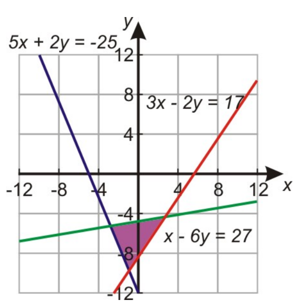
8. In Andrew's Furniture Shop, he assembles both bookcases and TV cabinets. Each type of furniture takes him about the same time to assemble. He figures he has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost him \$20 and the materials for each TV stand costs him \$45 . He has \$600 to spend on materials. Andrew makes a profit of \$60 on each bookcase and a profit of \$100 for each TV stand.
Find how many of each piece of furniture Andrew should make so that he maximizes his profit.

Review Answers

1. 

2. 

3. 

4. 

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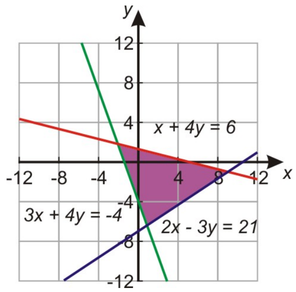

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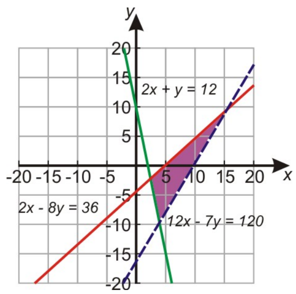


7. Maximum of $z = 0$ at point $(0,0)$, minimum of $z = -16$ at point $(-4,-4)$
8. Maximum profit of \$1,440 by making 9 bookcases and 9 TV stands.

Chapter 8

Exponential Functions

8.1 Exponent Properties Involving Products

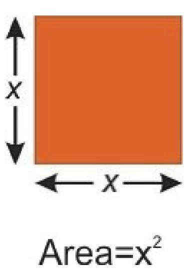
Learning Objectives

- Use the product of a power property.
- Use the power of a product property.
- Simplify expressions involving product properties of exponents.

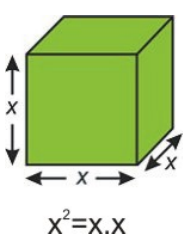
Introduction

In this chapter, we will discuss exponents and exponential functions. In Lessons 8.1, 8.2 and 8.3, we will be learning about the rules governing exponents. We will start with what the word exponent means.

Consider the area of the square shown right. We know that the area is given by:



But we also know that for any rectangle, Area = (width)   (height), so we can see that:



Similarly, the volume of the cube is given by:

$$\text{Volume} = \text{width} \cdot \text{depth} \cdot \text{height} = x \cdot x \cdot x$$

But we also know that the volume of the cube is given by Volume = x^3 so clearly

$$x^3 = x \cdot x \cdot x$$

You probably know that the **power** (the small number to the top right of the x) tells you how many x's to multiply together. In these examples the x is called the **base** and the **power** (or **exponent**) tells us how many **factors** of the **base** there are in the full expression.

$$x^2 = \underbrace{x \cdot x}_{\text{2 factors of x}} \qquad x^7 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\text{7 factors of x}}$$

$$x^3 = \underbrace{x \cdot x \cdot x}_{\text{3 factors of x}} \qquad x^n = \underbrace{x \cdot x \cdot \ldots\ldots \cdot x}_{\text{n factors of x}}$$

Example 1

Write in exponential form.

(a) $2 \cdot 2$

(b) $(-3)(-3)(-3)$

(c) $y \cdot y \cdot y \cdot y \cdot y$

(d) $(3a)(3a)(3a)(3a)$

Solution

(a) $2 \cdot 2 = 2^2$ *because we have* 2 *factors of* 2

(b) $(-3)(-3)(-3) = (-3)^3$ *because we have* 3 *factors of* (-3)

(c) $y \cdot y \cdot y \cdot y \cdot y = y^5$ *because we have* 5 *factors of* y

(d) $(3a)(3a)(3a)(3a) = (3a)^4$ *because we have* 4 *factors of* $3a$

When we deal with numbers, we usually just simplify. We'd rather deal with 16 than with 2^4. However, with variables, we need the exponents, because we'd rather deal with x^7 than with $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

Let's simplify Example 1 by evaluating the numbers.

Example 2

Simplify.

(a) $2 \cdot 2$

(b) $(-3)(-3)(-3)$

(c) $y \cdot y \cdot y \cdot y \cdot y$

(d) $(3a)(3a)(3a)(3a)$

Solution

(a) $2 \cdot 2 = 2^2 = 4$

(b) $(-3)(-3)(-3) = (-3)^3 = -27$

(c) $y \cdot y \cdot y \cdot y \cdot y = y^5$

(d) $(3a)(3a)(3a)(3a) = (3a)^4 = 3^4 \cdot a^4 = 81a^4$

Note: You must be careful when taking powers of negative numbers. Remember these rules.

(negative number) $\cdot$ (positive number) = negative number

(negative number) $\cdot$ (negative number) = positive number

For **even powers of negative numbers,** the answer is always positive. Since we have an even number of factors, we make pairs of negative numbers and all the negatives cancel out.

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} = +64$$

For **odd powers of negative numbers,** the answer is always negative. Since we have an odd number of factors, we can make pairs of negative numbers to get positive numbers but there is always an unpaired negative factor, so the answer is negative:

Ex: $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2} = -32$

Use the Product of Powers Property

What happens when we multiply one power of x by another? See what happens when we multiply x ***to the power*** 5 ***by*** x ***cubed.*** To illustrate better we will use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So $x^5 \cdot x^3 = x^8$. You may already see the pattern to multiplying powers, but let's confirm it with another example. We will multiply x ***squared by*** x ***to the power*** 4:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So $x^2 \cdot x^4 = x^6$. Look carefully at the powers and how many factors there are in each calculation. 5 factors of x times 3 factors of x equals $(5+3) = 8$ factors of x. 2 factors of x times 4 factors of x equals $(2+4) = 6$ factors of x.

You should see that when we take the product of two powers of x, the number of factors of x in the answer is the sum of factors in the terms you are multiplying. In other words the exponent of x in the answer is the sum of the exponents in the product.

Product rule for exponents: $x^n \cdot x^m = x^{n+m}$

Example 3

Multiply $x^4 \cdot x^5$.

Solution

$x^4 \cdot x^5 = x^{4+5} = x^9$

When multiplying exponents of the same base, it is a simple case of adding the exponents. It is important that when you use the product rule you avoid easy-to-make mistakes. Consider the following.

Example 4

Multiply $2^2 \cdot 2^3$.

Solution

$$2^2 \cdot 2^3 = 2^5 = 32$$

Note that when you use the product rule you DO NOT MULTIPLY BASES. In other words, you must avoid the common error of writing ~~$2^2 \cdot 2^3 = 4^5$~~. Try it with your calculator and check which is right!

Example 5

Multiply $2^2 \cdot 3^3$.

Solution

$2^2 \cdot 3^3 = 4 \cdot 27 = 108$

In this case, the bases are different. The product rule for powers ONLY APPLIES TO TERMS THAT HAVE THE SAME BASE. Common mistakes with problems like this include ~~$2^2 \cdot 3^3 = 6^5$~~.

Use the Power of a Product Property

We will now look at what happens when we raise a whole expression to a power. Let's take x ***to the power*** 4 and ***cube it.*** Again we will us the full factored form for each.

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4 \qquad \text{3 factors of } x \text{ to the power 4.}$$

$$\underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^{12}}$$

So $(x^4)^3 = x^{12}$. It is clear that when we raise a power of x to a new power, the powers multiply.

When we take an expression and raise it to a power, we are multiplying the existing powers of x by the power above the parenthesis.

Power rule for exponents: $(x^n)^m = x^{n \cdot m}$

Power of a product

If we have a product inside the parenthesis and a power on the parenthesis, then the power goes on each element inside. So that, for example, $(x^2y)^4 =)(x^2)^4 \cdot (y)^4 = x^8y^4$. Watch how it works the long way.

$$\underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y)}_{x^8y^4}$$

Power rule for exponents: $(x^n)^m = x^{nm}$ and $(x^ny^m)^p = x^{np}y^{mp}$

WATCH OUT! This does NOT work if you have a sum or difference inside the parenthesis. For example, $(x + y)^2 \neq x^2 + y^2$. This is a commonly made mistake. It is easily avoidable if you remember what an exponent means $(x + y)^2 = (x + y)(x + y)$. We will learn how to simplify this expression in a later chapter.

Let's apply the rules we just learned to a few examples.

When we have numbers, we just evaluate and most of the time it is not really important to use the product rule and the power rule.

Example 6

Simplify the following expressions.

(a) $3^4 \cdot 3^7$

(b) $2^6 \cdot 2$

(c) $(4^2)^3$

Solution

In each of the examples, we want to evaluate the numbers.

(a) Use the product rule first: $3^5 \cdot 3^7 = 3^{12}$

Then evaluate the result: $3^{12} = 531,441$

OR

We can evaluate each part separately and then multiply them. $3^5 \cdot 3^7 = 243 \cdot 2,187 = 531,441$.

Use the product rule first. $2^6 \cdot 2 = 2^7$

Then evaluate the result. $2^7 = 128$

OR

We can evaluate each part separately and then multiply them. $2^6 \cdot 2 = 64 \cdot 2 = 128$

(c) Use the power rule first. $(4^2)^3 = 4^6$

Then evaluate the result. $4^6 = 4096$

OR

We evaluate inside the parenthesis first. $(4^2)^3 = (16)^3$

Then apply the power outside the parenthesis. $(16)^3 = 4096$

When we have just one variable in the expression then we just apply the rules.

Example 7

Simplify the following expressions.

(a) $x^2 \cdot x^7$

(b) $(y^3)^5$

Solution

(a) Use the product rule. $x^2 \cdot x^7 = x^{2+7} = x^9$

(b) Use the power rule. $(y^3)^5 = y^{3 \cdot 5} = y^{15}$

When we have a mix of numbers and variables, we apply the rules to the numbers or to each variable separately.

Example 8

Simplify the following expressions.

(a) $(3x^2y^3) \cdot (4xy^2)$

(b) $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$

(c) $(2a^3b^3)^2$

Solution

(a) We group like terms together.

$$(3x^2y^3) \cdot (4xy^2) = (3 \cdot 4) \cdot (x^2 \cdot x) \cdot (y^3 \cdot y^2)$$

We multiply the numbers and apply the product rule on each grouping.

$$12x^3y^5$$

(b) We groups like terms together.

$$(4xyz) \cdot (x^2y^3) \cdot (2yz^4) = (4 \cdot 2) \cdot (x \cdot x^2) \cdot (y \cdot y^3 \cdot y) \cdot (z \cdot z^4)$$

We multiply the numbers and apply the product rule on each grouping.

$$8x^3y^5z^5$$

(c) We apply the power rule for each separate term in the parenthesis.

$$(2a^3b^3)^2 = 2^2 \cdot (a^3)^2 \cdot (b^3)^2$$

We evaluate the numbers and apply the power rule for each term.

$$4a^6b^6$$

In problems that we need to apply the product and power rules together, we must keep in mind the order of operation. Exponent operations take precedence over multiplication.

Example 9

Simplify the following expressions.

(a) $(x^2)^2 \cdot x^3$

(b) $(2x^2y) \cdot (3xy^2)^3$

(c) $(4a^2b^3)^2 \cdot (2ab^4)^3$

Solution

(a) $(x^2)^2 \cdot x^3$

We apply the power rule first on the first parenthesis.

$$(x^2)^2 \cdot x^3 = x^4 \cdot x^3$$

Then apply the product rule to combine the two terms.

$$x^4 \cdot x^3 = x^7$$

(b) $(2x^2y) \cdot (3xy^2)^3$

We must apply the power rule on the second parenthesis first.

$$(2x^2y) \cdot (3xy^2)^3 = (2x^2y) \cdot (27x^3y^6)$$

Then we can apply the product rule to combine the two parentheses.

$$(2x^2y) \cdot (27x^3y^6) = 54x^5y^7$$

(c) $(4a^2b^3)^2 \cdot (2ab^4)^3$

We apply the power rule on each of the parentheses separately.

$$(4a^2b^3)^2 \cdot (2ab^4)^3 = (16a^4b^6) \cdot (8a^3b^{12})$$

Then we can apply the product rule to combine the two parentheses.

$$(16a^4b^6) \cdot (8a^3b^{12}) = 128a^7b^{18}$$

Review Questions

Write in exponential notation.

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
2. $3x \cdot 3x \cdot 3x$
3. $(-2a)(-2a)(-2a)(-2a)$
4. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Find each number:

5. 5^4
6. $(-2)^6$
7. $(0.1)^5$
8. $(-0.6)^3$

Multiply and simplify.

9. $6^3 \cdot 6^6$
10. $2^2 \cdot 2^4 \cdot 2^6$
11. $3^2 \cdot 4^3$
12. $x^2 \cdot x^4$
13. $(-2y^4)(-3y)$
14. $(4a^2)(-3a)(-5a^4)$

Simplify.

15. $(a^3)^4$
16. $(xy)^2$
17. $(3a^2b^3)^4$
18. $(-2xy^4z^2)^5$
19. $(-8x)^3(5x)^2$
20. $(4a^2)(-2a^3)^4$
21. $(12xy)(12xy)^2$
22. $(2xy^2)(-x^2y)^2(3x^2y^2)$

Review Answers

1. 4^5
2. $(3x)^3$
3. $(-2a)^4$
4. $6^3x^2y^4$
5. 625
6. 64
7. 0.00001
8. −0.216
9. 10077696
10. 4096

11. 576
12. x^6
13. $6y^5$
14. $60a^7$
15. a^{12}
16. x^2y^2
17. $81a^8b^{12}$
18. $-32x^5y^{20}z^{10}$
19. $12800x^5$
20. $64a^{14}$
21. $1728x^3y^3$
22. $6x^7y^6$

8.2 Exponent Properties Involving Quotients

Learning Objectives

- Use the quotient of powers property.
- Use the power of a quotient property.
- Simplify expressions involving quotient properties of exponents.

Use the Quotient of Powers Property

You saw in the last section that we can use exponent rules to simplify products of numbers and variables. In this section, you will learn that there are similar rules you can use to simplify quotients. Let's take an example of a quotient, x^7 divided by x^4.

$$\frac{x^7}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

You should see that when we divide two powers of x, the number of factors of x in the solution is the difference between the factors in the numerator of the fraction, and the factors in the denominator. In other words, when dividing expressions with the same base, keep the base and subtract the exponent in the denominator from the exponent in the numerator.

Quotient Rule for Exponents: $\frac{x^n}{x^m} = x^{n-m}$

When we have problems with different bases, we apply the quotient rule separately for each base.

$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} \cdot \frac{\cancel{y} \cdot \cancel{y} \cdot y}{\cancel{y} \cdot \cancel{y}} = \frac{x \cdot x}{1} \cdot \frac{y}{1} = x^2y \qquad \text{OR} \qquad \frac{x^5y^3}{x^3y^2} = x^{5-3} \cdot y^{3-2} = x^2y$$

Example 1

Simplify each of the following expressions using the quotient rule.

(a) $\frac{x^{10}}{x^5}$

(b) $\frac{a^6}{a}$

(c) $\frac{a^5b^4}{a^3b^2}$

Solution

Apply the quotient rule.

(a) $\frac{x^{10}}{x^5} = x^{10-5} = x^5$

(b) $\frac{a^6}{a} = a^{6-1} = a^5$

(c) $\frac{a^5 b^4}{a^3 b^2} = a^{5-3} \cdot b^{4-2} = a^2 b^2$

Now let's see what happens if the exponent on the denominator is bigger than the exponent in the numerator.

Example 2

Divide. $x^4 \div x^7$ — Apply the quotient rule.

$$\frac{x^4}{x^7} = x^{4-7} = x^{-3}$$

A negative exponent!? What does that mean?

Let's do the division longhand by writing each term in factored form.

$$\frac{x^4}{x^6} = \frac{\not{x} \cdot \not{x} \cdot \not{x} \cdot \not{x}}{\not{x} \cdot \not{x} \cdot \not{x} \cdot \not{x} \cdot x \cdot x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$$

We see that when the exponent in the denominator is bigger than the exponent in the numerator, we still subtract the powers. This time we subtract the smaller power from the bigger power and we leave the x's in the denominator.

When you simplify quotients, to get answers with positive exponents you subtract the smaller exponent from the bigger exponent and leave the variable where the bigger power was.

- We also discovered what a negative power means $x^{-3} = \frac{1}{x^3}$. We'll learn more on this in the next section!

Example 3

Simplify the following expressions, leaving all powers positive.

(a) $\frac{x^2}{x^6}$

(b) $\frac{a^2 b^6}{a^5 b}$

Solution

(a) Subtract the exponent in the numerator from the exponent in the denominator and leave the x's in the denominator.

$$\frac{x^2}{x^6} = \frac{1}{x^{6-2}} = \frac{1}{x^4}$$

(b) Apply the rule on each variable separately.

$$\frac{a^2 b^6}{a^5 b} = \frac{1}{a^{5-2}} \cdot \frac{b^{6-1}}{1} = \frac{b^5}{a^3}$$

The Power of a Quotient Property

When we apply a power to a quotient, we can learn another special rule. Here is an example.

$$\left(\frac{x^3}{y^2}\right)^4 = \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) = \frac{(x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x)}{(y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y)} = \frac{x^{12}}{y^8}$$

Notice that the power on the outside of the parenthesis multiplies with the power of the x in the numerator and the power of the y in the denominator. This is called the power of a quotient rule.

Power Rule for Quotients $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

Simplifying Expressions Involving Quotient Properties of Exponents

Let's apply the rules we just learned to a few examples.

- When we have numbers with exponents and not variables with exponents, we evaluate.

Example 4

Simplify the following expressions.

(a) $\frac{4^5}{4^2}$

(b) $\frac{5^3}{5^7}$

(c) $\left(\frac{3^4}{5^2}\right)^2$

Solution

In each of the examples, we want to evaluate the numbers.

(a) Use the quotient rule first.

$$\frac{4^5}{4^2} = 4^{5-2} = 4^3$$

Then evaluate the result.

$$4^3 = 64$$

OR

We can evaluate each part separately and then divide.

$$\frac{1024}{16} = 64$$

(b) Use the quotient rule first.

$$\frac{5^3}{5^7} = \frac{1}{5^{7-3}} = \frac{1}{5^4}$$

Then evaluate the result.

$$\frac{1}{5^4} = \frac{1}{625}$$

OR

We can evaluate each part separately and then reduce.

$$\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$$

It makes more sense to apply the quotient rule first for examples (a) and (b). In this way the numbers we are evaluating are smaller because they are simplified first before applying the power.

(c) Use the power rule for quotients first.

$$\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4}$$

Then evaluate the result.

$$\frac{3^8}{5^4} = \frac{6561}{625}$$

OR

We evaluate inside the parenthesis first.

$$\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2$$

Then apply the power outside the parenthesis.

$$\left(\frac{81}{25}\right)^2 = \frac{6561}{625}$$

When we have just one variable in the expression, then we apply the rules straightforwardly.

Example 5: Simplify the following expressions:

(a) $\frac{x^{12}}{x^5}$

(b) $\left(\frac{x^4}{x}\right)^5$

Solution:

(a) Use the quotient rule.

$$\frac{x^{12}}{x^5} = x^{12-5} = x^7$$

(b) Use the power rule for quotients first.

$$\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5}$$

Then apply the quotient rule

$$\frac{x^{20}}{x^5} = x^{15}$$

OR

Use the quotient rule inside the parenthesis first.

$$\left(\frac{x^4}{x}\right)^5 = (x^3)^5$$

Then apply the power rule.

$$(x^3)^5 = x^{15}$$

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

Example 6

Simplify the following expressions.

(a) $\frac{6x^2y^3}{2xy^2}$

(b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

(a) We group like terms together.

$$\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$$

We reduce the numbers and apply the quotient rule on each grouping.

$$3xy >$$

(b) We apply the quotient rule inside the parenthesis first.

$$\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$$

Apply the power rule for quotients.

$$\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$$

In problems that we need to apply several rules together, we must keep in mind the order of operations.

Example 7

Simplify the following expressions.

(a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

(b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

(a) We apply the power rule first on the first parenthesis.

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction.

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^4$$

Apply the product rule to simplify.

$$x^4 \cdot x^2 = x^6$$

(b) Simplify inside the first parenthesis by reducing the numbers.

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then we can apply the power rule on the first parenthesis.

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together.

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

Apply the quotient rule on each fraction.

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Review Questions

Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^4}{3^{10}}$
4. $\left(\frac{2^2}{3^3}\right)^3$

Simplify the following expressions.

5. $\frac{a^3}{a^2}$
6. $\frac{x^5}{x^9}$
7. $\left(\frac{a^3b^4}{a^2b}\right)^3$
8. $\frac{x^6y^2}{x^2y^5}$
9. $\frac{6a^3}{2a^2}$
10. $\frac{15x^5}{5x}$
11. $\left(\frac{18a^4}{15a^{10}}\right)^4$
12. $\frac{25yx^6}{20y^5x^2}$

13. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
14. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
15. $\frac{(3ab)^2(4a^3b^4)^3}{(6a^2b)^4}$
16. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$

Review Answers

1. 5^4
2. $6^4 = 1296$
3. $\frac{1}{3^6} = \frac{1}{729}$
4. $\frac{2^6}{3^9} = \frac{64}{19683}$
5. a
6. $\frac{1}{x^4}$
7. a^3b^9
8. $\frac{x^4}{y^3}$
9. $3a$
10. $3x^4$
11. $\frac{1296}{625a^4}$
12. $\frac{5x^4}{4y^4}$
13. $\frac{x^6}{y^6}$
14. $\frac{15a^3}{4b^7}$
15. $\frac{4a^3b^{10}}{9}$
16. $3a^2c^4$

8.3 Zero, Negative, and Fractional Exponents

Learning Objectives

- Simplify expressions with zero exponents.
- Simplify expressions with negative exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Introduction

There are many interesting concepts that arise when contemplating the product and quotient rule for exponents. You may have already been wondering about different values for the exponents. For example, so far we have only considered positive, whole numbers for the exponent. So called **natural numbers** (or **counting numbers**) are easy to consider, but even with the everyday things around us we think about questions such as "is it possible to have a negative amount of money?" or "what would one and a half pairs of shoes look like?" In this lesson, we consider what happens when the exponent is not a natural number. We will start with "What happens when the exponent is zero?"

Simplify Expressions with Exponents of Zero

Let us look again at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and consider what happens when $n = m$. Let's take the example of x^4 divided by x^4.

$$\frac{x^4}{x^4} = x^{(4-4)} = x^0$$

Now we arrived at the quotient rule by considering how the factors of x cancel in such a fraction. Let's do that again with our example of x^4 divided by x^4.

$$\frac{x^4}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = 1$$

So $x^0 = 1$.

This works for any value of the exponent, not just 4.

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Since there is the same number of factors in the numerator as in the denominator, they cancel each other out and we obtain $x^0 = 1$. The zero exponent rule says that any number raised to the power zero is one.

Zero Rule for Exponents: $x^0 = 1, \ x \neq 0$

Simplify Expressions With Negative Exponents

Again we will look at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and this time consider what happens when $m > n$. Let's take the example of x^4 divided by x^6.

$\frac{x^4}{x^6} = x^{(4-6)} = x^{-2}$ for $x \neq 0$.

By the quotient rule our exponent for x is -2. But what does a negative exponent really mean? Let's do the same calculation long-hand by dividing the factors of x^4 by the factors of x^6.

$$\frac{x^4}{x^6} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$$

So we see that x to the power -2 is the same as one divided by x to the power $+2$. Here is the negative power rule for exponents.

Negative Power Rule for Exponents $\frac{1}{x^n} = x^{-n} \ x \neq 0$

You will also see negative powers applied to products and fractions. For example, here it is applied to a product.

$$\begin{aligned} (x^3y)^{-2} &= x^{-6}y^{-2} && \text{using the power rule} \\ x^{-6}y^{-2} &= \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} && \text{using the negative power rule separately on each variable} \end{aligned}$$

Here is an example of a negative power applied to a quotient.

$$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}} \qquad \text{using the power rule for quotients}$$

$$\frac{a^{-3}}{b^{-3}} = \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1} \qquad \text{using the negative power rule on each variable separately}$$

$$\frac{1}{a^3} \cdot \frac{b^3}{1} = \frac{b^3}{a^3} \qquad \text{simplifying the division of fractions}$$

$$\frac{b^3}{a^3} = \left(\frac{b}{a}\right)^3 \qquad \text{using the power rule for quotients in reverse.}$$

The last step is not necessary but it helps define another rule that will save us time. A fraction to a negative power is "flipped".

Negative Power Rule for Fractions $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n, \ x \neq 0, y \neq 0$

In some instances, it is more useful to write expressions without fractions and that makes use of negative powers.

Example 1

Write the following expressions without fractions.

(a) $\frac{1}{x}$

(b) $\frac{2}{x^2}$

(c) $\frac{x^2}{y^3}$

(d) $\frac{3}{xy}$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms in the denominator of the fractions.

(a) $\frac{1}{x} = x^{-1}$

(b) $\frac{2}{x^2} = 2x^{-2}$

(c) $\frac{x^2}{y^3} = x^2y^{-3}$

(d) $\frac{3}{xy} = 3x^{-1}y^{-1}$

Sometimes, it is more useful to write expressions without negative exponents.

Example 2

Write the following expressions without negative exponents.

(a) $3x^{-3}$

(b) $a^2b^{-3}c^{-1}$

(c) $4x^{-1}y^3$

(d) $\frac{2x^{-2}}{y^{-3}}$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms that have negative exponents.

(a) $3x^{-3} = \frac{3}{x^3}$

(b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

(c) $4x^{-1}y^3 = \frac{4y^3}{x}$

(d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

Example 3

Simplify the following expressions and write them without fractions.

(a) $\frac{4a^2b^3}{2a^5b}$

(b) $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

Solution

(a) Reduce the numbers and apply quotient rule on each variable separately.

$$\frac{4a^2b^3}{6a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

(b) Apply the power rule for quotients first.

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^2}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, use product rule on the x's and the quotient rule on the y's.

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

Example 4

Simplify the following expressions and write the answers without negative powers.

(a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

(b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

(a) Apply the quotient rule inside the parenthesis.

$$\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$$

Apply the power rule.

$$(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$$

(b) Apply the quotient rule on each variable separately.

$$\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$$

Simplify Expressions With Fractional Exponents

The exponent rules you learned in the last three sections apply to all powers. So far we have only looked at positive and negative integers. The rules work exactly the same if the powers are fractions or irrational numbers. Fractional exponents are used to express the taking of roots and radicals of something (square roots, cube roots, etc.). Here is an exmaple.

$\sqrt{a} = a^{1/2}$ and $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[5]{a^2} = \left(a^2\right)^{\frac{1}{5}} = a^{\frac{2}{5}} = a^{2/5}$

Roots as Fractional Exponents $\sqrt[m]{a^n} = a^{n/m}$

We will examine roots and radicals in detail in a later chapter. In this section, we will examine how exponent rules apply to fractional exponents.

Example 5

Simplify the following expressions.

(a) $a^{1/2} \cdot a^{1/3}$

(b) $(a^{1/3})^2$

(c) $\frac{a^{5/2}}{a^{1/2}}$

(d) $\left(\frac{x^2}{y^3}\right)^{1/3}$

Solution

(a) Apply the product rule.

$$a^{1/2} \cdot a^{1/3} = a^{\frac{1}{2}+\frac{1}{3}} = a^{5/6}$$

(b) Apply the power rule.

$$(a^{1/3})^2 = a^{2/3}$$

(c) Apply the quotient rule.

$$\frac{a^{5/2}}{a^{1/2}} = a^{\frac{5}{2}-\frac{1}{2}} = a^{4/2} = a^2$$

(d) Apply the power rule for quotients.

$$\left(\frac{x^2}{y^3}\right)^{1/3} = \frac{x^{2/3}}{y}$$

Evaluate Exponential Expressions

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS.**

Evaluate inside the **P**arenthesis.

Evaluate **E**xponents.

Perform **M**ultiplication and **D**ivision operations from left to right.

Perform **A**ddition and **S**ubtraction operations from left to right.

Example 6

Evaluate the following expressions to a single number.

(a) 5^0

(b) 7^2

(c) $\left(\frac{2}{3}\right)^3$

(d) 3^{-3}

(e) $16^{1/2}$

(f) $8^{-1/3}$

Solution

(a) $5^0 = 1$ Remember that a number raised to the power 0 is always 1.

(b) $7^2 = 7 \cdot 7 = 49$

(c) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

(d) $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

(e) $16^{1/2} = \sqrt{16} = 4$ Remember that an exponent of $\frac{1}{2}$ means taking the square root.

(f) $8^{-1/3} = \frac{1}{8^{1/3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ Remember that an exponent of $\frac{1}{3}$ means taking the cube root.

Example 7

Evaluate the following expressions to a single number.

(a) $3 \cdot 5^5 - 10 \cdot 5 + 1$

(b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$

Solution

(a) Evaluate the exponent.

$$3 \cdot 5^2 - 10 \cdot 6 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$$

Perform multiplications from left to right.

$$3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$$

Perform additions and subtractions from left to right.

$$75 - 50 + 1 = 26$$

(b) Treat the expressions in the numerator and denominator of the fraction like they are in parenthesis.

$$\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9 - 4)} = \frac{(32 - 75)}{5} = \frac{-43}{5}$$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$

Example 8

Evaluate the following expressions for $x = 2, y = -1$, $z = 3$.

(a) $2x^2 - 3y^3 + 4z$

(b) $(x^2 - y^2)^2$

(c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

(a) $2x^2 - 3y^+4z = 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3 = 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12 = 23$

(b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4 - 1)^2 = 3^2 = 9$

(c) $\left(\frac{3x^2-y^5}{4z}\right)^{-2} = \left(\frac{3 \cdot 2^2 \cdot (-1)^5}{4 \cdot 3}\right)^{-2} = \left(\frac{3 \cdot 4 \cdot (-1)}{12}\right)^{-2} = \left(\frac{-12}{12}\right)^{-2} = \left(\frac{-1}{1}\right)^{-2} = \left(\frac{1}{-1}\right)^2 = (-1)^2 = 1$

Review Questions

Simplify the following expressions, be sure that there aren't any negative exponents in the answer.

1. $x^{-1} \cdot y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{x^{-3}y^{-5}}{z^{-7}}$
5. $(x^{\frac{1}{2}}y^{-\frac{2}{3}})(x^2y^{\frac{1}{3}})$
6. $\left(\frac{a}{b}\right)^{-2}$
7. $(3a^{-2}b^2c^3)^3$
8. $x^{-3} \cdot x^3$

Simplify the following expressions so that there aren't any fractions in the answer.

9. $\frac{a^{-3}(a^5)}{a^{-6}}$
10. $\frac{5x^6y^2}{x^8y}$
11. $\frac{(4ab^6)^3}{(ab)^5}$
12. $\left(\frac{3x}{y^{1/3}}\right)^3$
13. $\frac{3x^2y^{3/2}}{xy^{1/2}}$
14. $\frac{(3x^3)(4x^4)}{(2y)^2}$
15. $\frac{a^{-2}b^{-3}}{c^{-1}}$
16. $\frac{x^{1/2}y^{5/2}}{x^{3/2}y^{3/2}}$

Evaluate the following expressions to a single number.

17. 3^{-2}
18. $(6.2)^0$
19. $8^{-4} \cdot 8^6$
20. $(16^{\frac{1}{2}})^3$
21. $x^24x^3y^44y^2$ if $x = 2$ and $y = -1$
22. $a^4(b^2)^3 + 2ab$ if $a = -2$ and $b = 1$
23. $5x^2 - 2y^3 + 3z$ if $x = 3$, $y = 2$, and $z = 4$
24. $\left(\frac{a^2}{b^3}\right)^{-2}$ if $a = 5$ and $b = 3$

Review Answers

1. $\frac{y^2}{x}$

2. $\frac{1}{x^4}$
3. x^4
4. $\frac{z^7}{x^3y^5}$
5. $\frac{x^{5/2}}{y^{1/3}}$
6. $\left(\frac{b}{a}\right)^2$ or $\frac{b^2}{a^2}$
7. $\frac{27b^6c^9}{a^6}$
8. 1
9. a^8
10. $5x^{-2}y$
11. $64a^{-2}b^{\frac{1}{3}}$
12. $27x^2y^{-1}$
13. $3xy$
14. $6x^7y^{-2}$
15. $a^{-2}b^{-3}c$
16. $x^{-1}y$
17. 0.111
18. 1
19. 64
20. 64
21. 512
22. 12
23. 41
24. 1.1664

8.4 Scientific Notation

Learning Objectives

- Write numbers in scientific notation.
- Evaluate expressions in scientific notation.
- Evaluate expressions in scientific notation using a graphing calculator.

Introduction – Powers of 10

Consider the number six hundred and forty three thousand, two hundred and ninety seven. We write it as 643, 297 and each digit's position has a "value" assigned to it. You may have seen a table like this before.

hundred-thousands	ten-thousands	thousands	hundreds	tens	units (ones)
6	4	3	2	9	7

We have seen that when we write an exponent above a number it means that we have to multiply a certain number of factors of that number together. We have also seen that a zero exponent always gives us one, and negative exponents make fractional answers. Look carefully at the table above. Do you notice that all the column headings are powers of ten? Here they are listed.

$$100,000 = 10^5$$
$$10,000 = 10^4$$
$$1,000 = 10^3$$
$$100 = 10^2$$
$$10 = 10^1$$

Even the "units" column is really just a power of ten. **Unit** means 1 and $1 = 10^0$.

If we divide 643,297 by 100,000 we get 6.43297. If we multiply this by 100,000 we get back to our original number. But we have just seen that 100,000 is the same as 10^5, so if we multiply 6.43297 by 10^5 we should also get our original answer. In other words

$$6.43297 \times 10^5 = 643,297$$

So we have found a new way of writing numbers! What do you think happens when we continue the powers of ten? Past the units column down to zero we get into decimals, here the exponent becomes negative.

Writing Numbers Greater Than One in Scientific Notation

Scientific notation numbers are always written in the following form.

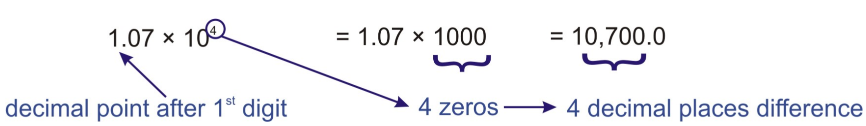
$$a \times 10b$$

Where $1 \leq a < 10$ and b, the exponent, is an integer. This notation is especially useful for numbers that are either very small or very large. When we use scientific notation to write numbers, the exponent on the 10 determines the position of the decimal point.

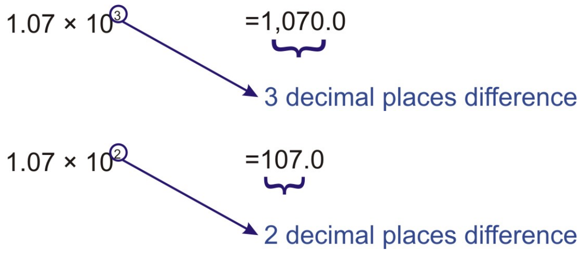
Look at the following examples.

$$1.07 \times 10^4 = 10,700$$
$$1.07 \times 10^3 = 1,070$$
$$1.07 \times 10^2 = 107$$
$$1.07 \times 10^1 = 10.7$$
$$1.07 \times 10^0 = 1.07$$
$$1.07 \times 10^{-1} = 0.107$$
$$1.07 \times 10^{-2} = 0.0107$$
$$1.07 \times 10^{-3} = 0.00107$$
$$1.07 \times 10^{-4} = 0.000107$$

Look at the first term of the list and examine the position of the decimal point in both expressions.



So the exponent on the ten acts to move the decimal point over to the right. An exponent of 4 moves it 4 places and an exponent of 3 would move it 3 places.



Example 1

Write the following numbers in scientific notation.

(a) 63

(b) $9,654$

(c) $653,937,000$

(d) $1,000,000,006$

(a) $63 = 6.3 \times 10 = 6.3 \times 10^1$

(b) $9,654 = 9.654 \times 1,000 = 9.654 \times 10^3$

(c) $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$

(d) $1,000,000,006 = 1.000000006 \times 1,000,000,000 = 1.000000006 \times 10^9$

Example 2

The Sun is approximately 93 *million miles for Earth. Write this distance in scientific notation.*

This time we will simply write out the number long-hand (with a decimal point) and count decimal places.

Solution

$$\underbrace{93,000,000,0}_{\text{7 decimal places}} = 9.3 \times 10^7 \text{ miles}$$

A Note on Significant Figures

We often combine scientific notation with rounding numbers. If you look at Example 2, the distance you are given has been rounded. It is unlikely that the distance is **exactly** 93 million miles! Looking back at the numbers in Example 1, if we round the final two answers to 2 significant figures (2 s.f.) they become:

$1(c)\ 6.5 \times 10^8$

$1(d)\ 1.0 \times 10^9$

Note that the zero after the decimal point has been left in for Example 1(d) to indicate that the result has been rounded. It is important to know when it is OK to round and when it is not.

Writing Numbers Less Than One in Scientific Notation

We have seen how we can use scientific notation to express large numbers, but it is equally good at expressing extremely small numbers. Consider the following example.

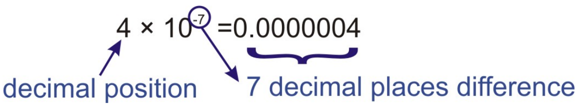
Example 3

The time taken for a light beam to cross a football pitch is 0.0000004 seconds. *Express this time in scientific notation.*

We will proceed in a similar way as before.

$$0.0000004 = 4 \times 0.0000001 = 4 \times \frac{1}{10,000,000} = 4 \times \frac{1}{10^7} = 4 \times 10^{-7}$$

So...



Just as a positive exponent on the ten moves the decimal point that many places to the right, a negative exponent moves the decimal place that many places to the left.

Example 4

Express the following numbers in scientific notation.

(a) 0.003

(b) 0.000056

(c) 0.00005007

(d) 0.00000000000954

Let's use the method of counting how many places we would move the decimal point before it is after the first non-zero number. This will give us the value for our negative exponent.

(a) $\underbrace{0.003}_{\text{3 decimal places}} = 3 \times 10^{-3}$

(b) $\underbrace{0.000056}_{\text{5 decimal places}} = 5.6 \times 10^{-5}$

(c) $\underbrace{0.00005007}_{\text{5 decimal places}} = 5.007 \times 10^{-5}$

(d) $\underbrace{0.00000000000954}_{\text{12 decimal places}} = 9.54 \times 10^{-12}$

Evaluating Expressions in Scientific Notation

When we are faced with products and quotients involving scientific notation, we need to remember the rules for exponents that we learned earlier. It is relatively straightforward to work with scientific notation problems if you remember to deal with all the powers of 10 together. The following examples illustrate this.

Example 5

Evaluate the following expressions and write your answer in scientific notation.

(a) $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$

(b) $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$

(c) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

The key to evaluating expressions involving scientific notation is to keep the powers of 10 together and deal with them separately. Remember that when we use scientific notation, the leading number **must be between** 1 **and** 10. We need to move the decimal point over one place to the left. See how this adds 1 to the exponent on the 10.

(a)

$$
\begin{aligned}
(3.2 \times 10^6) \cdot (8.7 \times 10^{11}) &= \underbrace{3.2 \times 8.7}_{27.84} \times \underbrace{10^6 \times 10^{11}}_{10^{17}} \\
(3.2 \times 10^6) \cdot (8.7 \times 10^{11}) &= 2.784 \times 10^1 \times 10^{17}
\end{aligned}
$$

Solution

$$(3.2 \times 10^6) \cdot (8.7 \times 10^{11}) = 2.784 \times 10^{18}$$

(b)

$$
\begin{aligned}
(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19}) &= \underbrace{5.2 \times 3.8}_{19.76} \times \underbrace{10^{-4} \times 10^{-19}}_{10^{-23}} \\
&= 1.976 \times 10^1 \times 10^{-23}
\end{aligned}
$$

Solution

$$(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19}) = 1.976 \times 10^{-22}$$

(c)

$$(1.7 \times 10^6) \cdot (2.7 \times 10^{-11}) = \underbrace{1.7 \times 2.7}_{4.59} \times \underbrace{10^6 \times 10^{-11}}_{10^{-5}}$$

Solution

$$(1.7 \times 10^6) \cdot (2.7 \times 10^{-11}) = 4.59 \times 10^{-5}$$

Example 6

Evaluate the following expressions. Round to 3 significant figures and write your answer in scientific notation.

(a) $(3.2 \times 10^6) \div (8.7 \times 10^{11})$

(b) $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$

(c) $(1.7 \times 10^6) \div (2.7 \times 10^{-11})$

It will be easier if we convert to fractions and THEN separate out the powers of 10.

(a)

$$
\begin{aligned}
(3.2 \times 10^6) \div (8.7 \times s10^{11}) &= \frac{3.2 \times 10^6}{8.7 \times 10^{11}} && \text{Next we separate the powers of 10.} \\
&= \frac{3.2}{8.7} \times \frac{10^6}{10^{11}} && \text{Evaluate each fraction (round to 3 s.f.):} \\
&= 0.368 \times 10(6 - 11) && \text{Remember how to write scientific notation!} \\
&= 3.68 \times 10^{-1} \times 10^{-5}
\end{aligned}
$$

Solution

$(3.2 \times 10^6) \div (8.7 \times 10^{11}) = 3.86 \times 10^{-6}$ *(rounded to 3 significant figures)*

(b)

$$
\begin{aligned}
(5.2 \times 10^{-4}) \div (3.8 \times 10^{19}) &= \frac{5.2 \times 10^{-4}}{3.8 \times 10^{-19}} && \text{Separate the powers of 10.} \\
&= \frac{5.2}{3.8} \times \frac{10^{-4}}{10^{-19}} && \text{Evaluate each fraction (round to 3 s.f.).} \\
&= 1.37 \times 10((-4) - (-19)) \\
&= 1.37 \times 10^{15}
\end{aligned}
$$

Solution

$(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19}) = 1.37 \times 10^{15}$ *(rounded to 3 significant figures)*

(c)

$$
\begin{aligned}
(1.7 \times 10^{6}) \div (2.7 \times 10^{-11}) &= \frac{1.7 \times 10^{6}}{2.7 \times 10^{-11}} && \text{Next we separate the powers of 10.} \\
&= \frac{1.7}{2.7} \times \frac{10^{6}}{10^{-11}} && \text{Evaluate each fraction (round to 3 s.f.).} \\
&= 0.630 \times 10^{(6-(-11))} && \text{Remember how to write scientific notation!} \\
&= 6.30 \times 10^{-1} \times 10^{17}
\end{aligned}
$$

Solution

$(1.7 \times 10^{6}) \div (2.7 \times 10^{-11}) = 6.30 \times 10^{16}$ *(rounded to 3 significant figures)*

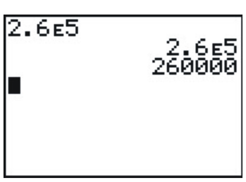
Note that the final zero has been left in to indicate that the result has been rounded.

Evaluate Expressions in Scientific Notation Using a Graphing Calculator

All scientific and graphing calculators have the ability to use scientific notation. It is extremely useful to know how to use this function.

To insert a number in scientific notation, use the [**EE**] button. This is [**2nd**] [**,**] on some TI models.

For example to enter 2.6×10^{5} enter 2.6 [**EE**] 5.



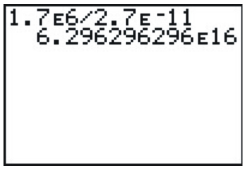
When you hit [**ENTER**] the calculator displays $2.6E5$ if it's set in **Scientific** mode OR it displays 260000 if it's set in **Normal** mode.

(To change the mode, press the 'Mode' key)

Example 7

Evaluate $(1.7 \times 10^{6}) \div (2.7 \times 10^{-11})$ *using a graphing calculator.*

[**ENTER**] 1.7 EE 6 ÷ 2.7 EE −11 and press [**ENTER**]



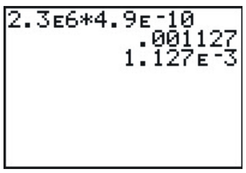
The calculator displays $6.296296296E16$ whether it is in Normal mode or Scientific mode. This is the case because the number is so big that it does not fit inside the screen in Normal mode.

Solution

$$(1.7 \times 10^6) \div (2.7 \times 10^{-11}) = 6.\bar{3} \times 10^{16}$$

Example 8

Evaluate $(2.3 \times 10^6) \times (4.9 \times 10^{-10})$ *using a graphing calculator.*



[**ENTER**] 2.3 EE 6×4.9 EE -10 and press [**ENTER**]

The calculator displays .001127 in Normal mode or $1.127E-3$ in Scientific mode.

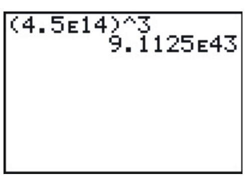
Solution

$$(2.3 \times 10^6) \times (4.9 \times 10^{-10}) = 1.127 \times 10^{-3}$$

Example 9

Evaluate $(4.5 \times 10^{14})^3$ *using a graphing calculator.*

[**ENTER**] $(4.5EE14)^3$ and press [**ENTER**].



The calculator displays 9.1125E43

Solution

$$(4.5 \times 10^{14})^3 = 9.1125 \times 10^{43}$$

Solve Real-World Problems Using Scientific Notation

Example 10

The mass of a single lithium atom is approximately one percent of one millionth of one billionth of one billionth of one kilogram. Express this mass in scientific notation.

We know that percent means we divide by 100, and so our calculation for the mass (in kg) is

$$\frac{1}{100} \times \frac{1}{1,000,000} \times \frac{1}{1,000,000,000} \times \frac{1}{1,000,000,000} = 10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9} \times 10^{-9}$$

Next, we use the product of powers rule we learned earlier in the chapter.

$$10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9} = 10^{((-2)+(-6)+(-9)+(-9))} = 10^{-26} \text{ kg.}$$

Solution

The mass of one lithium atom is approximately 1×10^{-26} kg .

Example 11

You could fit about 3 *million E. coli bacteria on the head of a pin. If the size of the pin head in question is* 1.2×10^{-5} m^2, *calculate the area taken up by one E. coli bacterium. Express your answer in scientific notation.*

Since we need our answer in scientific notation it makes sense to convert 3 million to that format first:

$$3,000,000 = 3 \times 10^6$$

Next, we need an expression involving our unknown. The area taken by one bacterium. Call this A.

$3 \times 10^6 \cdot A = 1.2 \times 10^{-5}$ Since 3 million of them make up the area of the pin-head.

Isolate A:

$A = \frac{1}{3 \times 10^6} \cdot 1.2 \times 10^{-5}$ Rearranging the terms gives

$A = \frac{1.2}{3} \cdot \frac{1}{10^6} \times 10^{-5}$ Then using the definition of a negative exponent

$A = \frac{1.2}{3} \times 10^{-6} \times 10^{-5}$ Evaluate combine exponents using the product rule.

$A = 0.4 \times 10^{-11}$ We cannot, however, leave our answer like this.

Solution

The area of one bacterium $A = 4.0 \times 10^{-12}$ m^2

Notice that we had to move the decimal point over one place to the right, subtracting 1 from the exponent on the 10.

Review Questions

1. Write the numerical value of the following.
   (a) 3.102×10^2
   (b) 7.4×10^4
   (c) 1.75×10^{-3}
   (d) 2.9×10^{-5}
   (e) 9.99×10^{-9}
2. Write the following numbers in scientific notation.

(a) 120,000
(b) 1,765,244
(c) 12
(d) 0.00281
(e) 0.000000027

3. The moon is approximately a sphere with radius $r = 1.08 \times 10^3$ miles. Use the formula Surface Area $= 4\pi r^2$ to determine the surface area of the moon, in square miles. Express your answer in scientific notation, rounded to 2 significant figures.
4. The charge on one electron is approximately 1.60×10^{-19} coulombs. One **Faraday** is equal to the total charge on 6.02×10^{23} electrons. What, in coulombs, is the charge on one Faraday?
5. Proxima Centauri, the next closest star to our Sun is approximately 2.5×10^{13} miles away. If light from Proxima Centauri takes 3.7×10^4 hours to reach us from there, calculate the speed of light in miles per hour. Express your answer in scientific notation, rounded to 2 significant figures.

Review Answers

1.
2. (a) 310.2
   (b) 74.000
   (c) 0.00175
   (d) 0.000029
   (e) 0.00000000999
3.
4. (a) 1.2×10^5
   (b) 1.765224×10^{10}
   (c) 1.2×10^1
   (d) 2.81×10^{-3}
   (e) 2.7×10^{-8}
5. 1.5×10^7 miles^2
6. 96,320 or 9.632×10^4
7. 6.8×10^8 miles per hour

8.5 Exponential Growth Functions

Learning Objectives

- Graph an exponential growth function.
- Compare graphs of exponential growth functions.
- Solve real-world problems involving exponential growth.

Introduction

Exponential functions are different than other functions you have seen before because now the variable appears as the exponent (or power) instead of the base. In this section, we will be working with functions where the base is a constant number and the exponent is the variable. Here is an example.

$$y = 2^x$$

This particular function describes something that doubles each time x increases by one. Let's look at a particular situation where this might occur.

A colony of bacteria has a population of three thousand at noon on Sunday. During the next week, the colony's population doubles every day. What is the population of the bacteria colony at noon on Saturday?

Let's make a table of values and calculate the population each day.

Day	0(Sunday)	1(Monday)	2(Tuesday)	3(Wednesday)	4(Thursday)	5(Friday)	6(Satur
Population (in thousands)	3	6	12	24	48	96	192

To get the population of bacteria for the next day we simply multiply the current day's population by 2.

We start with a population of 3 (thousand):	$P = 3$
To find the population on Monday we double	$P = 3 \cdot 2$
The population on Tuesday will be double that on Monday	$P = 3 \cdot 2 \cdot 2$
The population on Wednesday will be double that on Tuesday	$P = 3 \cdot 2 \cdot 2 \cdot 2$
Thursday is double that on Wednesday	$P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Friday is double that on Thursday	$P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Saturday is double that on Friday	$P = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

You can see that this function describes a population that is multiplied by 2 each time a day passes.

If we define x as the number of days since Sunday at noon, then we can write the following.

$P = 3.2^x$ *This is a formula that we can use to calculate the population on any day.*

For instance, the population on Saturday at noon will be $P = 3.2^6 = 3.64 = 192$ (thousand) bacteria.

We used $x = 6$, since Saturday at noon is six days after Sunday at noon.

In general exponential function takes the form:

$y = A \cdot b^x$ *where A is the initial amount and b is the factor that the amount gets multiplied by each time x is increased by one.*

Graph Exponential Functions

Let's start this section by graphing some exponential functions. Since we don't yet know any special properties of exponential functions, we will graph using a table of values.

Example 1

Graph the equation using a table of values $y = 2^x$.

Solution

Let's make a table of values that includes both negative and positive values of x.

To evaluate the positive values of x, we just plug into the function and evaluate.

$$x = 1, \qquad y = 2^1 = 2$$
$$x = 2, \qquad y = 2^2 = 2 \cdot 2 = 4$$
$$x = 3, \qquad y = 2^3 = 2 \cdot 2 \cdot 2 = 8$$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

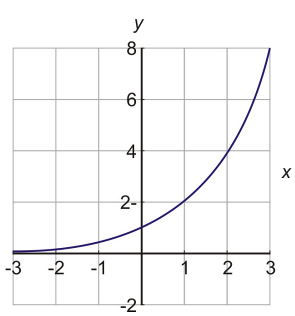
For $x = 0$, we must remember that a number to the power 0 is always 1.

$$x = 0, \qquad y = 2^0 = 1$$

To evaluate the negative values of x, we must remember that x to a negative power means one over x to the same positive power.

$$x = -1, \qquad y = 2-1 = \frac{1}{2^1} = \frac{1}{2}$$
$$x = -2, \qquad y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$x = -3, \qquad y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

When we plot the points on the coordinate axes we get the graph below. Exponentials always have this basic shape. That is, they start very small and then, once they start growing, they grow faster and faster, and soon they become extremely big!



You may have heard people say that something is growing **exponentially.** This implies that the growth is very quick. An exponential function actually starts slow, but then grows faster and faster all the time. Specifically, our function y above doubled each time we increased x by one.

This is the definition of exponential growth. There is a consistent fixed period during which the function will double or triple, or quadruple. The change is always a fixed proportion.

Compare Graphs of Exponential Growth Functions

Let's graph a few more exponential functions and see what happens as we change the constants in the functions. The basic shape of the exponential function should stay the same. But, it may become steeper or shallower depending on the constants we are using.

We mentioned that the general form of the exponential function is $y = A \cdot b^x$ where A is the initial amount and b is the factor that the amount gets multiplied by each time x is increased by one. Let's see what happens for different values of A.

Example 2

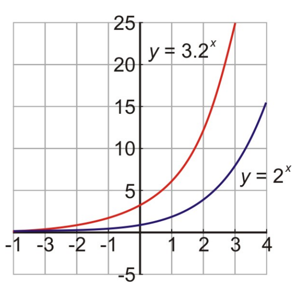
Graph the exponential function $y = 3 \cdot 2^x$ *and compare with the graph of* $y = 2^x$.

Solution

Let's make a table of values for $y = 3 \cdot 2^x$.

x	$y = 3 \cdot^x$
-2	$y = 3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	$y = 3 \cdot 2^0 = 3$
1	$y = 3 \cdot 2^1 = 6$
2	$y = 3 \cdot 2^2 = 3 \cdot 4 = 12$
3	$y = 3 \cdot 2^3 = 3 \cdot 8 = 24$

Now let's use this table to graph the function.



We can see that the function $y = 3 \cdot 2^x$ is bigger than function $y = 2^x$. In both functions, the value of y doubled every time x increases by one. However, $y = 3 \cdot 2^x$ "starts" with a value of 3, while $y = 2^x$ "starts" with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger as its values of y keep getting doubled.

You might think that if the initial value A is less than one, then the corresponding exponential function would be less than $y = 2^x$. This is indeed correct. Let's see how the graphs compare for $A = \frac{1}{3}$.

Example 3

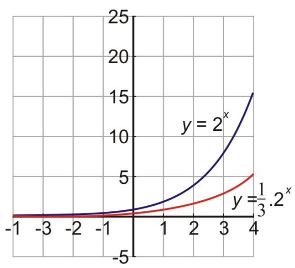
Graph the exponential function $y = \frac{1}{3} \cdot 2^x$ *and compare with the graph of* $y = 2^x$.

Solution

Let's make a table of values for $y = \frac{1}{3} \cdot 2^x$.

x	$y = \frac{1}{3} \cdot 2^x$
-2	$y = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{2^2} = \frac{1}{12}$
-1	$y = \frac{1}{3} \cdot 2^{-1} = \frac{1}{3} \cdot \frac{1}{2^1} = \frac{1}{6}$
0	$y = \frac{1}{3} \cdot 2^0 = \frac{1}{3}$
1	$y = \frac{1}{3} \cdot 2^1 = \frac{2}{3}$
2	$y = \frac{1}{3} \cdot 2^2 = \frac{1}{3} \cdot 4 = \frac{4}{3}$
3	$y = \frac{1}{3} \cdot 2^3 = \frac{1}{3} \cdot 8 = \frac{8}{3}$

Now let's use this table to graph the function.



As expected, the exponential function $y = \frac{1}{3} \cdot 2^x$ is smaller that the exponential function $y = 2^x$.

Now, let's compare exponential functions whose bases are different.

The function $y = 2^x$ has a base of 2. That means that the value of y doubles every time x is increased by 1.

The function $y = 3^x$ has a base of 3. That means that the value of y triples every time x is increased by 1.

The function $y = 5^x$ has a base of 5. That means that the value of y gets multiplies by a factor of 5 every time x is increased by 1.

The function $y = 10^x$ has a base of 10. That means that the value of y gets multiplied by a factor of 10 every time x is increased by 1.

What do you think will happen as the base number is increased? Let's find out.

Example 4

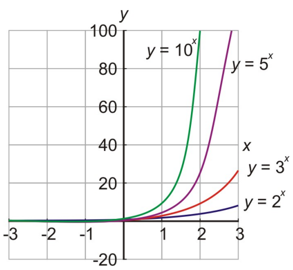
Graph the following exponential functions of the same graph $y = 2^x$, $y = 3^x$, $y = 5^x$, $y = 10^x$.

Solution

To graph these functions we should start by making a table of values for each of them.

x	$y = 2^x$	$y = 3^x$	$y = 5^x$	$y = 10^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$	$\frac{1}{100}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$
0	1	1	1	1
1	2	3	5	10
2	4	9	25	100
3	8	27	125	1000

Now let's graph these functions.



Notice that for $x = 0$ the values for all the functions are equal to 1. This means that the initial value of the functions is the same and equal to 1. Even though all the functions start at the same value, they increase at different rates. We can see that the bigger the base is the faster the values of y will increase. It makes sense that something that triples each time will increase faster than something that just doubles each time.

Finally, let's examine what the graph of an exponential looks like if the value of A is negative.

Example 5

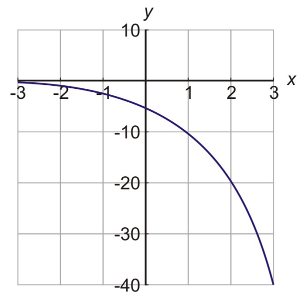
Graph the exponential function $y = -5 \cdot 2^x$.

Solution

Let's make a table of values.

x	$y = -5 \cdot 2^x$
-2	$-\frac{5}{4}$
-1	$-\frac{5}{2}$
0	-5
1	-10
2	-20
3	-40

Now let's graph the function.



This result should not be unexpected. Since the initial value is negative and doubles with time, it makes sense that the value of y increasess, but in a negative direction. Notice that the shape of the graph remains that of a typical exponential function, but is now a mirror image about the horizontal axis (i.e. upside down).

Solve Real-World Problems Involving Exponential Growth

We will now examine some real-world problems where exponential growth occurs.

Example 6

The population of a town is estimated to increase by 15% *per year. The population today is* 20 *thousand. Make a graph of the population function and find out what the population will be ten years from now.*

Solution

First, we need to write a function that describes the population of the town. The general form of an exponential function is.

$$y = A \cdot b^x$$

Define y as the population of the town.

Define x as the number of years from now.

A is the initial population, so $A = 20$ (thousand)

Finally, we must find what b is. We are told that the population increases by 15% each year.

To calculate percents, it is necessary to change them into decimals. 15% is equivalent to 0.15.

15% of A is equal to $0.15A$. This represents the increase in population from one year to the next.

In order to get the total population for the following year we must add the current population to the increase in population. In other words $A + 0.15A = 1.15A$. We see from this that the population must be multiplied by a factor of 1.15 each year.

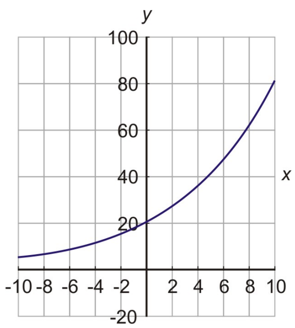
This means that the base of the exponential is $b = 1.15$.

The formula that describes this problem is $y = 20 \cdot (1.15)^x$

Now let's make a table of values.

x	$y = 20 \cdot (1.15)^x$
-10	4.9
-5	9.9
0	20
5	40.2
10	80.9

Now let's graph the function.



Notice that we used negative values of x in our table of values. Does it make sense to think of negative time? In this case $x = -5$ represents what the population was five years ago, so it can be useful information. The question asked in the problem was *"What will be the population of the town ten years from now?"*

To find the population exactly, we use $x = 10$ in the formula. We found $y = 20 \cdot (1.15)^{10} = 89,911$.

Example 7

Peter earned \$1500 *last summer. If he deposited the money in a bank account that earns* 5% *interest compounded yearly, how much money will he have after five years?*

Solution

This problem deals with interest which is compounded yearly. This means that each year the interest is calculated on the amount of money you have in the bank. That interest is added to the original amount and next year the interest is calculated on this new amount. In this way, you get paid interest on the interest.

Let's write a function that describes the amount of money in the bank. The general form of an exponential function is

$$y = A \cdot b^x$$

Define y as the amount of money in the bank.

Define x as the number of years from now.

A is the initial amount, so $A = 1500$.

Now we must find what b is.

We are told that the interest is 5% each year.

Change percents into decimals 5% is equivalent to 0.05.

5% of A is equal to $0.05A$ *This represents the interest earned per year.*

In order to get the total amount of money for the following year, we must add the interest earned to the initial amount.

$$A + 0.05A = 1.05A$$

We see from this that the amount of money must be multiplied by a factor of 1.05 each year.

This means that the base of the exponential is $b = 1.05$

The formula that describes this problem is $y = 1500 \cdot (1.05)^x$

To find the total amount of money in the bank at the end of five years, we simply use $x = 5$ in our formula.

Answer $y = 1500 \cdot (1.05)^5 = \$1914.42$

Review Questions

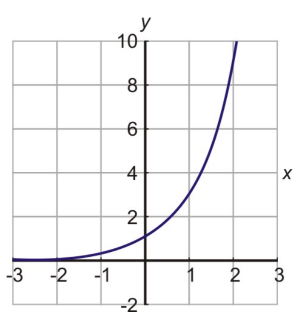
Graph the following exponential functions by making a table of values.

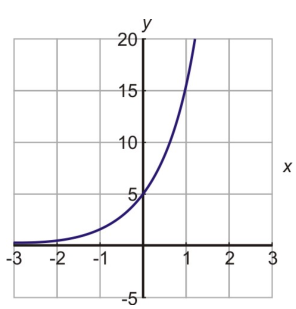
1. $y = 3^x$
2. $y = 5 \cdot 3^x$
3. $y = 40 \cdot 4^x$
4. $y = 3 \cdot 10^x$

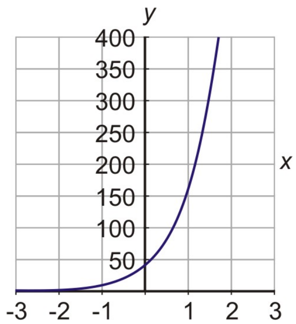
Solve the following problems.

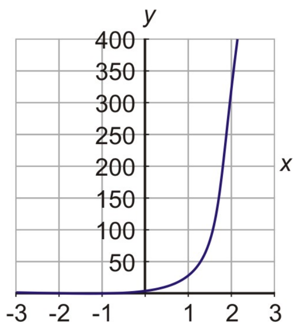
5. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person. Assume that everyone who receives the letter sends it to ten new people and that it takes a week for each cycle. How many people receive the letter on the sixth week?
6. Nadia received \$200 for her 10^{th} birthday. If she saves it in a bank with a 7.5% interest compounded yearly, how much money will she have in the bank by her 21^{st} birthday?

Review Answers

1. 

2. 

3. 

4. 

5. $10,000,000$
6. \$443.12

8.6 Exponential Decay Functions

Learning Objectives

- Graph an exponential decay function.
- Compare graphs of exponential decay functions.
- Solve real-world problems involving exponential decay.

Introduction

In the last section, we looked at graphs of exponential functions. We saw that exponentials functions describe a quantity that doubles, triples, quadruples, or simply gets multiplied by the same factor. All the functions we looked at in the last section were exponentially increasing functions. They started small and then became large very fast. In this section, we are going to look at exponentially decreasing functions. An example of such a function is a quantity that gets decreased by one half each time. Let's look at a specific example.

For her fifth birthday, Nadia's grandmother gave her a full bag of candy. Nadia counted her candy and found out that there were 160 *pieces in the bag. As you might suspect Nadia loves candy so she ate half the candy on the first day. Her mother told her that if she eats it at that rate it will be all gone the next day and she will not have anymore until her next birthday. Nadia devised a clever plan. She will always eat half of the candy that is left in the bag each day. She thinks that she will get candy every day and her candy will never run out. How much candy does Nadia have at the end of the week? Would the candy really last forever?*

Let's make a table of values for this problem.

Day	0	1	2	3	4	5	6	7
No. of Candies	160	80	40	20	10	5	2.5	1.25

You can see that if Nadia eats half the candies each day, then by the end of the week she only has 1.25 candies left in her bag.

Let's write an equation for this exponential function.

Nadia started with 160 pieces.	$y = 160$
After the first she has $\frac{1}{2}$ of that amount.	$y = 160 \cdot \frac{1}{2}$
After the second day she has $\frac{1}{2}$ of the last amount.	$y = 160 \cdot \frac{1}{2} \cdot \frac{1}{2}$

You see that in order to get the amount of candy left at the end of each day we keep multiplying by $\frac{1}{2}$.

We can write the exponential function as

$$y = 160 \cdot \frac{1}{2}^x$$

Notice that this is the same general form as the exponential functions in the last section.

$$y = A \cdot b^x$$

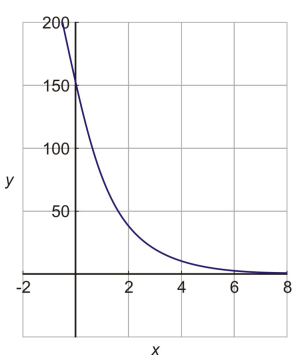
Here $A = 160$ is the initial amount and $b = \frac{1}{2}$ is the factor that the quantity gets multiplied by each time. The difference is that now b is a fraction that is less than one, instead of a number that is greater than one.

This is a good rule to remember for exponential functions.

If b is greater than one, then the exponential function increased, but

If b is less than one (but still positive), then the exponential function decreased

Let's now graph the candy problem function. The resulting graph is shown below.



So, will Nadia's candy last forever? We saw that by the end of the week she has 1.25 candies left so there doesn't seem to be much hope for that. But if you look at the graph you will see that the graph never really gets to zero.

Theoretically there will always be some candy left, but she will be eating very tiny fractions of a candy every day after the first week!

This is a fundamental feature of an exponential decay function. Its value get smaller and smaller and approaches zero but it never quite gets there. In mathematics we say that the function **asymptotes** to the value zero. This means that it approaches that value closer and closer without ever actually getting there.

Graph an Exponential Decay Function

The graph of an exponential decay function will always take the same basic shape as the one in the previous figure. Let's graph another example by making a table of values.

Example 1

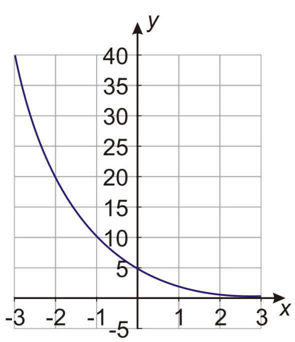
Graph the exponential function $y = 5 \cdot \left(\frac{1}{2}\right)^x$

Solution

Let's start by making a table of values.

x	$y = 5 \cdot \left(\frac{1}{2}\right)^x$
-3	$y = 5 \cdot \left(\frac{1}{2}\right)^{-3} = 5.2^3 = 40$
-2	$y = 5 \cdot \left(\frac{1}{2}\right)^{-2} = 5.2^2 = 20$
-1	$y = 5 \cdot \left(\frac{1}{2}\right)^{-1} = 5.2^1 = 10$
0	$y = 5 \cdot \left(\frac{1}{2}\right)^{0} = 5.1 = 5$
1	$y = 5 \cdot \left(\frac{1}{2}\right)^{1} = \frac{5}{2}$
2	$y = 5 \cdot \left(\frac{1}{2}\right)^{2} = \frac{5}{4}$

Now let's graph the function.



Remember that a fraction to a negative power is equivalent to its reciprocal to the same positive power.

We said that an exponential decay function has the same general form as an exponentially increasing function, but that the base b is a positive number less than one. When b can be written as a fraction, we can use the Property of Negative Exponents that we discussed in Section 8.3 to write the function in a different form.

For instance, $y = 5 \cdot \left(\frac{1}{2}\right)^x$ is equivalent to $5 \cdot 2^{-x}$.

These two forms are both commonly used so it is important to know that they are equivalent.

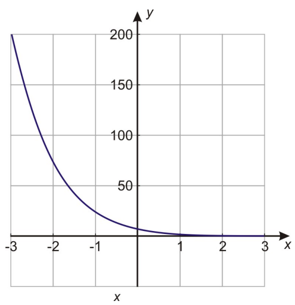
Example 2

Graph the exponential function $y = 8 \cdot 3^{-x}$.

Solution

Here is our table of values and the graph of the function.

x	$y = 8 \cdot 3^{-x}$
-3	$y = 8.3^{-(-3)} = 8 \cdot 3^3 = 216$
-2	$y = 8.3^{-(-2)} = 8 \cdot 3^2 = 72$
-1	$y = 8.3^{-(-1)} = 8 \cdot 3^1 = 24$
0	$y = 8 \cdot 3^0 = 8$
1	$y = 8 \cdot 3^{-1} = \frac{8}{3}$
2	$y = 8 \cdot 3^{-2} = \frac{8}{9}$



Compare Graphs of Exponential Decay Functions

You might have noticed that an exponentially decaying function is very similar to an exponentially increasing function. The two types of functions behave similarly, but they are backwards from each other.

The increasing function starts very small and increases very quickly and ends up very, very big. While the decreasing function starts very big and decreases very quickly to soon become very, very small. Let's graph two such functions together on the same graph and compare them.

Example 3

Graph the functions $y = 4^x$ *and* $y = 4^{-x}$ *on the same coordinate axes.*

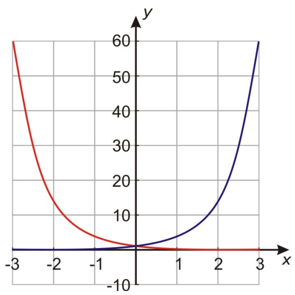
Solution

Here is the table of values and the graph of the two functions.

Looking at the values in the table we see that the two functions are "backwards" of each other in the sense that the values for the two functions are reciprocals.

x	$y = 4^x$	$y = 4^{-x}$
-3	$y = 4^{-3} = \frac{1}{64}$	$y = 4^{-(-3)} = 64$
-2	$y = 4^{-2} = \frac{1}{16}$	$y = 4^{-(-2)} = 16$
-1	$y = 4^{-1} = \frac{1}{4}$	$y = 4^{-(-1)} = 4$
0	$y = 4^0 = 1$	$y = 4^0 = 1$
1	$y = 4^1 = 4$	$y = 4^{-1} = \frac{1}{4}$
2	$y = 4^2 = 16$	$y = 4^{-2} = \frac{1}{16}$
3	$y = 4^3 = 64$	$y = 4^{-3} = \frac{1}{64}$

Here is the graph of the two functions. Notice that the two functions are mirror images of each others if the mirror is placed vertically on the y-axis.



Solve Real-World Problems Involving Exponential Decay

Exponential decay problems appear in several application problems. Some examples of these are **half-life problems**, and **depreciation problems**. Let's solve an example of each of these problems.

Example 4 Half-Life

A radioactive substance has a half-life of one week. In other words, at the end of every week the level of radioactivity is half of its value at the beginning of the week. The initial level of radioactivity is 20 *counts per second.*

a) Draw the graph of the amount of radioactivity against time in weeks.

b) Find the formula that gives the radioactivity in terms of time.

c) Find the radioactivity left after three weeks

Solution

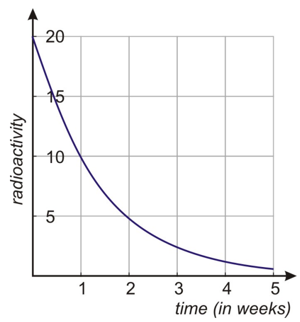
Let's start by making a table of values and then draw the graph.

Table 8.1:

time	radioactivity
0	20

Table 8.1: (continued)

time	radioactivity
1	10
2	5
3	2.5
4	1.25
5	0.625



Exponential decay fits the general formula

$$y = A \cdot b^x$$

In this case

y is the amount of radioactivity

x is the time in weeks

$A = 20$ is the starting amount

$b = \frac{1}{2}$ since the substance losses half its value each week

The formula for this problem is: $y = 20 \cdot \left(\frac{1}{2}\right)^x$ or $y = 20 \cdot 2^{-x}$.

Finally, to find out how much radioactivity is left after three weeks, we use $x = 3$ in the formula we just found.

$$y = 20 \cdot \left(\frac{1}{2}\right)^3 = \frac{20}{8} = 2.5$$

Example 5 Depreciation

The cost of a new car is \$32,000. *It depreciates at a rate of* 15% *per year. This means that it looses* 15% *of each value each year.*

Draw the graph of the car's value against time in year.

Find the formula that gives the value of the car in terms of time.

Find the value of the car when it is four years old.

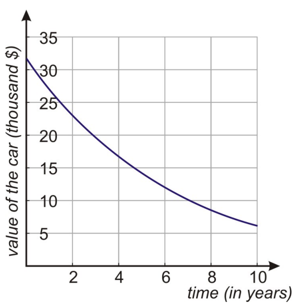
Solution

Let's start by making a table of values. To fill in the values we start with 32,000 at time $t = 0$. Then we multiply the value of the car by 85% for each passing year. (Since the car looses 15% of its value, that means that it keeps 85% of its value). Remember that 85% means that we multiply by the decimal 0.85.

Table 8.2:

Time	Value(Thousands)
0	32
1	27.2
2	23.1
3	19.7
4	16.7
5	14.2

Now draw the graph



Let's start with the general formula

$$y = A \cdot b^x$$

In this case:

y is the value of the car

x is the time in years

$A = 32$ is the starting amount in thousands

$b = 0.85$ since we multiply the amount by this factor to get the value of the car next year

The formula for this problem is $y = 32 \cdot (0.85)^x$.

Finally, to find the value of the car when it is four years old, we use $x = 4$ in the formula we just found.

$y = 32 \cdot (0.85)^4 = 16.7$ thousand dollars or $\$16,704$ if we don't round.

Review Questions

Graph the following exponential decay functions.

1. $y = \frac{1}{5}^x$
2. $y = 4 \cdot \left(\frac{2}{3}\right)^x$
3. $y = 3^{-x}$
4. $y = \frac{3}{4} \cdot 6^{-x}$

Solve the following application problems.

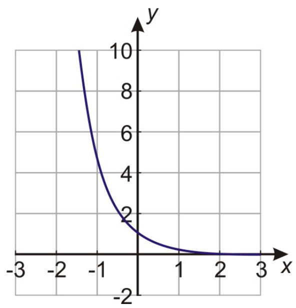
5. The cost of a new ATV (all-terrain vehicle) is $7200. It depreciates at 18% per year. Draw the graph of the vehicle's value against time in years.

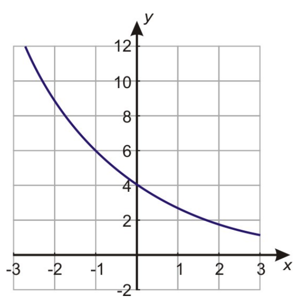
Find the formula that gives the value of the ATV in terms of time.
Find the value of the ATV when it is ten year old.

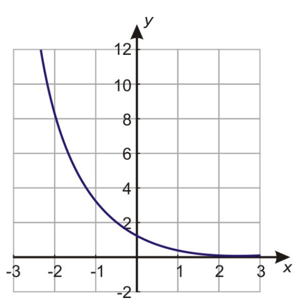
6. A person is infected by a certain bacterial infection. When he goes to the doctor the population of bacteria is 2 million. The doctor prescribes an antibiotic that reduces the bacteria population to $\frac{1}{4}$ of its size each day.

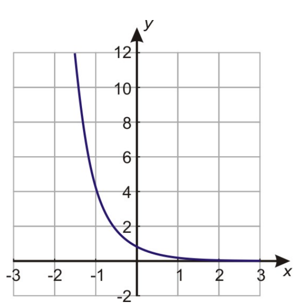
   (a) Draw the graph of the size of the bacteria population against time in days.
   (b) Find the formula that gives the size of the bacteria population in term of time.
   (c) Find the size of the bacteria population ten days after the drug was first taken.
   (d) Find the size of the bacteria population after 2 weeks (14 days)

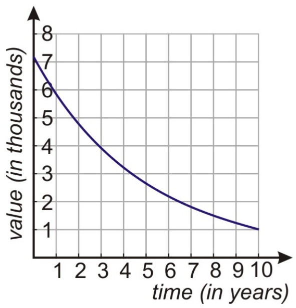
Review Answers

1. 

2. 

3. 

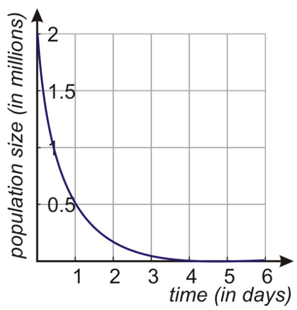
4. 

5. 

Formula $y = 7200 \cdot (0.82)^x$

At $x = 10$, $y = \$989.62$

6.

7. (a) 

(b) Formula $y = 2,000,000 \cdot 4 - x$ or $y = 2,000,000 \cdot (0.25)^x$

(c) At $x = 5$, $y = 1953$ bacteria

(d) At $x = 10$, $y = 1.9$ (≈ 2 bacteria)

(e) At $x = 14$, $y = 0.007$ (bacteria effectively gone)

8.7 Geometric Sequences and Exponential Functions

Learning Objectives

- Identify a geometric sequence
- Graph a geometric sequence.
- Solve real-world problems involving geometric sequences.

Introduction

Consider the following question.

Which would you prefer, being given one million dollars, or one penny the first day, double that penny the next day, and then double the previous day's pennies and so on for a month?

At first glance it's easy to say "Give me the million please!"

However, let's do a few calculations before we decide in order to see how the pennies add up. You start with a penny the first day and keep doubling each day. Doubling means that we keep multiplying by 2 each day for one month (30 days).

On the 1^{st} day we get	1 penny $= 2^0$ pennies	Look at the exponent on the 2.
On the 2^{nd} day we get	2 pennies $= 2^1$ pennies	Can you see the pattern?
On the 3^{rd} day we get	4 pennies $= 2^2$ pennies	The exponent increases by 1 each day.
On the 4^{th} day we get	8 pennies $= 2^3$ pennies	So we calculate that ...
On the 30^{th} day we get	$= 2^{29}$ pennies	

$2^29 = 536,870,912$ pennies or $\$5,368,709$ which is well over 5 times greater than one million dollars.

So even just considering the pennies given on the final day, the pennies win!

The previous problem is an example of a geometric sequence. In this section, we will find out what a geometric sequence is and how to solve problems involving geometric sequences.

Identify a Geometric Sequence

The problem above is an example of a **geometric sequence.** A geometric sequence is a sequence of numbers in which each number in the sequence is found by multiplying the previous number by a fixed amount called the **common ratio.** In other words, the ratio between a term and the previous term is always the same. In the previous example the common ratio was 2, as the number of pennies doubled each day.

The common ratio, r, in any geometric sequence can be found by dividing any term by the preceding term.

Here are some examples of geometric sequences and their common ratios.

$4, 16, 64, 256, \ldots$	$r = 4$	(divide $16 \div 4$ to get 4)
$15, 30, 60, 120, \ldots$	$r = 2$	(divide $30 \div 15$ to get 2)
$11, \frac{11}{2}, \frac{11}{4}, \frac{11}{8}, \frac{11}{16}, \ldots$	$r = \frac{1}{2}$	(divide $\frac{1}{2} \div 11$ to get $\frac{1}{2}$)
$25, -5, 1, -\frac{1}{5}, -\frac{1}{25}, \ldots$	$r = -\frac{1}{5}$	(divide $1 \div (-5)$to get $-\frac{1}{5}$)

If we know the common ratio, we can find the next term in the sequence just by multiplying the last term by it. Also, if there are any terms missing in the sequence, we can find them by multiplying the terms before the gap by the common ratio.

Example 1

Fill is the missing terms in the geometric sequences.

a) $1, \_\_\_\_, 25, 125, \_\_\_\_$

b) $20, \_\_\_\_, 5, \_\_\_\_, 1.25$

Solution

a) First we can find the common ratio by dividing 125 by 25 to obtain $r = 5$.

To find the 1^{st} missing term we multiply 1 by the common ratio $1 \cdot 5 = 5$

To find the 2^{nd} missing term we multiply 125 by the common ratio $125 \cdot 5 = 625$

Answer Sequence (a) becomes $1, 5, 25, 125, 625$.

b) We first need to find the common ratio, but we run into difficulty because we have no terms next to each other that we can divide.

However, we know that to get from 20 to 5 in the sequence we must multiply 20 by the common ratio twice. We multiply it once to get to the second term in the sequence and again to get to five. So we can say

$20 \cdot r \cdot r = 5$ or $20 \cdot r^2 = 5$

Divide both sides by 20 and find $r^2 = \frac{5}{20} = \frac{1}{4}$ or $r = \frac{1}{2}$ (because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$).

To get the 1^{st} missing term we multiply 20 by $\frac{1}{2}$ and get $20 \cdot \frac{1}{2} = 10$.

To get the 2^{nd} missing term we multiply 5 by $\frac{1}{2}$ and get: $5 \cdot \frac{1}{2} = 2.5$.

Answer

Sequence (b) becomes $20, 10, 5, 2.5, 1.25$.

You see that we can find any term in a geometric sequence simply by multiplying the last term by the common ratio. Then, we keep multiplying by the common ratio until we get to a term in the sequence that we want. However, if we want to find a term that is a long way from the start it becomes tedious to keep multiplying over and over again. There must be a better way to do this.

Because we keep multiplying by the same number, we can use exponents to simplify the calculation. For example, let's take a geometric sequence that starts with the number 7 and has common ratio of 2.

The 1^{st} term is:	7
We obtain the 2^{nd} term by multiplying by 2.	$7 \cdot 2$
We obtain the 3^{rd} term by multiplying by 2 again.	$7 \cdot 2 \cdot 2$
We obtain the 4^{th} term by multiplying by 2 again.	$7 \cdot 2 \cdot 2 \cdot 2$
The n^{th} term would be	$7 \cdot 2^{n-1}$

The n^{th} term is $7 \cdot 2^{n-1}$ because the 7 is multiplied by one factor of two for the 2^{nd} term, two factors of 2 for the third term and always by one less factor of 2 than the term's place in the sequence. In general terms we write geometric sequence with n terms like this

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

The general formula for finding specific terms in a geometric sequence is

n^{th} term in a geometric sequence $a_n = ar_1 r^{n-1} (a_1 = \text{first term}, \ r = \text{common ratio})$

Example 2

For each of these geometric sequences, find the eighth term in the sequence.

a) $1, 2, 4, \ldots$

b) $16, -8, 4, -2, 1, \ldots$

Solution

a) First we need to find the common ratio $r = \frac{2}{1} = 2$.

The eighth term is given by the formula $2 = 1 \cdot 2^7 = 128$.

In other words, to get the eighth term we started with the first term which is 1 and multiplied by 2 seven times.

b) The common ratio is $r = \frac{-8}{16} = \frac{-1}{2}$

The eighth term in the sequence is: $a_g = a_1 r^7 = 16 \cdot \left(\frac{-1}{2}\right)^7 = 16 \cdot \frac{(-1)^7}{2^7} = 16 \cdot \frac{-1}{2^7} = \frac{-16}{128} = -\frac{1}{8}$

Look again at the terms in *b*).

When a **common ratio is negative** the terms in the sequence alternate **positive, negative, positive, negative** all the way down the list. When you see this, you know the common ratio is negative.

Graph a Geometric Sequence

Geometric sequences and exponential functions are very closely related. You just learned that to get to the next term in a geometric sequence you multiply the last term in the sequence by the common ratio. In Sections 8.5 and 8.6, you learned that an exponential function is multiplied by the same factor every time the independent value is increased by one unit. As a result, geometric sequences and exponential functions look very similar.

The fundamental difference between the two concepts is that a geometric sequence is **discrete** while an exponential function is **continuous.**

Discrete means that the sequence has values only at distinct points (the 1^{st} term, 2^{nd} term, etc)

Continuous means that the function has values for all possible values of x. The integers are included, but also all the numbers in between.

As a result of this difference, we use a geometric series to describe quantities that have values at discrete points, and we use exponential functions to describe quantities that have values that change continuously.

Here are two examples one discrete and one continuous.

Example 3 Discrete sequence

An ant walks past several stacks of Lego blocks. There is one block in the first stack, 3 *blocks in the* 2^{nd} *stack and* 9 *blocks in the* 3^{rd} *stack. In fact, in each successive stack there are triple the number of blocks than there were in the previous stack.*

In this example, each stack has a distinct number of blocks and the next stack is made by adding a certain number of whole pieces all at once. More importantly, however, there are no values of the sequence **between** the stacks. You cannot ask how high the stack is between the 2^{nd} and 3^{rd} stack, as no stack exists at that position!

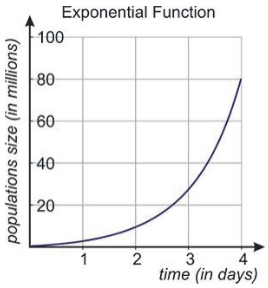
Example 4 Continuous Function

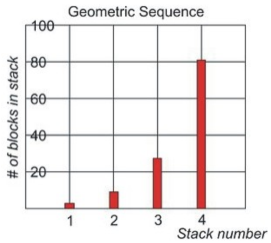
A population of bacteria in a Petri dish increases by a factor of three every 24 hours. *The starting population is* 1 million *bacteria. This means that on the first day the population increases to* 3 million *on the second day to* 9 million *and so on.*

In this example, the population of bacteria is continuous. Even though we only measured the population every 24 hours we know that it does not get from 1 million to 3 million all at once but that the population changes bit by bit over the 24 hours. In other words, the bacteria are always there, and you can, if you so wish, find out what the population is at any time during a 24 hours period.

When we graph an exponential function, we draw the graph with a solid curve to signify that the function has values at any time during the day. On the other hand, when we graph a geometric sequence, we draw discrete points to signify that the sequence only has value at those points but not in between.

Here are graphs for the two examples we gave before:





Solve Real-World Problems Involving Geometric Sequences

Let's solve two application problems involving geometric sequences.

Example 5 Grains of rice on a chessboard

A courtier presented the Indian king with a beautiful, hand-made chessboard. The king asked what he would like in return for his gift and the courtier surprised the king by asking for one grain of rice on the first square, two grains on the second, four grains on the third, etc. The king readily agreed and asked for the rice to be brought. (From Meadows et al. 1972, p.29 via Porritt 2005) How many grains of rice does the king have to put on the last square?

[Wikipedia; GNU-FDL]

Solution

A chessboard is an 8×8 square grid, so it contains a total of 64 squares.

The courtier asked for one grain of rice on the first square, 2 grains of rice on the second square, 4 grains of rice on the third square and so on.

We can write this as a geometric sequence.

$$1, 2, 4, \ldots$$

The numbers double each time, so the common ratio is $r = 2$.

The problem asks how many grains of rice the kind needs to put on the last square. What we need to find is the 64^{th} term in the sequence.

This means multiplying the starting term, 1, by the common ratio 64 times in a row. Instead of doing this, let's use the formula we found earlier.

$a_n = a_1 r^{n-1}$, where a_n is the n^{th} term, a_1 is the first term and r is the common ratio.

$a_{64} = 1 \cdot 2^{63} = 9,223,372,036,854,775,808$ grains of rice.

Second Half of the Chessboard

The problem we just solved has real applications in business and technology. In technology the strategy, strategy, the **Second Half of the Chessboard** is a phrase, coined by a man named Ray Kurzweil, in reference to the point where an exponentially growing factor begins to have a significant economic impact on an organization's overall business strategy.

The total number of grains of rice on the **first half** of the chessboard is $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + \ldots + 2,147,483,648$, for a total of exactly $4,294,967,295$ grains of rice, or about $100,000$ kg of rice, with the mass of one grain of rice being roughly 25 mg. This total amount is about $\frac{1}{1,000,000}^{th}$ of total rice production in India in year 2005 and was considered economically viable to the emperor of India.

The total number of grains of rice on the **second half** of the chessboard is $2^{32} + 2^{33} + 2^{34} + \ldots + 2^{63}$, for a total of $18,446,744,069,414,584,320$ grains of rice. This is about 460 billion tons, or 6 times the entire weight of all living matter on Earth. The king did not realize what he was agreeing. Next time maybe he should read the fine print! [Wikipedia; GNU-FDL]

Example 6 Bouncing Ball

A super-ball has a 75% *rebound ratio. When you drop it from a height of* 20 feet *, it bounces and bounces and bounces...*

(a) How high does the ball bounce after it strikes the ground for the third time?

(b) How high does the ball bounce after it strikes the ground for the seventeenth time?

Solution

75% rebound ratio means that after the ball bounces on the ground, it reaches a maximum height that is 75% or (3/4) of its previous maximum height. We can write a geometric sequence that gives the maximum heights of the ball after each bounce with the common ratio of $r = \frac{3}{4}$.

$$20, 20 \cdot \frac{3}{4}, 20 \cdot \left(\frac{3}{4}\right)^2, 20 \cdot \left(\frac{3}{4}\right)^3 \ldots$$

a) The ball starts at a height of 20 feet, after the first bounce it reaches a height of $20 \cdot \frac{3}{4} = 15$ feet

After the second bounce, it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^2 = 11.25$ feet

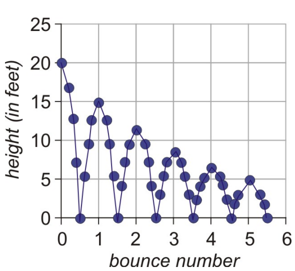
After the third bounce, it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^3 = 8.44$ feet

Notice that the height after the first bounce corresponds to the second term in the sequence, the height after the second bounce corresponds to the third term in the sequence and so on.

b) This means that the height after the seventeenth bounce corresponds to the 18^{th} term in the sequence. You can find the height by using the formula for the 18^{th} term:

$$a_{18} = 20 \cdot \left(\frac{3}{4}\right)^{17} = 0.15 \text{ feet}$$

Here is a graph that represents this information.



Review Questions

Determine the first five terms of each geometric sequence.

1. $a_1 = 2$, $r = 3$
2. $a_1 = 90$, $r = -\frac{1}{3}$
3. $a_1 = 6$, $r = -2$

Find the missing terms in each geometric series:

4. $3, \_\_\_\_, 48, 192, \_\_\_\_$
5. $81, \_\_\_\_, \_\_\_\_, \_\_\_\_, 1$
6. $\frac{9}{4}, \_\_\_\_, \_\_\_\_, \frac{2}{3}, \_\_\_\_$

Find the indicated term of each geometric series.

7. $a_1 = 4$, $r = 2$ Find a_6.
8. $a_1 = -7, r = -\frac{3}{4}$ Find a_4.
9. $a_1 = -10$, $r = -3$ Find a_{10}.
10. Anne goes bungee jumping off a bridge above water. On the initial jump, the bungee cord stretches by 120 feet. On the next bounce, the stretch is 60% of the original jump and each additional bounce stretches the rope by 60% of the previous stretch.
    (a) What will the rope stretch be on the third bounce?
    (b) What will be the rope stretch be on the 12^{th} bounce?

Review Answers

1. $2, 6, 18, 54, 162$
2. $90, -30, 10, -\frac{10}{3}, \frac{10}{9}$
3. $6, -12, 24, -48, 96$
4. $3, 12, 48, 192, 768$
5. $81, 27, 9, 3, 1$
6. $\frac{9}{4}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}$
7. $a_6 = 128$
8. $a_4 = 2.95$
9. $a_{10} = 196830$
10.
11. (a) 43.2 feet
    (b) 0.44 feet

8.8 Problem-Solving Strategies

Learning Objectives

- Read and understand given problem situations.
- Make tables and identify patterns.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

Problem solving appears everywhere, in your regular life as well as in all jobs and careers. Of course, in this manual we concentrate on solving problems that involve algebra. From previous sections, remember our problem solving plan.

Step 1

Understand the problem.

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2

Devise a plan – Translate.

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solving your problem.

Step 3

Carry out the plan – Solve.

This is where you solve the equation you came up with in Step 2.

Step 4

Look – Check and Interpret.

Check to see if you used all your information and that the answer makes sense.

Examples of Exponential Word Problems

In this section, we will be applying this problem solving strategy to solving real-world problems where exponential functions appear. Compound interest, loudness of sound, population increase, population decrease or radioactive decay are all applications of exponential functions. In these problems, we will use the methods of constructing a table and identifying a pattern to help us devise a plan for solving the problems.

Example 1 Compound Interest

Suppose \$4000 *is invested at* 6% *interest compounded annually. How much money will there be in the bank at the end of five years? At the end of* 20 years?

Solution

Step 1

Read the problem and summarize the information.

\$4000 is invested at 6% interest compounded annually

We want to know how much money we have after five years.

Assign variables.

Let $x =$ time in years

Let $y =$ amount of money in investment account

Step 2

Look for a pattern.

We start with \$4000 and each year we apply a 6% interest on the amount in the bank

Start		\$4000
1^{st} year	Interest	$= 4000 \times (0.06) = \$240$
	This is added to the previous amount	$= \$4000 + \$4000 \times (0.06)$
		$= \$4000(1 + 0.06)$
		$= \$4000(1.06)$
		$= \$4240$
2^{nd} year	Previous amount + interest on the previous amount.	$= \$4240(1 + 0.06)$
		$= \$4240(1.06)$
		$= \$4494.40$

The pattern is that each year we multiply the previous amount by the factor of 1.06.

Let's fill in a table of values.

Time (Years)	0	1	2	3	4	5
Investments Amount(\$)	4000	4240	4494.4	4764.06	5049.90	5352.9

Answer We see that at the end of five years we have \$5352.90 in the investment account.

Step 3 In the case of 5 years, we don't need an equation to solve the problem. However, if we want the amount at the end of 20 years, we get tired of multiplying by 1.06, and we want a formula.

Since we take the original investment and keep multiplying by the same factor of 1.06, that means we can use exponential notation.

$$y = 4000 \cdot (1.06)^x$$

To find the amount after 5 years we use $x = 5$ in the equation.

$$y = 4000 \cdot (1.06)^5 = \$5352.90$$

To find the amount after 20 years we use $x = 20$ in the equation.

$$y = 4000 \cdot (1.06)^{20} = \$12828.54$$

Step 4 Looking back over the solution, we see that we obtained the answers to the questions we were asked and the answers make sense.

To check our answers we can plug in some low values of x to see if they match the values in the table:

$$x = 0, \qquad y = 4000 \cdot (1.06)^0 = 4000$$
$$x = 1, \qquad y = 4000 \cdot (1.06)^1 = 4240$$
$$x = 2, \qquad y = 4000 \cdot (1.06)^2 = 4494.4$$

The answers make sense because after the first year the amount goes up by \$240 (6% of \$4000).

The amount of increase gets larger each year and that makes sense because the interest is 6% of an amount that is larger and larger every year.

Example 2 Population decrease

In 2002, the population of school children in a city was 90,000. *This population decreases at a rate of* 5% *each year. What will be the population of school children in year 2010?*

Solution

Step 1

Read the problem and summarize the information.

In 2002, population => 90,000.

Rate of decrease = 5% each year.

What is the population in year 2010?

Assign variables.

Let x = time since 2002 (in years)

Let y = population of school children

Step 2

Look for a pattern.

Let's start in 2002.

Population = 90,000

Rate of decrease is 5% each year, so we need to find the amount of increase by $90,000 \times 0.05$ and subtract this increase from the original number $90,000 - 90,000 \times 0.05 = 90,000(1 - 0.05) = 90,000 \times 0.95$.

In 2003	Population $= 90,000 \times 0.95$
In 2004	Population $= 90,000 \times 0.95 \times 0.95$

The pattern is that for each year we multiply by a factor of 0.95

Let's fill in a table of values:

Year	2002	2003	2004	2005	2006	2007
Population	90,000	85,500	81,225	77,164	73,306	69,640

Step 3

Let's find a formula for this relationship.

Since we take the original population and keep multiplying by the same factor of 0.95, this pattern fits to an exponential formula.

$$y = 90000 \cdot (0.95)^x$$

To find the population in year 2010, plug in $x = 8$ (number of years since 2002)

$y = 90000 \cdot (0.95)^8 = 59,708$ school children

Step 4

Looking back over the solution, we see that we answered the question we were asked and that it makes sense.

The answer makes sense because the numbers decrease each year as we expected. We can check that the formula is correct by plugging in the values of x from the table to see if the values match those given by the formula.

Year 2002, $x = 0$	Population $= y = 90000 \cdot (0.95)^0 = 90,000$
Year 2003, $x = 1$	Population $= y = 90000 \cdot (0.95)^1 = 85,500$
Year 2004, $x = 2$	Population $= y = 90000 \cdot (0.95)^2 = 81,225$

Example 3 Loudness of sound

Loudness is measured in decibels (dB). *An increase in loudness of* 10 *decibels means the sound intensity increases by a factor of* 10. *Sound that is barely audible has a decibel level of* 0 dB *and an intensity level of* 10^{-12} W/m^2. *Sound painful to the ear has a decibel level of* 130 dB *and an intensity level of* 10 W/m^2.

(a) The decibel level of normal conversation is 60 dB. *What is the intensity of the sound of normal conversation?*

(b) The decibel level of a subway train entering a station is 100 dB. *What is the intensity of the sound of the train?*

Solution:

Step 1

Read the problem and summarize the information.

For 10 decibels, sound intensity increases by a factor of 10.

Barely audible sound = 0 dB = 10^{-12} W/m^2

Ear-splitting sound = 130 dB = 10 W/m^2

Find intensity at 60 dB and find intensity at 100 dB.

Assign variables.

Let x = sound level in decibels (dB)

Let y = intensity of sound (W/m^2)

Step 2

Look for a pattern.

Let's start at 0 dB

For 0 dB	Intensity $= 10^{-12}$ W/m^2

For each decibel the intensity goes up by a factor of ten.

For 10 dB	Intensity $= 10^{-12} \times 10$ W/m^2
For 20 dB	Intensity $= 10^{-12} \times 10 \times 10$ W/m^2
For 30 dB	Intensity $= 10^{-12} \times 10 \times 10 \times 10$ W/m^2

The pattern is that for each 10 decibels we multiply by a factor of 10.

Let's fill in a table of values.

Loudness (dB)	0	10	20	30	40	50
Intensity (W/m^2)	10^{-12}	10^{-11}	10^{-10}	10^{-9}	10^{-8}	10^{-7}

Step 3

Let's find a formula for this relationship.

Since we take the original sound intensity and keep multiplying by the same factor of 10, that means we can use exponential notation.

$$y = 10^{-12} \cdot 10^{\frac{x}{10}}$$

The power is $\frac{x}{10}$, since we go up by 10 dB each time.

To find the intensity at 60 dB we use $x = 60$ in the equation.

$$y = 10^{-12} \cdot (10)^{\left(\frac{60}{10}\right)} = 10^{-12} \cdot (10)^6 = 10^{-6} \text{ W/m}^2$$

To find the intensity at 100 dB we use $x = 100$ in the equation.

$$y = 10^{-12} \cdot (10)^{\left(\frac{100}{10}\right)} = 10^{-12} \cdot (10)^{10} = 10^{-2} \text{ W/m}^2$$

Step 4

Looking back over the solution, we see that we did not use all the information we were given. We still have the fact that a decibel level of 130 dB has an intensity level of 10 W/m^2.

We can use this information to see if our formula is correct. Use $x = 130$ in our formula.

$$y = 10^{-12} \cdot (10)^{\left(\frac{130}{10}\right)} = 10^{-12} \cdot (10)^{13} = 10 \text{ W/m}^2$$

The formula confirms that a decibel level of 130 dB corresponds to an intensity level of 10 W/m^2.

Review Questions

Apply the problem-solving techniques described in this section to solve the following problems.

1. **Half-life** Suppose a radioactive substance decays at a rate of 3.5% per hour. What percent of the substance is left after 6 hours?
2. **Population decrease** In 1990, a rural area has 1200 bird species. If species of birds are becoming extinct at the rate of 1.5% per decade (ten years), how many bird species will there be left in year 2020?
3. **Growth** Nadia owns a chain of fast food restaurants that operated 200 stores in 1999. If the rate of increase is 8% annually, how many stores does the restaurant operate in 2007?
4. **Investment** Peter invests \$360 in an account that pays 7.25% compounded annually. What is the total amount in the account after 12 years?

Review Answers

1. $100(.965)^x = 100(.965)^6 = 80.75\%$
2. $1200(.985)^x = 1200(.985)^3 = 1147$
3. $200(1.08)^x = 200(1.08)^8 = 370$
4. $360(1.0725)^x = 360(1.0725)^{12} = \$833.82$

Chapter 9

Factoring Polynomials; More on Probability

9.1 Addition and Subtraction of Polynomials

Learning Objectives

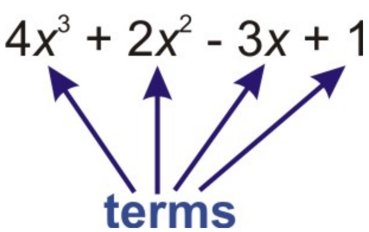
- Write a polynomial expression in standard form.
- Classify polynomial expression by degree
- Add and subtract polynomials
- Problem solving using addition and subtraction of polynomials

Introduction

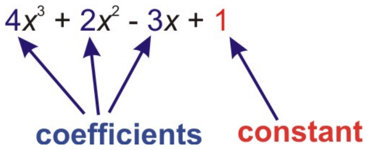
So far we have seen functions described by straight lines (linear functions) and functions where the variable appeared in the exponent (exponential functions). In this section we will introduce polynomial functions. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial.

$$4x^3 + 2x^2 - 3x + 1$$

Each part of the polynomial that is added or subtracted is called a **term** of the polynomial. The example above is a polynomial with *four terms*.



The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant**.



In this case, the coefficient of x^3 is 4, the coefficient of x^2 is 2, the coefficient of x is -3 and the constant is 1.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a **degree.** This is the power of the variable in that term.

$4x^3$ Has a degree of 3 and is called a **cubic term or** 3^{rd} **order term.**

$2x^2$ Has a degree of 2 and is called the **quadratic term or** 2^{nd} **order term**.

$-3x$ Has a degree of 1 and is called the **linear term or** 1^{st} **order term.**

1 Has a degree of 0 and is called the **constant.**

By definition, **the degree of the polynomial** is the same as the degree of the term with the highest degree. This example is a polynomial of degree 3, which is also called a "cubic" polynomial. (Why do you think it is called a cubic?).

Polynomials can have more than one variable. Here is another example of a polynomial.

$$t^4 - 6s^3t^2 - 12st + 4s^4 - 5$$

This is a polynomial because all exponents on the variables are positive integers. This polynomial has five terms. Let's look at each term more closely. **Note:** *The degree of a term is the sum of the powers on each variable in the term.*

t^4 Has a degree of 4, so it's a 4^{th} order term

$-6s^3t^2$ Has a degree of 5, so it's a 5^{th} order term.

-12^{st} Has a degree of 2, so it's a 2^{nd} order term

$4s^4$ Has a degree of 4, so it's a 4^{th} order term

-5 Is a constant, so its degree is 0.

Since the highest degree of a term in this polynomial is 5, then this is polynomial of degree 5 or a 5^{th} order polynomial.

A polynomial that has only one term has a special name. It is called a **monomial** (*mono means one*). A monomial can be a constant, a variable, or a product of a constant and one or more variables. You can see that each term in a polynomial is a monomial. A polynomial is the sum of monomials. Here are some examples of monomials.

$$\underbrace{b^2}_{\text{This is a monomial}} \qquad \underbrace{8}_{\text{So is this}} \qquad \underbrace{-2ab^2}_{\text{and this}} \qquad \underbrace{\frac{1}{4}x^4}_{\text{and this}} \qquad \underbrace{-29xy}_{\text{and this}}$$

Example 1

For the following polynomials, identify the coefficient on each term, the degree of each term and the degree of the polynomial.

a) $x^5 - 3x^3 + 4x^2 - 5x + 7$

b) $x^4 - 3x^3y^2 + 8x - 12$

Solution

a) $x^5 - 3x^3 + 4x^2 - 5x + 7$

The coefficients of each term in order are $1, -3, 4, -5$ and the constant is 7.

The degrees of each term are $5, 3, 2, 1,$ and 0. Therefore, the degree of the polynomial is 5.

b) $x^4 - 3x^3y^2 + 8x - 12$

The coefficients of each term in order are $1, -3, 8$ and the constant is -12.

The degrees of each term are $4, 5, 1,$ and 0. Therefore, the degree of the polynomial is 5.

Example 2

Identify the following expressions as polynomials or non-polynomials.

a) $5x^2 - 2x$

b) $3x^2 - 2x^{-2}$

c) $x\sqrt{x} - 1$

d) $\frac{5}{x^3+1}$

e) $4x^{1/3}$

f) $4xy^2 - 2x^2y - 3 + y^3 - 3x^3$

Solution

(a) $5x^2 - 2x$ This ***is*** a polynomial.

(b) $3x^2 - 2x^{-2}$ This is ***not*** a polynomial because it has a negative exponent.

(c) $x\sqrt{x} - 1$ This is ***not*** a polynomial because is has a square root.

(d) $\frac{5}{x^3+1}$ This is ***not*** a polynomial because the power of x appears in the denominator.

(e) $4x^{1/3}$ This is ***not*** a polynomial because it has a fractional exponent.

(f) $4xy^2 - 2x^y - 3 + y^3 - 3x^3$ This ***is*** a polynomial.

You saw that each term in a polynomial has a degree. The degree of the highest term is also the degree of the polynomial. Often, we arrange the terms in a polynomial so that the term with the highest degree is first and it is followed by the other terms in order of decreasing power. This is called **standard form.**

The following polynomials are in standard form.

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4b^3 - 2a^3b^3 + 3a^4b - 5ab^2 + 2$$

The first term of a polynomial in standard form is called the **leading term** and the coefficient of the leading term is called the **leading coefficient.**

The first polynomial above has a leading term of $4x^4$ and a leading coefficient of 4.

The second polynomial above has a leading term of a^4b^3 and a leading coefficient of 1.

Example 3

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

(a) $7 - 3x^3 + 4x$

(b) $ab - a^3 + 2b$

(c) $-4b + 4 + b^2$

Solution

(a) $7 - 3x^3 + 4x$ is rearranged as $-3x^3 + 4x + 7$. The leading term is $-3x^3$ and the leading coefficient is -3.

(b) $ab - a^3 + 2b$ is rearranged as $-a^3 + ab + 2b$. The leading term is $-a^3$ and the leading coefficient is -1.

(c) $-4b + 4 + b^2$ is rearranged as $b^2 - 4b + 4$. The leading term is b^2 and the leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

$2x^2y$ and $5x^2y$ are like terms.

$6x^2y$ and $6xy^2$ are not like terms.

If we have a polynomial that has like terms, we simplify by combining them.

$$x^2 + \underbrace{6xy}_{\nearrow} - \underbrace{4xy}_{\nwarrow} + y^2$$

Like terms

This sample polynomial simplified by combining the like terms $6xy - 4xy = 2xy$. We write the simplified polynomial as

$$x^2 + 2xy + y^2$$

Example 4

Simplify the following polynomials by collecting like terms and combining them.

(a) $2x - 4x^2 + 6 + x^2 - 4 + 4x$

(b) $a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$

Solution

(a) $2x - 4x^2 + 6 + x^2 - 4 + 4x$

Rearrange the terms so that like terms are grouped together

$= (-4x^2 + x^2) + (2x + 4x) + (6 - 4)$

Combine each set of like terms by adding or subtracting the coefficients

$= -3x^2 + 6x + 2$

(b) $a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$

Rearrange the terms so that like terms are grouped together:

$$= (a^3b^3 - a^3b^3) + (-5ab^4 + 3ab^4) + 2a^3b - a^2b$$

Combine each set of like terms:

$$= 0 - 2ab^4 + 2a^3b - a^2b$$
$$= -2ab^4 + 2a^3b - a^2b$$

Add and Subtract Polynomials

Polynomial addition

To add two or more polynomials, write their sum and then simplify by combining like terms.

Example 5

Add and simplify the resulting polynomials.

(a) Add $3x^2 - 4x + 7$ and $2x^3 - 4x^2 - 6x + 5$.

(b) Add $x^2 - 2xy + y^2$ and $2y^2 - 4x^2$ and $10xy + y^3$.

Solution:

(a) Add $3x^2 - 4x + 7$ and $2x^3 - 4x^2 - 6x + 5$

$$\begin{aligned} & = (3x^2 - 4x + 7) + (2x^3 - 4x^2 - 6x + 5) \\ \text{Group like terms} \quad & = 2x^3 + (3x^2 - 4x^2) + (-4x - 6x) + (7 + 5) \\ \text{Simplify} \quad & = 2x^3 - x^2 - 10x + 12 \end{aligned}$$

(b) Add $x^2 - 2xy + y^2$ and $2y^2 - 3x^2$ and $10xy + y^3$

$$\begin{aligned} & = (x^2 - 2xy + y^2) + (2y^2 - 3x^2) + (10xy + y^3) \\ \text{Group like terms} \quad & = (x^2 - 3x^2) + (y^2 + 2y^2) + (-2xy + 10xy) + y^3) \\ \text{Simplify} \quad & = 2x^2 + 3y^2 + 8xy + y^3 \end{aligned}$$

Polynomial subtraction

To subtract one polynomial from another, add the opposite of each term of the polynomial you are subtracting.

Example 6

Subtract and simplify the resulting polynomials.

a) Subtract $x^3 - 3x^2 + 8x + 12$ from $4x^2 + 5x - 9$.

b) Subtract $5b^2 - 2a^2$ from $4a^2 - 8ab - 9b^2$.

Solution

a)

$$\begin{aligned} (4x^2 + 5x - 9) - (x^3 - 3x^2 + 8x + 12) & = (4x^2 + 5x - 9) + (-x^3 + 3x^2 - 8x - 12) \\ \text{Group like terms} & = -x^3 - (4x^2 + 3x^2) + (5x - 8x) + (-9 - 12) \\ \text{Simplify} & = -x^3 + 7x^2 - 3x - 21 \end{aligned}$$

b)

$$(4a^2 - 8ab - 9b^2) - (5b^2 - 2a^2) = (4a^2 - 8ab - 9b^2) + (-5b^2 + 2a^2)$$
$$\text{Group like terms} = (4a^2 + 2a^2) + (-9b^2 - 5b^2) - 8ab$$
$$\text{Simplify} = 6a^2 - 14b^2 - 8ab$$

Note: An easy way to check your work after adding or subtracting polynomials is to substitute a convenient value in for the variable, and check that your answer and the problem both give the same value. For example, in part (b) of Example 6, if we let $a = 2$ and $b = 3$, then we can check as follows.

Given	Solution
$(4a^2 - 8ab - 9b^2) - (5b^2 - 2a^2)$	$6a^2 - 14b^2 - 8ab$
$(4(2)^2 - 8(2)(3) - 9(3)^2) - (5(3)^2 - 2(2)^2)$	$6(2)^2 - 14(3)^2 - 8(2)(3)$
$(4(4) - 8(2)(3) - 9(9)) - (5(9) - 2(4))$	$6(4) - 14(9) - 8(2)(3)$
$(-113) - 37$	$24 - 126 - 48$
-150	-150

Since both expressions evaluate to the same number when we substitute in arbitrary values for the variables, we can be reasonably sure that our answer is correct. Note, when you use this method, do not choose 0 or 1 for checking since these can lead to common problems.

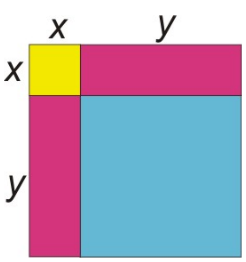
Problem Solving Using Addition or Subtraction of Polynomials

An application of polynomials is their use in finding areas of a geometric object. In the following examples, we will see how the addition or subtraction of polynomials might be useful in representing different areas.

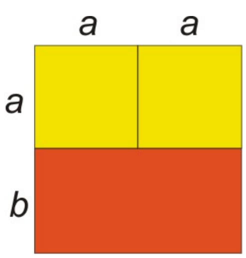
Example 7

Write a polynomial that represents the area of each figure shown.

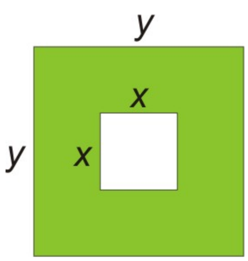
a)



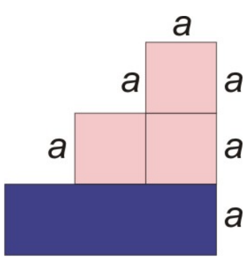
b)



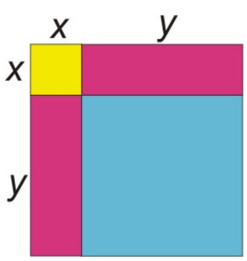
c)



d)



Solutions



a) This shape is formed by two squares and two rectangles.

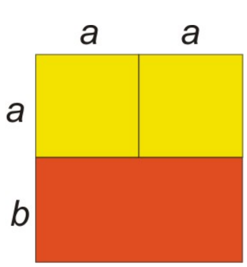
The blue square has area: $y \cdot y = y^2$

The yellow square has area: $x \cdot x = x^2$

The pink rectangles each have area: $x \cdot y = xy$

To find the total area of the figure we add all the separate areas.

$$\begin{aligned}\text{Total area} &= y^2 + x^2 + xy + xy \\ &= y^2 + x^2 + 2xy\end{aligned}$$



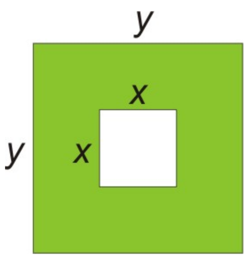
b) This shape is formed by two squares and one rectangle.

The yellow squares each have an area: $a \cdot a = a^2$.

The orange rectangle has area: $2a \cdot b = 2ab$.

To find the total area of the figure we add all the separate areas.

$$\text{Total area} = a^2 + a^2 + 2ab$$
$$= 2a^2 + 2ab$$

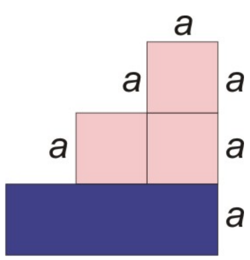


c) To find the area of the green region we find the area of the big square and subtract the area of the little square.

The big square has area $y \cdot y = y^2$.

The little square has area $x \cdot x = x^2$.

Area of the green region $= y^2 - x^2$



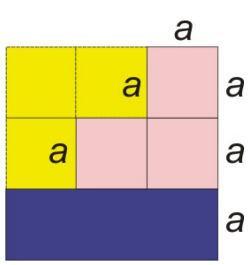
d) To find the area of the figure we can find the area of the big rectangle and add the areas of the pink squares.

The pink squares each have area: $a \cdot a = a^2$.

The blue rectangle has area: $3a \cdot a = 3a^2$.

To find the total area of the figure we add all the separate areas.

$$\text{Total area} = a^2 + a^2 + a^2 + 3a^2 = 6a^2$$



Another way to find this area is to find the area of the big square and subtract the areas of the three yellow squares.

The big square has area: $3a \cdot 3a = 9a^2$.

The yellow squares each have areas: $a \cdot a = a^2$.

To find the total area of the figure we subtract:

$$\begin{aligned}\text{Area} &= 9a^2 - (a^2 + a^2 + a^2)\\ &= 9a^2 - 3a^2\\ &= 6a^2\end{aligned}$$

Review Questions

Indicate which expressions are polynomials.

1. $x^2 + 3x^{1/2}$
2. $\frac{1}{3}x^2y - 9y^2$
3. $3x^{-3}$
4. $\frac{2}{3}t^2 - \frac{1}{t^2}$

Express each polynomial in standard form. Give the degree of each polynomial.

5. $3 - 2x$
6. $8 - 4x + 3x^3$
7. $-5 + 2x - 5x^2 + 8x^3$
8. $x^2 - 9x^4 + 12$

Add and simplify.

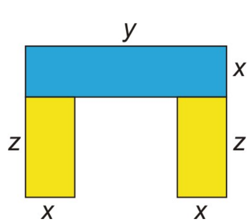
9. $(x + 8) + (-3x - 5)$
10. $(-2x^2 + 4x - 12) + (7x + x^2)$
11. $(2a^2b - 2a + 9) + (5a^2b - 4b + 5)$
12. $(6.9a^2 - 2.3b^2 + 2ab) + (3.1a - 2.5b^2 + b)$

Subtract and simplify.

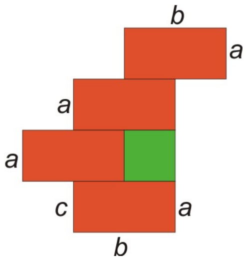
13. $(-t + 15t^2) - (5t^2 + 2t - 9)$
14. $(-y^2 + 4y - 5) - (5y^2 + 2y + 7)$
15. $(-5m^2 - m) - (3m^2 + 4m - 5)$
16. $(2a^2b - 3ab^2 + 5a^2b^2) - (2a^2b^2 + 4a^2b - 5b^2)$

Find the area of the following figures.

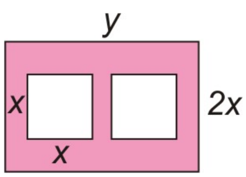
17.



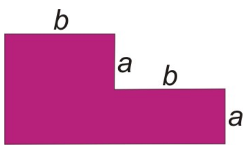
18.



19.



20.



Review Answers

1. No
2. yes
3. no
4. no
5. $-2x+3$; Degree $= 1$
6. $3x^3-4x+8$; Degree $= 3$
7. $8x^3-5x^2+2x-5$; Degree $= 3$
8. $-9x^4+x^2+12$; Degree $= 4$
9. $-2x+3$
10. $-x^2+11x-12$
11. $7a^2b-2a-4b+14$
12. $6.9a^2-4.8b^2+2ab+3.1a+b$
13. $-3t+9$
14. $-6y^2+2y-12$
15. $-8m^2-5m+5$
16. $-2a^2b-3ab^2+3a^2b^2+5b^2$
17. Area $= 2xz-xy$
18. Area $= 4ab+ac$
19. $2xy-2x^2$
20. Area $= 3ab$

9.2 Multiplication of Polynomials

Learning Objectives

- Multiply a polynomial by a monomial
- Multiply a polynomial by a binomial
- Solve problems using multiplication of polynomials

Introduction

When multiplying polynomials we must remember the exponent rules that we learned in the last chapter.

The Product Rule $x^n.x^m = x^n + m$

This says that if we multiply expressions that have the same base, we just add the exponents and keep the base unchanged.

If the expressions we are multiplying have coefficients and more than one variable, we multiply the coefficients just as we would any number and we apply the product rule on each variable separately.

$$(2x^2y^3) \cdot (3x^2y) = (2.3) \cdot (x^2 + 2).(y^3 + 1) = 6x^4y^4$$

Multiplying a Polynomial by a Monomial

We begin this section by multiplying a monomial by a monomial. As you saw above, we need to multiply the coefficients separately and then apply the exponent rules to each variable separately. Let's try some examples.

Example 1

Multiply the following monomials.

a) $(2x^2)(5x^3)$

b) $(-3y^4)(2y^2)$

c) $(3xy^5)(-6x^4y^2)$

d) $(-12a^2b^3c^4)(-3a^2b^2)$

Solution

a) $(2x^2)(5x^3) = (2 \cdot 5) \cdot (x^2 \cdot x^3) = 10x^{2+3} = 10x^5$

b) $(-3y^4)(2y^2) = (-3 \cdot 2) \cdot (y^4 \cdot y^2) = -6y^{4+2} = -6y^6$

c) $(3xy^5)(-6x^4y^2) = 18x^{1+4}y^{5+2} = -18x^5y^7$

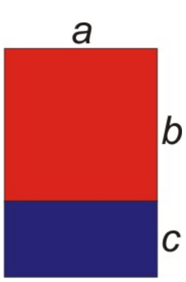
d) $(-12a^2b^3c^4)(-3a^2b^2) = 36a^{2+2}b^{3+2}c^4 = 36a^4b^5c^4$

To multiply a polynomial by a monomial, we use the **Distributive Property**.

This says that

$$a(b + c) = ab + ac$$

This property is best illustrated by an area problem. We can find the area of the big rectangle in two ways.



One way is to use the formula for the area of a rectangle.

$$\text{Area of the big rectangle} = \text{length} \cdot \text{width}$$
$$\text{Length} = a, \text{Width} = b + c$$
$$\text{Area} = a \cdot (b \cdot c)$$

The area of the big rectangle can also be found by adding the areas of the two smaller rectangles.

$$\text{Area of red rectangle} = ab$$
$$\text{Area of blue rectangle} = ac$$
$$\text{Area of big rectangle} = ab + ac$$

This means that $a(b + c) = ab + ac$. It shows why the Distributive Property works.

This property is useful for working with numbers and also with variables.

For instance, to solve this problem, you would add 2 and 7 to get 9 and then multiply by 5 to get 45. But there is another way to do this.

$$5(2 + 7) = 5 \cdot 2 + 5 \cdot 7$$

It means that each number in the parenthesis is multiplied by 5 separately and then the products are added together.

$$5(2 + 7) = 5 \cdot 2 + 5 \cdot 7 = 10 + 35 = 45$$

In general, if we have a number or variable in front of a parenthesis, this means that each term in the parenthesis is multiplied by the expression in front of the parenthesis. The distributive property works no matter how many terms there are inside the parenthesis.

$$a(b + c + d + e + f + \ldots) = ab + ac + ad + ae + af + \ldots$$

The "..." means "and so on".

Let's now apply this property to multiplying polynomials by monomials.

Example 2

Multiply

a) $3(x^2 + 3x - 5)$

b) $4x(3x^2 - 7)$

c) $-7y(4y^2 - 2y + 1)$

Solution

a) $3(x^2 + 3x - 5) = 3(x^2) + 3(3x) - 3(5) = 3x^2 + 9x - 15$

b) $4x(3x^2 - 7) = (4x)(3x^2) + (4x)(-7) = 12x^3 - 28x$

c) $-7y(4y^2 - 2y + 1) = (-7y)(4y^2) + (-7y)(-2y) + (-7y)(1) = -28y^3 + 14y^2 - 7y$

Notice that the use of the Distributive Property simplifies the problems to just multiplying monomials by monomials and adding all the separate parts together.

Example 3

Multiply

a) $2x^3(-3x^4 + 2x^3 - 10x^2 + 7x + 9)$

b) $-7a^2bc^3(5a^2 - 3b^2 - 9c^2)$

Solution

a)

$$\begin{aligned} 2x^3(-3x^4 + 2x^3 - 10x^2 + 7x + 9) &= (2x^3)(-3x^4) + (2x^3)(2x^3) + (2x^3)(-10x^2) + (2x^3)(7x) + (2x^3)(9) \\ &= -6x^7 + 4x^6 - 20x^5 + 14x^4 + 18x^3 \end{aligned}$$

b)

$$\begin{aligned} -7a^2bc^3(5a^2 - 3b^2 - 9c^2) &= (-7a^2bc^3)(5a^2) + (-7a^2bc^3)(-3b^2) + (-7a^2bc^3)(-9c^2) \\ &= -35a^4bc^3 + 21a^2b^3c^3 + 63a^2bc^5 \end{aligned}$$

Multiply a Polynomial by a Binomial

Let's start by multiplying two binomials together. A binomial is a polynomial with two terms, so a product of two binomials will take the form.

$$(a + b)(c + d)$$

The Distributive Property also applies in this situation. Let's think of the first parenthesis as one term. The Distributive Property says that the term in front of the parenthesis multiplies with each term inside the parenthesis separately. Then, we add the results of the products.

$$(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$$

Let's rewrite this answer as $c \cdot (a + b) + d \cdot (a + b)$

We see that we can apply the distributive property on each of the parenthesis in turn.

$$c \cdot (a + b) + d \cdot (a + b) = c \cdot a + c \cdot b + d \cdot a + d \cdot b \text{ (or } ca + cb + da + db)$$

What you should notice is that when multiplying any two polynomials, **every term in one polynomial is multiplied by every term in the other polynomial**.

Let's look at some examples of multiplying polynomials.

Example 4

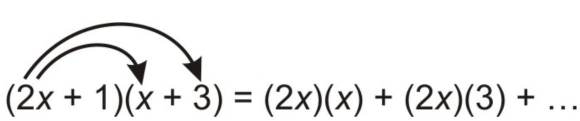
Multiply and simplify $(2x + 1)(x + 3)$

Solution

We must multiply each term in the first polynomial with each term in the second polynomial.

Let's try to be systematic to make sure that we get all the products.

First, multiply the first term in the first parenthesis by all the terms in the second parenthesis.



We are now done with the first term.

Now we multiply the second term in the first parenthesis by all terms in the second parenthesis and add them to the previous terms.

$(2x+1)(x+3) = (2x)(x) + (2x)(3) + (1)(x) + (1)(3)$

We are done with the multiplication and we can simplify.

$$\begin{aligned}(2x)(x) + (2x)(3) + (1)(x) + (1)(3) &= 2x^2 + 6x + x + 3 \\ &= 2x^2 + 7x + 3\end{aligned}$$

This way of multiplying polynomials is called **in-line** multiplication or **horizontal** multiplication.

Another method for multiplying polynomials is to use **vertical** multiplication similar to the vertical multiplication you learned with regular numbers. Let's demonstrate this method with the same example.

$$\begin{array}{rrrrr} & 2x & + & 1 & \\ & x & + & 3 & \\ \hline & 6x & + & 3 & \leftarrow \text{Multiply each term on top by} 3 \\ \text{Multiply each term on top by} x \rightarrow 2x^2 + & x & & & \\ \hline 2x^2 \ + & 7x & + & 3 & \leftarrow \text{Arrange like terms on top of each other and add vertically} \end{array}$$

This method is typically easier to use although it does take more space. Just make sure that like terms are together in vertical columns so you can combine them at the end.

Example 5

Multiply and simplify

(a) $(4x-5)(x-20)$

(b) $(3x-2)(3x+2)$

(c) $(3x^2+2x-5)(2x-3)$

(d) $(x^2-9)(4x^4+5x^2-2)$

Solution

a) $(4x-5)(x-20)$

Horizontal multiplication

$$\begin{aligned}(4x-5)(x-20) &= (4x)(x) + (4x)(-20) + (-5)(x) + (-5)(-20) \\ &= 4x^2 - 80x - 5x + 100 = 4x^2 - 85x + 100\end{aligned}$$

Vertical multiplication

Arrange the polynomials on top of each other with like terms in the same columns.

$$\begin{array}{rrrrr}
 & & 4x & - & 5 \\
 & & x & - & 20 \\
\hline
 & & -80x & + & 100 \\
4x^2 & - & 5x & & \\
\hline
4x^2 & - & 85x & + & 100
\end{array}$$

Both techniques result in the same answer, $4x^2 - 85x + 100$.

For the last question, we'll show the solution with vertical multiplication because it may be a technique you are not used to. Horizontal multiplication will result in the exact same terms and the same answer.

b) $(3x-2)(3x+2)$

$$\begin{array}{rrrrr}
 & & 3x & - & 2 \\
 & & 3x & + & 2 \\
\hline
 & & 6x & - & 4 \\
9x^2 & - & 6x & & \\
\hline
9x^2 & + & 0x & - & 4
\end{array}$$

Answer $9x^2 - 4$

(c) $(3x^2 + 2x - 5)(2x - 3)$

It's better to place the smaller polynomial on the bottom:

$$\begin{array}{rrrr}
 & 3x^2 & +2x & -5 \\
\underline{} & \underline{} & \underline{2x} & \underline{-3} \\
 & -9x^2 & -6x & +15 \\
\underline{6x^3} & \underline{+4x^2} & \underline{-10x} & \underline{} \\
6x^3 & -5x^2 & -16x & +15
\end{array}$$

Answer $6x^3 - 5x^2 - 16x + 15$

(d) $(x^2 - 9)(4x^4 + 5x^2 - 2)$

Set up the multiplication vertically and leave gaps for missing powers of x:

$$\begin{array}{rrrr}
 & 4x^4 & +5x^2 & -2 \\
\underline{} & \underline{} & \underline{x^2} & \underline{-9} \\
 & -36x^4 & -45x^2 & +18 \\
\underline{4x^6} & \underline{+5x^4} & \underline{-2x^2} & \underline{} \\
4x^6 & -31x^4 & -47x^2 & +18
\end{array}$$

Answer $4x^6 - 31x^4 - 47x^2 + 18$

Multimedia Link The following video shows how multiplying two binomials together is related to the distributive property. Khan Academy Multiplying Expressions (7:59) .

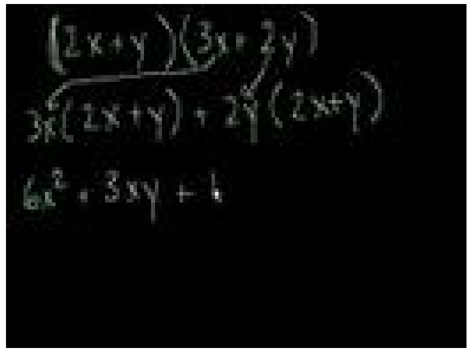


Figure 9.1: (Ax+By)(Ax+By) (Watch on Youtube)

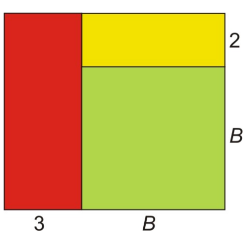
Solve Real-World Problems Using Multiplication of Polynomials

In this section, we will see how multiplication of polynomials is applied to finding the areas and volumes of geometric shapes.

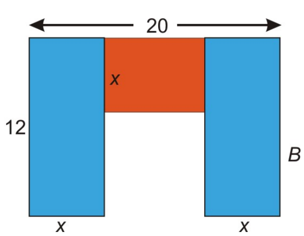
Example 6

Find the areas of the following figures

a)

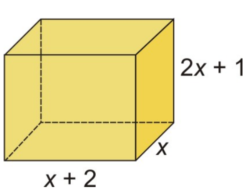


b)

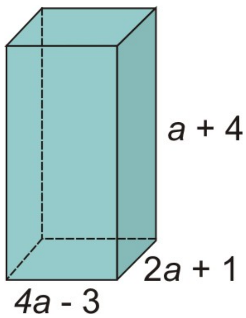


Find the volumes of the following figures

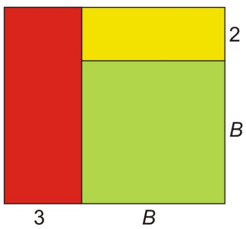
c)



d)



Solution



a) We use the formula for the area of a rectangle.

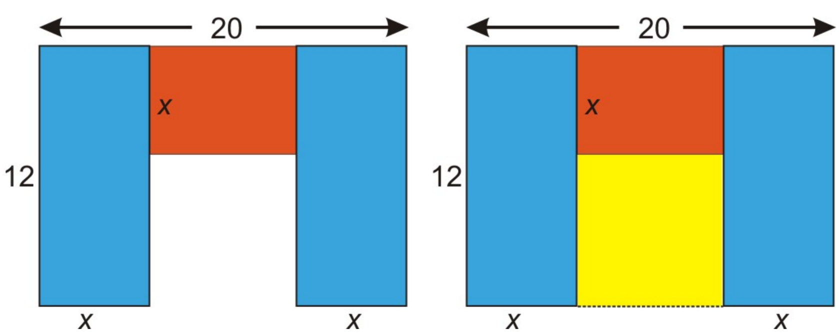
$$\text{Area} = \text{length} \cdot \text{width}$$

For the big rectangle

$$\text{Length} = b + 3, \ \text{Width} = b + 2$$

$$\begin{aligned}\text{Area} &= (b+3)(b+2)\\ &= b^2 + 2b + 3b + 6\\ &= b^2 + 5b + 6\end{aligned}$$

b) Let's find the area of the big rectangle in the second figure and subtract the area of the yellow rectangle.

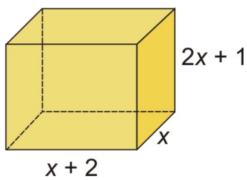


$$\text{Area of big rectangle} = 20(12) = 240$$

$$\begin{aligned}\text{Area of yellow rectangle} &= (12 - x)(20 - 2x)\\ &= 240 - 24x - 20x + 2x^2\\ &= 240 - 44x + 2x^2\\ &= 2x^2 - 44x + 240\end{aligned}$$

The desired area is the difference between the two.

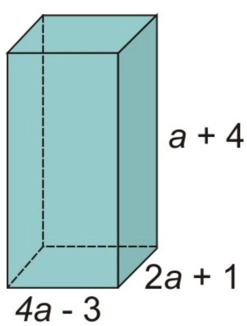
$$\begin{aligned}\text{Area} &= 240 - (2x^2 - 44x + 240)\\ &= 240 + (-2x^2 + 44x - 240)\\ &= 240 - 2x^2 + 44x - 240\\ &= -2x^2 + 44x\end{aligned}$$



c) The volume of this shape = (area of the base) · (height).

$$\begin{aligned}\text{Area of the base } &= x(x+2)\\ &= x^2 + 2x\end{aligned}$$

$$\begin{aligned}\text{Height} &= 2x + 1\\ \text{Volume} &= (x^2 + 2x)(2x + 1)\\ &= 2x^3 + x^2 + 4x^2 + 2x\\ &= 2x^3 + 5x^2 + 2x\end{aligned}$$



d) The volume of this shape = (area of the base) · (height).

$$\begin{aligned}\text{Area of the base} &= (4a - 3)(2a + 1)\\ &= 8a^2 + 4a - 6a - 3\\ &= 8a^2 - 2a - 3\end{aligned}$$

$$\begin{aligned}\text{Height} &= a + 4\\ \text{Volume} &= (8a^2 - 2a - 3)(a + 4)\end{aligned}$$

Let's multiply using the vertical method:

$$\begin{array}{rrrr} & 8a^2 & -\ 2a & -\ 3 \\ & & a & +\ 4 \\ \hline & 32a^2 & -\ 8a & -\ 12 \\ 8a^3 & -\ 2a^2 & -\ 3a & \\ \hline 8a^3 & +\ 30a^2 & -\ 11a & -\ 12 \end{array}$$

Answer Volume $= 8a^3 + 30a^2 - 11a - 12$

Review Questions

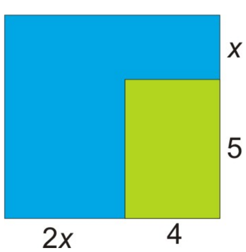
Multiply the following monomials.

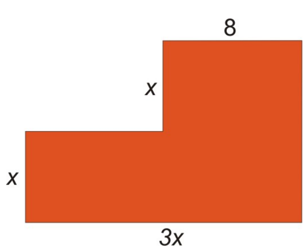
1. $(2x)(-7x)$
2. $(-5a^2b)(-12a^3b^3)$
3. $(3xy^2z^2)(15x^2yz^3)$

Multiply and simplify.

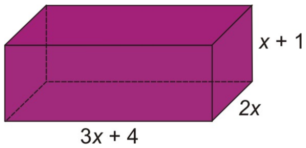
4. $2x(4x - 5)$
5. $9x^3(3x^2 - 2x + 7)$
6. $-3a^2b(9a^2 - 4b^2)$
7. $(x - 3)(x + 2)$
8. $(a^2 + 2)(3a^2 - 4)$
9. $(7x - 2)(9x - 5)$
10. $(2x - 1)(2x^2 - x + 3)$
11. $(3x + 2)(9x^2 - 6x + 4)$
12. $(a^2 + 2a - 3)(a^2 - 3a + 4)$

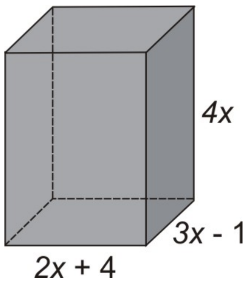
Find the areas of the following figures.

13. 

14. 

Find the volumes of the following figures.

15. 

16. 

Review Answers

1. $-14x^2$
2. $60a^5b^4$
3. $45x^3y^3z^5$
4. $8x^2 - 10x$
5. $27x^5 - 18x^4 + 63x^3$
6. $-27a^4b + 12a^2b^3$
7. $x^2 - x - 6$
8. $3a^4 + 2a^2 - 8$
9. $63x^2 - 53x + 10$
10. $4x^3 - 4x^2 + 7x - 3$
11. $27x^3 + 8$
12. $a^4 - a^3 - 5a^2 + 17a - 12$
13. $(2x + 4)(x + 6) = 2x^2 + 16x + 24$
14. $x(3x + 8) = 3x^2 + 8x$
15. $6x^3 + 14x^2 + 8x$
16. $24x^3 - 28x^2 - 12x$

9.3 Special Products of Polynomials

Learning Objectives

- Find the square of a binomial
- Find the product of binomials using sum and difference formula
- Solve problems using special products of polynomials

Introduction

We saw that when we multiply two binomials we need to make sure that each term in the first binomial multiplies with each term in the second binomial. Let's look at another example.

Multiply two linear (i.e. with degree = 1) binomials:

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic (i.e. with degree = 2)polynomial with four terms.

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get:

$$2x^2+11x+12$$

This is a quadratic or 2^{nd} degree**trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we will talk about some special products of binomials.

Find the Square of a Binomial

A special binomial product is the **square of a binomial**. Consider the following multiplication.

$$(x+4)(x+4)$$

Since we are multiplying the same expression by itself that means that we are squaring the expression. This means that:

$(x+4)(x+4)=(x+4)^2$

Let's multiply: $(x+4)(x+4)=x^2+4x+4x+16$

And combine like terms: $=x^2+8x+16$

Notice that the middle terms are the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2=(a+b)(a+b)&=a^2+ab+ab+b^2\\&=a^2+2ab+b^2\end{aligned}$$

It looks like the middle terms are the same again. So far we have squared the sum of binomials. Let's now square a difference of binomials.

$$\begin{aligned}(a-b)^2=(a-b)(a-b)&=a^2-ab-ab+b^2\\&=a^2-2ab+b^2\end{aligned}$$

We notice a pattern when squaring binomials. To square a binomial, add the square of the first term, add or subtract twice the product of the terms, and the square of the second term. You should remember these formulas:

Square of a Binomial

$(a+b)^2=a^2+2ab+b^2$ and $(a–b)^2=a^2-2ab-b^2$

Remember! A polynomial that is raised to an exponent means that we multiply the polynomial by itself however many times the exponent indicates. For instance

$$(a+b)^2 = (a+b)(a+b)$$

Don't make the common mistake $(a+b)^2 = a^2 + b^2$. To see why $(a+b)^2 \neq a^2 + b^2$ try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example 1

Square each binomial and simplify.

(a) $(x+10)^2$

(b) $(2x-3)^2$

(c) $(x^2+4)^2$

(d) $(5x-2y)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2$

If we let $a = x$ and $b = 10$, then

$$\begin{array}{ccccccccccc}
(a^2 & + & b) & = & a^2 & + & 2a & & b & + & b^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\end{array}$$

$$\begin{aligned}
(x + 10)^2 &= (x)^2 + 2(x)(10) + (10)^2 \\
&= x^2 + 20x + 100
\end{aligned}$$

b) $(2x-3)^2$

If we let $a = 2x$ and $b = 3$, then

$$\begin{aligned}
(a-b)^2 &= a^2 - 2ab + b^2 \\
(2x-3)^2 &= (2x)^2 - 2(2x)(3) + (3)^2 \\
&= 4x^2 - 12x + 9
\end{aligned}$$

c) $(x+4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}
(x^2+4)^2 &= (x^2)^2 + 2(x^2)(4) + (4)^2 \\
&= x^4 + 8x^2 + 16
\end{aligned}$$

d) $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}
(5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2 \\
&= 25x^2 - 20xy + 4y^2
\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2 - 4x + 4x - 16\\ &= x^2 - 16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they cancel out when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms.

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab - b^2\\ &= a^2 - b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula

$$(a+b)(a-b) = a^2 - b^2$$

Let's apply this formula to a few examples.

Example 2

Multiply the following binomials and simplify.

(a) $(x+3)(x-3)$

(b) $(5x+9)(5x-9)$

(c) $(2x^3+7)(2x^3-7)$

(d) $(4x+5y)(4x-5y)$

Solution

(a) Let $a = x$ and $b = 3$, then

$$\begin{array}{ccccccc}(a & + & b)(a & - & b) = & a^2 & - \; b^2\\ \downarrow & & \downarrow \;\; \downarrow & & \downarrow & \downarrow & \downarrow\\ (x & + & 3)(x & - & 3) = & (x)^2 & - \; (3)^2\\ & & & & = & x^2 & - \; 9\end{array}$$

(b) Let $a = 5x$ and $b = 9$, then

$$\begin{array}{ccccccc}(a & + & b)(a & - & b) = & a^2 & - \; b^2\\ \downarrow & & \downarrow \;\; \downarrow & & \downarrow & \downarrow & \downarrow\\ (5x & + & 9)(5x & - & 9) = & (5x)^2 & - \; (9)^2\\ & & & & = & 25x^2 & - \; 81\end{array}$$

(c) Let $a = 2x^3$ and $b = 7$, then

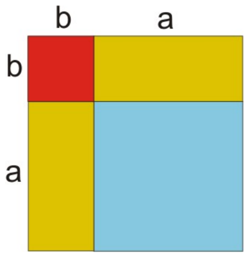
$$(2x^3+7)(2x^3-7) = (2x^3)^2 - (7)^2$$
$$= 4x^6 - 49$$

(d) Let $a = 4x$ and $b = 5y$, then

$$(4x+5y)(4x-5y) = (4x)^2 - (5y)^2$$
$$= 16x^2 - 25y^2$$

Solve Real-World Problems Using Special Products of Polynomials

Let's now see how special products of polynomials apply to geometry problems and to mental arithmetic.



Example 3

Find the area of the following square

Solution

The area of the square = side × side

$$\text{Area} = (a+b)(a+b)$$
$$= a^2 + 2ab + b^2$$

Notice that this gives a visual explanation of the square of binomials product.

Area of the big square	= area of the blue square	+ 2(area of yellow rectangle)	+ area of red square
$(a+b)^2$	$= a^2$	$+ 2ab$	$+ b^2$

The next example shows how to use the special products in doing fast mental calculations.

Example 4

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

(a) 43×57

(b) 112×88

(c) 45^2

(d) 481×309

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

(a) Rewrite $43 = (50 - 7)$ and $57 = (50 + 7)$.

Then $43 \times 57 = (50 - 7)(50 + 7) = (50)^2 - (7)^2 = 2500 - 49 = 2,451$

(b) Rewrite $112 = (100 + 12)$ and $88 = (100 - 12)$

Then $112 \times 88 = (100 + 12)(100 - 12) = (100)^2 - (12)^2 = 10,000 - 144 = 9,856$

(c) $45^2 = (40 + 5)^2 = (40)^2 + 2(40)(5) + (5)^2 = 1600 + 400 + 25 = 2,025$

(d) Rewrite $481 = (400 + 81)$ and $319 = (400 - 81)$

Then, $481 \times 319 = (400 + 81)(400 - 81) = (400)^2 - (81)^2$

$(400)^2$ is easy - it equals $160,000$

$(81)^2$ is not easy to do mentally. Let's rewrite it as $81 = 80 + 1$

$(81)^2 = (80 + 1)^2 = (80)^2 + 2(80)(1) + (1)^2 = 6400 + 160 + 1 = 6,561$

Then, $481 \times 309 = (400)^2 - (81)^2 = 160,000 - 6,561 = 153,439$

Review Questions

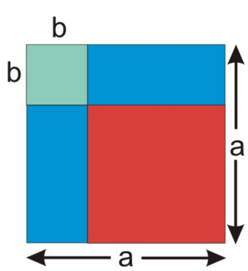
Use the special product for squaring binomials to multiply these expressions.

1. $(x + 9)^2$
2. $(3x - 7)^2$
3. $(4x^2 + y^2)^2$
4. $(8x - 3)^2$

Use the special product of a sum and difference to multiply these expressions.

5. $(2x - 1)(2x + 1)$
6. $(x - 12)(x + 12)$
7. $(5a - 2b)(5a + 2b)$
8. $(ab - 1)(ab + 1)$

Find the area of the orange square in the following figure. It is the lower right shaded box.

9. 

Multiply the following numbers using the special products.

10. 45×55
11. 56^2
12. 1002×998
13. 36×44

Review Answers

1. $x^2 + 18x + 81$
2. $9x^2 - 42x + 49$
3. $16x^4 + 8x^2y^2 + y^4$
4. $64x^2 - 48x + 9$
5. $4x^2 - 1$
6. $x^2 - 144$
7. $25a^2 - 4b^2$
8. $a^2b^2 - 1$
9. Area $= (a - b)^2 = a^2 - 2ab + b^2$
10. $(50 - 5)(50 + 5) = 2475$
11. $(50 + 6)^2 = 3136$
12. $(1000 + 2)(1000 - 2) = 999,996$
13. $(40 - 4)(40 + 4) = 1584$

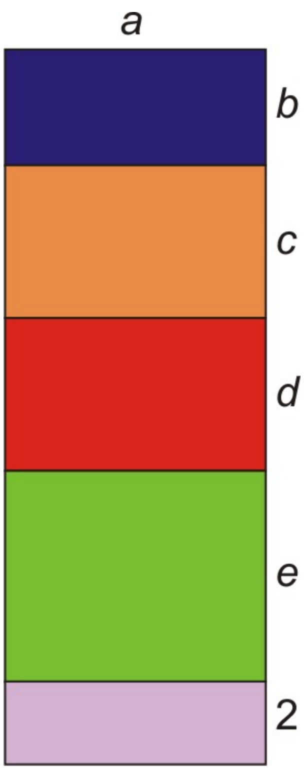
9.4 Polynomial Equations in Factored Form

Learning Objectives

- Use the zero-product property
- Find greatest common monomial factor
- Solve simple polynomial equations by factoring

Introduction

In the last few sections, we learned how to multiply polynomials. We did that by using the Distributive Property. All the terms in one polynomial must be multiplied by all terms in the other polynomial. In this section, you will start learning how to do this process in reverse. The reverse of distribution is called **factoring**.



Let's look at the areas of the rectangles again: Area = length · width. The total area of the figure on the right can be found in two ways.

Method 1 Find the areas of all the small rectangles and add them

Blue rectangle $= ab$

Orange rectangle $= ac$

Red rectangle $= ad$

Green rectangle $= ae$

Pink rectangle $= 2a$

Total area $= ab + ac + ad + ae + 2a$

Method 2 Find the area of the big rectangle all at once

$$\text{Length} = a$$
$$\text{Width} = b + c + d + e + 2$$

$$\text{Area} = a(b + c + d + e = 2)$$

Since the area of the rectangle is the same no matter what method you use then the answers are the same:

$$ab + ac + ad + ae + 2a = a(b + c + d + e + 2)$$

Factoring means that you take the factors that are common to all the terms in a polynomial. Then, multiply them by a parenthesis containing all the terms that are left over when you divide out the common factors.

Use the Zero-Product Property

Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:

$$6x^4 + 7x^3 - 26x^2 + 17x + 30$$

Notice that the degree of the polynomials is four. It is written in standard form because the terms are written in order of decreasing power.

Factored form means that the polynomial is written as a product of different factors. The factors are also polynomials, usually of lower degree. Here is the same polynomial in factored form.

$$\underbrace{x-1}_{1^{st}\ factor}\ \underbrace{x+2}_{2^{nd}\ factor}\ \underbrace{2x-3}_{3^{rd}\ factor}\ \underbrace{3x+5}_{4^{th}\ factor}$$

Notice that each factor in this polynomial is a binomial. Writing polynomials in factored form is very useful because it helps us solve polynomial equations. Before we talk about how we can solve polynomial equations of degree 2 or higher, let's review how to solve a linear equation (degree 1).

Example 1

Solve the following equations

a) $x - 4 = 0$

b) $3x - 5 = 0$

Solution

Remember that to solve an equation you are trying to find the value of x:

a)

$$\begin{aligned} x - 4 &= 0 \\ +4 &= +4 \\ \hline x &= 4 \end{aligned}$$

b)

$$\begin{aligned} 3x - 5 &= 0 \\ +5 &= +5 \\ \hline 3x &= 5 \\ \frac{3x}{3} &= \frac{5}{3} \\ x &= \frac{5}{3} \end{aligned}$$

Now we are ready to think about solving equations like $2x^2 + 5x = 42$. Notice we can't isolate x in any way that you have already learned. But, we can subtract 42 on both sides to get $2x^2 + 5x - 42 = 0$. Now, the left hand side of this equation can be factored!

Factoring a polynomial allows us to break up the problem into easier chunks. For example, $2x^2 + 5x - 42 = (x+6)(2x-7)$. So now we want to solve: $(x+6)(2x-7) = 0$

How would we solve this? If we multiply two numbers together and their product is zero, what can we say about these numbers? The only way a product is zero is if one or both of the terms are zero. This property is called the **Zero-product Property**.

How does that help us solve the polynomial equation? Since the product equals 0, then either of the terms or factors in the product must equal zero. We set each factor equal to zero and we solve.

$$(x+6)=0 \qquad \text{OR} \qquad (2x-7)=0$$

We can now solve each part individually and we obtain:

$$\begin{array}{lcl} x+6=0 & \text{or} & 2x-7=0 \\ & & 2x=7 \\ x=-6 & \text{or} & x=-\dfrac{7}{2} \end{array}$$

Notice that the solution is $x = -6$ **OR** $x = 7/2$. The **OR** says that either of these values of x would make the product of the two factors equal to zero. Let's plug the solutions back into the equation and check that this is correct.

$$\begin{aligned} &\text{Check } x = -6; \\ &(x+6)(2x-7) = \\ &(-6+6)(2(6)-7) = \\ &(0)(5) = 0 \end{aligned}$$

$$\begin{aligned} &\text{Check } x = 7/2 \\ &(x+6)(2x-7) = \\ &\left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right) = \\ &\left(\frac{19}{2}\right)(7-7) = \\ &\left(\frac{19}{2}\right)(0) = 0 \end{aligned}$$

Both solutions check out. You should notice that the product equals to zero because each solution makes one of the factors simplify to zero. Factoring a polynomial is very useful because the Zero-product Property allows us to break up the problem into simpler separate steps.

If we are not able to factor a polynomial the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-product Property only works when a product equals to zero. For example, if you multiplied two numbers and the answer was nine you could not say that each of the numbers was nine. In order to use the property, you must have the factored polynomial equal to zero.

Example 2

Solve each of the polynomials

a) $(x-9)(3x+4)=0$

b) $x(5x-4)=0$

c) $4x(x+6)(4x-9)=0$

Solution

Since all polynomials are in factored form, we set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4)=0$ can be split up into two linear equations

$$\begin{array}{ccc} x-9=0 & \text{or} & 3x+4=0 \\ & & 3x=-4 \\ x=9 & \text{or} & x=-\frac{4}{3} \end{array}$$

b) $x(5x-4)=0$ can be split up into two linear equations

$$\begin{array}{ccc} & & 5x-4=0 \\ x=0 & \text{or} & 5x=4 \\ & & x=\frac{4}{5} \end{array}$$

c) $4x(x+6)(4x-9)=0$ can be split up into three linear equations.

$$\begin{array}{ccccc} 4x=0 & & & & 4x-9=0 \\ x=\frac{0}{4} & \text{or} & x+6=0 & \text{or} & 4x=9 \\ x=0 & & x=-6 & & x=\frac{9}{4} \end{array}$$

Find Greatest Common Monomial Factor

Once we get a polynomial in factored form, it is easier to solve the polynomial equation. But first, we need to learn how to factor. There are several factoring methods you will be learning in the next few sections. In most cases, factoring takes several steps to complete because we want to **factor completely**. That means that we factor until we cannot factor anymore.

Let's start with the simplest case, finding the greatest monomial factor. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form.

$$ax+bx+cx+dx$$

A common factor can be a number, a variable or a combination of numbers and variables that appear in all terms of the polynomial. We are looking for expressions that divide out evenly from each term in the polynomial. Notice that in our example, the factor x appears in all terms. Therefore **x is a common factor**

$$ax+bx+cx+dx$$

Since x is a common factor, we factor it by writing in front of a parenthesis:

$$x(\quad)$$

Inside the parenthesis, we write what is left over when we divide x from each term.

$$x(a+b+c+d)$$

Let's look at more examples.

Example 3

Factor

a) $2x+8$

b) $15x-25$

c) $3a+9b+6$

Solution

a) We see that the factor 2 divides evenly from both terms.

$2x+8=2(x)+2(4)$

We factor the 2 by writing it in front of a parenthesis.

2( )

Inside the parenthesis, we write what is left from each term when we divide by 2.

$2(x+4)$ This is the factored form.

b) We see that the factor of 5 divides evenly from all terms.

Rewrite $15x-25=5(3x)-5(5)$

Factor 5 to get $5(3x-5)$

c) We see that the factor of 3 divides evenly from all terms.

Rewrite $3a+9b+6=3(a)+3(3b)+3(2)$

Factor 3 to get $3(a+3b+2)$

Here are examples where different powers of the common factor appear in the polynomial

Example 4

Find the greatest common factor

a) a^3-3a^2+4a

b) $12a^4-5a^3+7a^2$

Solution

a) Notice that the factor a appears in all terms of a^3-3a^2+4a but each term has a different power of a. The common factor is the lowest power that appears in the expression. In this case the factor is a.

Let's rewrite $a^3-3a^2+4a=a(a^2)+a(-3a)+a(4)$

Factor a to get $a(a^2-3a+4)$

b) The factor a appears in all the term and the lowest power is a^2.

We rewrite the expression as $12a^4-5a^3+7a^2=12a^2\cdot a^2-5a\cdot a^2+7\cdot a^2$

Factor a^2 to get $a^2(12a^2-5a+7)$

Let's look at some examples where there is more than one common factor.

Example 5:

Factor completely

a) $3ax + 9a$

b) $x^3y + xy$

c) $5x^3y - 15x^2y^2 + 25xy^3$

Solution

a) Notice that 3 is common to both terms.

When we factor 3 we get $3(ax + 3a)$

This is not completely factored though because if you look inside the parenthesis, we notice that a is also a common factor.

When we factor a we get $3 \cdot a(x + 3)$

This is the answer because there are no more common factors.

A different option is to factor ***all*** common factors at once.

Since both 3 and a are common we factor the term $3a$ and get $3a(x + 3)$.

b) Notice that both x and y are common factors.

Let's rewrite the expression $x^3y + xy = xy(x^2) + xy(1)$

When we factor xy we obtain $xy(x^2 + 1)$

c) The common factors are $5xy$.

When we factor $5xy$ we obtain $5xy(x^2 - 3xy + 5y^2)$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1

If necessary, **re-write** the equation in standard form such that:

Polynomial expression $= 0$

Step 2

Factor the polynomial completely

Step 3

Use the zero-product rule to set **each factor equal to zero**

Step 4

Solve each equation from step 3

Step 5

Check your answers by substituting your solutions into the original equation

Example 6

Solve the following polynomial equations

a) $x^2 - 2x = 0$

b) $2x^2 = 5x$

c)$9x^2y - 6xy = 0$

Solution:

a) $x^2 - 2x = 0$

Rewrite this is not necessary since the equation is in the correct form.

Factor The common factor is x, so this factors as: $x(x-2) = 0$.

Set each factor equal to zero.

$$x = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check Substitute each solution back into the original equation.

$$\begin{array}{llll} x = 0 & \Rightarrow & (0)^2 - 2(0) = 0 & \text{works out} \\ x = 2 & \Rightarrow & (2)^2 - 2(2) = 4 - 4 = 0 & \text{works out} \end{array}$$

Answer $x = 0, x = 2$

b) $2x^2 = 5x$

Rewrite $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$.

Factor The common factor is x, so this factors as: $x(2x-5) = 0$.

Set each factor equal to zero:.

$$x = 0 \qquad \text{or} \qquad 2x - 5 = 0$$

Solve

$$\underline{x = 0} \qquad \text{or} \qquad \begin{aligned} 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Check Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 & \text{works out} \\ x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} & \text{works out} \end{array}$$

Answer $x = 0, x = 5/2$

c) $9x^2y - 6xy = 0$

Rewrite Not necessary

Factor The common factor is $3xy$, so this factors as $3xy(3x-2) = 0$.

Set each factor equal to zero.

$3 = 0$ is never true, so this part does not give a solution

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solve

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad \begin{aligned} 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

Check Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 9(0)y - 6(0)y = 0 - 0 = 0 \qquad \text{works out}$$
$$y = 0 \Rightarrow 9x^2(0) - 6x = 0 - 0 = 0 \qquad \text{works out}$$
$$\frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 y - 6 \cdot \frac{2}{3}y = 9 \cdot \frac{4}{9}y - 4y = 4y - 4y = 0 \qquad \text{works out}$$

Answer $x = 0, y = 0, x = 2/3$

Review Questions

Factor the common factor in the following polynomials.

1. $3x^3 - 21x$
2. $5x^6 + 15x^4$
3. $4x^3 + 10x^2 - 2x$
4. $-10x^6 + 12x^5 - 4x^4$
5. $12xy + 24xy^2 + 36xy^3$
6. $5a^3 - 7a$
7. $45y^{12} + 30y^{10}$
8. $16xy^2z + 4x^3y$

Solve the following polynomial equations.

9. $x(x + 12) = 0$
10. $(2x + 1)(2x - 1) = 0$
11. $(x - 5)(2x + 7)(3x - 4) = 0$
12. $2x(x + 9)(7x - 20) = 0$
13. $18y - 3y^2 = 0$
14. $9x^2 = 27x$
15. $4a^2 + a = 0$
16. $b^2 - 5/3b = 0$

Review Answers

1. $3x(x^2 - 7)$
2. $5x^4(x^2 + 3)$
3. $2x(2x^2 + 5x - 1)$

4. $2x^4(-5x^2 + 6x - 2)$
5. $12xy(1 + 2y + 3y^2)$
6. $a(5a^2 - 7)$
7. $15y^{10}(3y^2 + 2)$
8. $4xy(4yz + x^2)$
9. $x = 0, x = -12$
10. $x = -1/2, x = 1/2$
11. $x = 5, x = -7/2, x = 4/3$
12. $x = 0, x = -9, x = 20/7$
13. $y = 0, y = 6$
14. $x = 0, x = 3$
15. $a = 0, a = -1/4$
16. $b = 0, b = 5/3$

9.5 Factoring Quadratic Expressions

Learning Objectives

- Write quadratic equations in standard form.
- Factor quadratic expressions for different coefficient values.
- Factor when $a = -1$.

Write Quadratic Expressions in Standard Form

Quadratic polynomials are polynomials of 2^{nd} degree. The standard form of a quadratic polynomial is written as

$$ax^2 + bx + c$$

Here a, b, and c stand for constant numbers. Factoring these polynomials depends on the values of these constants. In this section, we will learn how to factor quadratic polynomials for different values of a, b, and c. In the last section, we factored common monomials, so you already know how to factor quadratic polynomials where $c = 0$.

For example for the quadratic $ax^2 + bx$, the common factor is x and this expression is factored as $x(ax + b)$. When all the coefficients are not zero these expressions are also called **Quadratic Trinomials**, since they are polynomials with three terms.

Factor when a = 1, b is Positive, and c is Positive

Let's first consider the case where $a = 1$, b is positive and c is positive. The quadratic trinomials will take the following form.

$$x^2 + bx + c$$

You know from multiplying binomials that when you multiply two factors $(x + m)(x + n)$ you obtain a quadratic polynomial. Let's multiply this and see what happens. We use The Distributive Property.

$$(x + m)(x + n) = x^2 + nx + mx + mn$$

To simplify this polynomial we would combine the like terms in the middle by adding them.

$$(x+m)(x+n) = x^2 + (n+m)x + mn$$

To factor we need to do this process in reverse.

We see that	$x^2 + (n+m)x + mn$
Is the same form as	$x^2 + bx + c$

This means that we need to find two numbers m and n where

$$n+m = b \qquad \text{and} \qquad mn = c$$

To factor $x^2 + bx + c$, the answer is the product of two parentheses.

$(x+m)(x+n)$

so that $n+m = b$ and $mn = c$

Let's try some specific examples.

Example 1

Factor $x^2 + 5x + 6$

Solution We are looking for an answer that is a product of two binomials in parentheses.

$$(x + \_\_\_\_)(x + \_\_\_\_)$$

To fill in the blanks, we want two numbers m and n that multiply to 6 and add to 5. A good strategy is to list the possible ways we can multiply two numbers to give us 6 and then see which of these pairs of numbers add to 5. The number six can be written as the product of.

$6 = 1 \cdot 6$	and	$1+6=7$		
$6 = 2 \cdot 3$	and	$2+3=5$	$\leftarrow$	This is the correct choice.

So the answer is $(x+2)(x+3)$.

We can check to see if this is correct by multiplying $(x+2)(x+3)$.

$$\begin{array}{rrrrr} & & x & + & 2 \\ & & x & + & 3 \\ \hline & & 3x & + & 6 \\ x^2 & + & 2x & & \\ \hline x^2 & + & 5x & + & 9 \end{array}$$

The answer checks out.

Example 2

Factor $x^2 + 7x + 12$

Solution

We are looking for an answer that is a product of two parentheses $(x + \_\_\_\_)(x + \_\_\_\_)$.

The number 12 can be written as the product of the following numbers.

$$\begin{array}{llll} 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 & \\ 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 & \\ 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 & \leftarrow \quad \text{This is the correct choice.} \end{array}$$

The answer is $(x+3)(x+4)$.

Example 3

Factor $x^2 + 8x + 12$.

Solution

We are looking for an answer that is a product of the two parentheses $(x + \_\_\_\_)(x + \_\_\_\_)$.

The number 12 can be written as the product of the following numbers.

$$\begin{array}{llll} 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 & \\ 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 & \leftarrow \quad \text{This is the correct choice.} \\ 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 & \end{array}$$

The answer is $(x+2)(x+6)$.

Example 4

Factor $x^2 + 12x + 36$.

Solution

We are looking for an answer that is a product of the two parentheses $(x + \_\_\_\_)(x + \_\_\_\_)$.

The number 36 can be written as the product of the following numbers.

$$\begin{array}{llll} 36 = 1 \cdot 36 & \text{and} & 1 + 36 = 37 & \\ 36 = 2 \cdot 18 & \text{and} & 2 + 18 = 20 & \\ 36 = 3 \cdot 12 & \text{and} & 3 + 12 = 15 & \\ 36 = 4 \cdot 9 & \text{and} & 4 + 9 = 13 & \\ 36 = 6 \cdot 6 & \text{and} & 6 + 6 = 12 & \leftarrow \quad \text{This is the correct choice} \end{array}$$

The answer is $(x+6)(x+6)$.

Factor when a = 1, b is Negative and c is Positive

Now let's see how this method works if the middle coefficient (b) is negative.

Example 5

Factor $x^2 - 6x + 8$.

Solution

We are looking for an answer that is a product of the two parentheses $(x + \_\_\_\_)(x + \_\_\_\_)$.

The number 8 can be written as the product of the following numbers.

$8 = 1 \cdot 8$ and $1 + 8 = 9$ Notice that these are two different choices.

But also,

$$\begin{array}{llll} 8 = (-1) \cdot (-8) & \text{and} & -1 + (-8) = -9 & \text{Notice that these are two different choices.} \\ 8 = 2 \cdot 4 & \text{and} & 2 + 4 = 6 & \end{array}$$

But also,

$$8 = (-2) \cdot (-4) \qquad \text{and} \qquad -2 + (-4) = -6 \qquad \leftarrow \qquad \text{This is the correct choice.}$$

The answer is $(x-2)(x-4)$

We can check to see if this is correct by multiplying $(x-2)(x-4)$.

$$\begin{array}{rrr} & x & -\ 2 \\ & x & -\ 4 \\ \hline & -4x & +\ 8 \\ x^2 & -\ 2x & \\ \hline x^2 & -\ 6x & +\ 8 \end{array}$$

The answer checks out.

Example 6

Factor $x - 17x + 16$

Solution

We are looking for an answer that is a product of two parentheses: $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$.

The number 16 can be written as the product of the following numbers:

$$\begin{array}{lllll} 16 = 1 \cdot 16 & \text{and} & 1 + 16 = 17 & & \\ 16 = (-1) \cdot (-16) & \text{and} & -1 + (-16) = -17 & \leftarrow & \text{This is the correct choice.} \\ 16 = 2 \cdot 8 & \text{and} & 2 + 8 = 10 & & \\ 16 = (-2) \cdot (-8) & \text{and} & -2 + (-8) = -10 & & \\ 16 = 4 \cdot 4 & \text{and} & 4 + 4 = 8 & & \\ 16 = (-4) \cdot (-4) & \text{and} & -4 + (-4) = -8 & & \end{array}$$

The answer is $(x-1)(x-16)$.

Factor when a = 1 and c is Negative

Now let's see how this method works if the constant term is negative.

Example 7

Factor $x^2 + 2x - 15$

Solution

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$.

In this case, we must take the negative sign into account. The number -15 can be written as the product of the following numbers.

$-15 = -1 \cdot 15$ and $-1 + 15 = 14$ Notice that these are two different choices.

And also,

$-15 = 1 \cdot (-15)$ and $1 + (-15) = -14$ Notice that these are two different choices.

$-15 = -3 \cdot 5$	and	$-3 + 5 = 2$	$\leftarrow$	This is the correct choice.
$-15 = 3 \cdot (-5)$	and	$3 + (-5) = -2$		

The answer is $(x - 3)(x + 5)$.

We can check to see if this is correct by multiplying $(x - 3)(x + 5)$.

$$\begin{array}{rrr} & x & -\ 3 \\ & x & +\ 5 \\ \hline & 5x & -\ 15 \\ x^2 & -\ 3x & \\ \hline x^2 & +\ 2x & -\ 15 \end{array}$$

The answer checks out.

Example 8

Factor $x^2 - 10x - 24$

Solution

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$.

The number -24 can be written as the product of the following numbers.

$-24 = -1 \cdot 24$	and	$-1 + 24 = 23$		
$-24 = 1 \cdot (-24)$	and	$1 + (-24) = -23$		
$-24 = -2 \cdot 12$	and	$-2 + 12 = 10$		
$-24 = 2 \cdot (-12)$	and	$2 + (-12) = -10$	$\leftarrow$	This is the correct choice.
$-24 = -3 \cdot 8$	and	$-3 + 8 = 5$		
$-24 = 3 \cdot (-8)$	and	$3 + (-8) = -5$		
$-24 = -4 \cdot 6$	and	$-4 + 6 = 2$		
$-24 = 4 \cdot (-6)$	and	$4 + (-6) = -2$		

The answer is $(x - 12)(x + 2)$.

Example 9

Factor $x^2 + 34x - 35$

Solution

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$

The number -35 can be written as the product of the following numbers:

$$
\begin{array}{llll}
-35 = -1 \cdot 35 & \text{and} & -1 + 35 = 34 & \leftarrow \text{This is the correct choice.} \\
-35 = 1 \cdot (-35) & \text{and} & 1 + (-35) = -34 & \\
-35 = -5 \cdot 7 & \text{and} & -5 + 7 = 2 & \\
-35 = 5 \cdot (-7) & \text{and} & 5 + (-7) = -2 &
\end{array}
$$

The answer is $(x - 1)(x + 35)$.

Factor when a = - 1

When $a = -1$, the best strategy is to factor the common factor of -1 from all the terms in the quadratic polynomial. Then, you can apply the methods you have learned so far in this section to find the missing factors.

Example 10

Factor $x^2 + x + 6$.

Solution

First factor the common factor of -1 from each term in the trinomial. Factoring -1 changes the signs of each term in the expression.

$$-x^2 + x + 6 = -(x^2 - x - 6)$$

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$

Now our job is to factor $x^2 - x - 6$.

The number -6 can be written as the product of the following numbers.

$$
\begin{array}{lllll}
-6 = -1 \cdot 6 & \text{and} & -1 + 6 = 5 & & \\
-6 = 1 \cdot (-6) & \text{and} & 1 + (-6) = -5 & & \\
-6 = -2 \cdot 3 & \text{and} & -2 + 3 = 1 & & \\
-6 = 2 \cdot (-3) & \text{and} & 2 + (-3) = -1 & \leftarrow & \text{This is the correct choice.}
\end{array}
$$

The answer is $-(x - 3)(x + 2)$.

To Summarize,

A quadratic of the form $x + bx + c$ factors as a product of two parenthesis $(x + m)(x + n)$.

- If b and c are positive then both m and n are positive
  - Example $x^2 + 8x + 12$ factors as $(x + 6)(x + 2)$.
- If b is negative and c is positive then both m and n are negative.
  - Example $x^2 - 6x + 8$ factors as $(x - 2)(x - 4)$.
- If c is negative then either m is positive and n is negative or vice-versa
  - Example $x^2 + 2x - 15$ factors as $(x + 5)(x - 3)$.

- Example $x^2 + 34x - 35$ factors as $(x + 35)(x - 1)$.

- If $a = -1$, factor a common factor of -1 from each term in the trinomial and then factor as usual. The answer will have the form $-(x + m)(x + n)$.
  - Example $-x^2 + x + 6$ factors as $-(x - 3)(x + 2)$.

Review Questions

Factor the following quadratic polynomials.

1. $x^2 + 10x + 9$
2. $x^2 + 15x + 50$
3. $x^2 + 10x + 21$
4. $x^2 + 16x + 48$
5. $x^2 - 11x + 24$
6. $x^2 - 13x + 42$
7. $x^2 - 14x + 33$
8. $x^2 - 9x + 20$
9. $x^2 + 5x - 14$
10. $x^2 + 6x - 27$
11. $x^2 + 7x - 78$
12. $x^2 + 4x - 32$
13. $x^2 - 12x - 45$
14. $x^2 - 5x - 50$
15. $x^2 - 3x - 40$
16. $x^2 - x - 56$
17. $-x^2 - 2x - 1$
18. $-x^2 - 5x + 24$
19. $-x^2 + 18x - 72$
20. $-x^2 + 25x - 150$
21. $x^2 + 21x + 108$
22. $-x^2 + 11x - 30$
23. $x^2 + 12x - 64$
24. $x^2 - 17x - 60$

Review Answers

1. $(x + 1)(x + 9)$
2. $(x + 5)(x + 10)$
3. $(x + 7)(x + 3)$
4. $(x + 12)(x + 4)$
5. $(x - 3)(x - 8)$
6. $(x - 7)(x - 6)$
7. $(x - 11)(x - 3)$
8. $(x - 5)(x - 4)$
9. $(x - 2)(x + 7)$
10. $(x - 3)(x + 9)$
11. $(x - 6)(x + 13)$
12. $(x - 4)(x + 8)$

13. $(x-15)(x+3)$
14. $(x-10)(x+5)$
15. $(x-8)(x+5)$
16. $(x-8)(x+7)$
17. $-(x+1)(x+1)$
18. $-(x-3)(x+8)$
19. $-(x-6)(x-12)$
20. $-(x-15)(x-10)$
21. $(x+9)(x+12)$
22. $-(x-5)(x-6)$
23. $(x-4)(x+16)$
24. $(x-20)(x+3)$

9.6 Factoring Special Products

Learning Objectives

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Solve quadratic polynomial equation by factoring.

Introduction

When you learned how to multiply binomials we talked about two special products.

The Sum and Difference Formula	$(a+b)(a-b)=a^2-b^2$
The Square of a Binomial Sormula	$(a+b)^2=a^2+2ab+b^2$
	$(a-b)^2=a^2-2ab+b^2$

In this section we will learn how to recognize and factor these special products.

Factor the Difference of Two Squares

We use the sum and difference formula to factor a difference of two squares. A difference of two squares can be a quadratic polynomial in this form.

$$a^2-b^2$$

Both terms in the polynomial are perfect squares. In a case like this, the polynomial factors into the sum and difference of the square root of each term.

$$a^2-b^2=(a+b)(a-b)$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example 1

Factor the difference of squares.

a) $x^2 - 9$

b) $x^2 - 100$

c) $x^2 - 1$

Solution

a) Rewrite as $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

$$\text{The difference of squares formula is} \qquad a^2 - b^2 = (a+b)(a-b)$$
$$\text{Let's see how our problem matches with the formula} \qquad x^2 - 3^2 = (x+3)(x-3)$$

The answer is $x^2 - 9 = (x+3)(x-3)$.

We can check to see if this is correct by multiplying $(x+3)(x-3)$.

$$\begin{array}{rrr} & x & +\ 3 \\ & x & -\ 3 \\ \hline & -\,3x & -\ 9 \\ x^2 & +\ 3x & \\ \hline x^2 & +\ 0x & -\ 9 \end{array}$$

The answer checks out.

We could factor this polynomial without recognizing that it is a difference of squares. With the methods we learned in the last section we know that a quadratic polynomial factors into the product of two binomials.

$$(x \pm \_\_\_\_)(x \pm \_\_\_\_)$$

We need to find two numbers that multiply to -9 and add to 0, since the middle term is missing.

We can write -9 as the following products

$$\begin{array}{rcrcl} -9 = -1 \cdot 9 & \text{and} & -1 + 9 = 8 & & \\ -9 = 1 \cdot (-9) & \text{and} & 1 + (-9) = -8 & & \\ -9 = 3 \cdot (-3) & \text{and} & 3 + (-3) = 0 & \leftarrow & \text{This is the correct choice} \end{array}$$

We can factor $x^2 - 9$ as $(x+3)(x-3)$, which is the same answer as before.

You can always factor using methods for factoring trinomials, but it is faster if you can recognize special products such as the difference of squares.

b) Rewrite $x^2 - 100$ as $x^2 - 10^2$. This factors as $(x+10)(x-10)$.

c) Rewrite $x^2 - 1$ as $x^2 - 1^2$. This factors as $(x+1)(x-1)$.

Example 2

Factor the difference of squares.

a) $16x^2 - 25$

b) $4x^2 - 81$

c) $49x^2 - 64$

Solution

a) Rewrite $16x^2 - 25$ as $(4x)^2 - 5^2$. This factors as $(4x+5)(4x-5)$.

b) Rewrite $4x^2 - 81$ as $(2x)^2 - 9^2$. This factors as $(2x+9)(2x-9)$.

c) Rewrite $49x^2 - 64$ as $(7x)^2 - 8^2$. This factors as $(7x+8)(7x-8)$.

Example 3

Factor the difference of squares:

a) $x^2 - y^2$

b) $9x^2 - 4y^2$

c) $x^2y^2 - 1$

Solution

a) $x^2 - y^2$ factors as $(x+y)(x-y)$.

b) Rewrite$9x^2 - 4y^2$ as $(3x)^2 - (2y)^2$. This factors as $(3x+2y)(3x-2y)$.

c) Rewrite as $x^2y^2 - 1$ as $(xy)^2 - 1^2$. This factors as $(xy+1)(xy-1)$.

Example 4

Factor the difference of squares.

a) $x^4 - 25$

b) $16x^4 - y^2$

c) $x^2y^8 - 64z^2$

Solution

a) Rewrite $x^4 - 25$ as $(x^2)^2 - 5^2$. This factors as $(x^2+5)(x^2-5)$.

b) Rewrite $16x^4 - y^2$ as $(4x^2)^2 - y^2$. This factors as $(x^2+5)(x^2-5)$.

c) Rewrite $x^2y^8 - 64z^2$ as $(xy^2)^2 - (8z)$. This factors as $(xy^2+8z)(xy^2-8z)$.

Factor Perfect Square Trinomials

We use the **Square of a Binomial Formula** to factor perfect square trinomials. A perfect square trinomial has the following form.

$$a^2 + 2ab + b^2 \qquad \text{or} \qquad a^2 - 2ab + b^2$$

In these special kinds of trinomials, the first and last terms are perfect squares and the middle term is twice the product of the square roots of the first and last terms. In a case like this, the polynomial factors into perfect squares.

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

In these problems, the key is figuring out what the a and b terms are. Let's do some examples of this type.

Example 5

Factor the following perfect square trinomials.

a) $x^2 + 8x + 16$

b) $x^2 - 4x + 4$

c) $x^2 + 14x + 49$

Solution

a) $x^2 + 8x + 16$

The first step is to recognize that this expression is actually perfect square trinomials.

1. Check that the first term and the last term are perfect squares. They are indeed because we can re-write:

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 8x + 4^2.$$

2. Check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 2 \cdot 4 \cdot x + 4^2$$

This means we can factor $x^2 + 8x + 16$ as $(x + 4)^2$.

We can check to see if this is correct by multiplying $(x + 4)(x + 4)$.

$$\begin{array}{ccccc} & & x & + & 4 \\ & & x & - & 4 \\ \hline & & 4x & + & 16 \\ x^2 & + & 4x & & \\ \hline x^2 & + & 8x & + & 16 \end{array}$$

The answer checks out.

We could factor this trinomial without recognizing it as a perfect square. With the methods we learned in the last section we know that a trinomial factors as a product of the two binomials in parentheses.

$$(x \pm \underline{\quad\quad})(x \pm \underline{\quad\quad})$$

We need to find two numbers that multiply to 16 and add to 8. We can write 16 as the following products.

$$\begin{array}{rcrcl} 16 = 1 \cdot 16 & \text{and} & 1 + 16 = 17 & & \\ 16 = 2 \cdot 8 & \text{and} & 2 + 8 = 10 & & \\ 16 = 4 \cdot 4 & \text{and} & 4 + 4 = 8 & \leftarrow & \text{This is the correct choice.} \end{array}$$

We can factor $x^2 + 8x + 16$ as $(x + 4)(x + 4)$ which is the same as $(x + 4)^2$.

You can always factor by the methods you have learned for factoring trinomials but it is faster if you can recognize special products.

b) Rewrite $x^2 - 4x + 4$ as $x^2 + 2 \cdot (-2) \cdot x + (-2)^2$.

We notice that this is a perfect square trinomial and we can factor it as: $(x - 2)^2$.

c) Rewrite $x^2 + 14x + 49$ as $x^2 + 2 \cdot 7 \cdot x + 7^2$.

We notice that this is a perfect square trinomial as we can factor it as: $(x+7)^2$.

Example 6

Factor the following perfect square trinomials.

a) $4x^2 + 20x + 25$

b) $9x^2 - 24x + 16$

c) $x + 2xy + y^2$

Solution

a) Rewrite $4x^2 + 20x + 25$ as $(2x)^2 + 2.5 \cdot (2x) + 5^2$

We notice that this is a perfect square trinomial and we can factor it as $(2x+5)^2$.

b) Rewrite $9x^2 - 24x + 16$ as $(3x)^2 + 2 \cdot (-4) \cdot (3x) + (-4)^2$.

We notice that this is a perfect square trinomial as we can factor it as $(3x-4)^2$.

We can check to see if this is correct by multiplying $(3x-4)^2 = (3x-4)(3x-4)$.

$$\begin{array}{rrrrr} & & 3x & + & 4 \\ & & 3x & - & 4 \\ \hline & & -12x & + & 16 \\ 9x^2 & - & 12x & & \\ \hline 9x^2 & - & 24x & + & 16 \end{array}$$

The answer checks out.

c) $x + 2xy + y^2$

We notice that this is a perfect square trinomial as we can factor it as $(x+y)^2$.

Solve Quadratic Polynomial Equations by Factoring

With the methods we learned in the last two sections, we can factor many kinds of quadratic polynomials. This is very helpful when we want to solve polynomial equations such as

$$ax^2 + bx + c = 0$$

Remember that to solve polynomials in expanded form we use the following steps:

Step 1

If necessary, **rewrite** the equation in standard form so that

Polynomial expression = 0.

Step 2

Factor the polynomial completely.

Step 3

Use the Zero-Product rule to **set each factor equal to zero**.

Step 4

Solve each equation from Step 3.

Step 5

Check your answers by substituting your solutions into the original equation.

We will do a few examples that show how to solve quadratic polynomials using the factoring methods we just learned.

Example 7

Solve the following polynomial equations.

a) $x^2 + 7x + 6 = 0$

b) $x^2 - 8x = -12$

c) $x^2 = 2x + 15$

Solution

a) $x^2 + 7x + 6 = 0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor We can write 6 as a product of the following numbers.

$6 = 1 \cdot 6$	and	$1 + 6 = 7$	$\leftarrow$	This is the correct choice.
$6 = 2 \cdot 3$	and	$2 + 3 = 5$		

$x^2 + 7x + 6 = 0$ factors as $(x + 1)(x + 6) = 0$

Set each factor equal to zero

$$x + 1 = 0 \quad \text{or} \quad x + 6 = 0$$

Solve

$$x = -1 \quad \text{or} \quad x = -6$$

Check Substitute each solution back into the original equation.

$x = -1$	$(-1)^2 + 7(-1) + 6 = 1 - 7 + 6 = 0$	Checks out.
$x = -6$	$(-6)^2 + 7(-6) + 6 = 36 - 42 + 6 = 0$	Checks out.

b) $x^2 - 8x = -12$

Rewrite $x^2 - 8x = -12$ is rewritten as $x^2 - 8x + 12 = 0$.

Factor We can write 12 as a product of the following numbers.

$12 = 1 \cdot 12$	and	$1 + 12 = 13$		
$12 = -1 \cdot (-12)$	and	$-1 + (-12) = -13$		
$12 = 2 \cdot 6$	and	$2 + 6 = 8$		
$12 = -2 \cdot (-6)$	and	$-2 + (-6) = -8$	$\leftarrow$	This is the correct choice.
$12 = 3 \cdot 4$	and	$3 + 4 = 7$		
$12 = -3 \cdot (-4)$	and	$-3 + (-4) = -7$		

$x^2 - 8x + 12 = 0$ factors as $(x - 2)(x - 6) = 0$

Set each factor equal to zero.

$$x - 2 = 0 \qquad \text{or} \qquad x - 6 = 0$$

Solve.

$$x = 2 \qquad \text{or} \qquad x = 6$$

Check Substitute each solution back into the original equation.

$x = 2$	$(2)^2 - 8(2) = 4 - 16 = -12$	Checks out.
$x = 2$	$(6)^2 - 8(6) = 36 - 48 = -12$	Checks out.

c) $x^2 = 2x + 15$

Rewrite $x = 2x + 15$ is re-written as $x - 2x - 15 = 0$.

Factor We can write -15 as a product of the following numbers.

$-15 = 1 \cdot (-15)$	and	$1 + (-15) = -14$		
$-15 = -1 \cdot (15)$	and	$-1 + (15) = 14$		
$-15 = -3 \cdot 5$	and	$-3 + 5 = 2$		
$-15 = 3 \cdot (-5)$	and	$3 + (-5) = -2$	$\leftarrow$	This is the correct choice.

$x^2 - 2x - 15 = 0$ factors as $(x + 3)(x - 5) = 0$.

Set each factor equal to zero

$$x + 3 = 0 \qquad \text{or} \qquad x - 5 = 0$$

Solve

$$x = -3 \qquad \text{or} \qquad x = 5$$

Check Substitute each solution back into the original equation.

$x = -3$	$(-3)^2 = 2(-3) + 15 \Rightarrow 9 = 0$	Checks out.
$x = 5$	$(5)^2 = 2(5) + 15 \Rightarrow 25 = 25$	Checks out.

Example 8

Solve the following polynomial equations.

a) $x^2 - 12x + 36 = 0$

b) $x^2 - 81 = 0$

c) $x^2 + 20x + 100 = 0$

Solution

a) $x^2 - 12x + 36 = 0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor: Re-write $x^2 - 12x + 36$ as $x^2 - 2 \cdot (-6)x + (-6)^2$.

We recognize this as a difference of squares. This factors as $(x-6)^2 = 0 or (x-6)(x-6) = 0$.

Set each factor equal to zero

$$x - 6 = 0 \qquad \text{or} \qquad x - 6 = 0$$

Solve

$$x = 6 \qquad \text{or} \qquad x = 6$$

Notice that for a perfect square the two solutions are the same. This is called a **double root**.

Check Substitute each solution back into the original equation.

$$x = 6 \qquad 6^2 - 12(6) + 36 = 36 - 72 + 36 + 0 \qquad \text{Checks out.}$$

b) $x^2 - 81 = 0$

Rewrite This is not necessary since the equation is in the correct form already

Factor Rewrite $x^2 - 81 = 0$ as $x^2 - 9^2 = 0$.

We recognize this as a difference of squares. This factors as $(x-9)(x+9) = 0$.

Set each factor equal to zero.

$$x - 9 = 0 \qquad \text{or} \qquad x + 9 = 0$$

Solve:

$$x = 9 \qquad \text{or} \qquad x = -9$$

Check: Substitute each solution back into the original equation.

$$x = 9 \qquad 9^2 - 81 = 81 - 81 = 0 \qquad \text{Checks out.}$$
$$x = -9 \qquad (-9)^2 - 81 = 81 - 81 = 0 \qquad \text{Checks out.}$$

c) $x^2 + 20x + 100 = 0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor Rewrite $x^2 + 20x + 100 = 0$ as $x^2 + 2 \cdot 10 \cdot x + 10^2$

We recognize this as a perfect square.This factors as: $(x+10)^2 = 0$ or $(x+10)(x+10) = 0$.

Set each factor equal to zero.

$$x + 10 = 0 \qquad \text{or} \qquad x + 10 = 0$$

Solve.

$$x = -10 \qquad \text{or} \qquad x = -10 \qquad \text{This is a double root.}$$

Check Substitute each solution back into the original equation.

$$x = 10 \qquad (-10)^2 + 20(-10) + 100 = 100 - 200 + 100 = 0 \qquad \text{Checks out.}$$

Review Questions

Factor the following perfect square trinomials.

1. $x^2 + 8x + 16$
2. $x^2 - 18x + 81$
3. $-x^2 + 24x - 144$
4. $x^2 + 14x + 49$
5. $4x^2 - 4x + 1$
6. $25x^2 + 60x + 36$
7. $4x^2 - 12xy + 9y^2$
8. $x^4 + 22x^2 + 121$

Factor the following difference of squares.

9. $x^2 - 4$
10. $x^2 - 36$
11. $-x^2 + 100$
12. $x^2 - 400$
13. $9x^2 - 4$
14. $25x^2 - 49$
15. $-36x^2 + 25$
16. $16x^2 - 81y^2$

Solve the following quadratic equation using factoring.

17. $x^2 - 11x + 30 = 0$
18. $x^2 + 4x = 21$
19. $x^2 + 49 = 14x$
20. $x^2 - 64 = 0$
21. $x^2 - 24x + 144 = 0$
22. $4x^2 - 25 = 0$
23. $x^2 + 26x = -169$
24. $-x^2 - 16x - 60 = 0$

Review Answers

1. $(x + 4)^2$
2. $(x - 9)^2$
3. $-(x - 12)^2$
4. $(x + 7)^2$
5. $(2x - 1)^2$
6. $(5x + 6)^2$
7. $(2x - 3y)^2$
8. $(x^2 + 11)^2$
9. $(x + 2)(x - 2)$
10. $(x + 6)(x - 6)$
11. $-(x + 10)(x - 10)$

12. $(x+20)(x-20)$
13. $(3x+2)(3x-2)$
14. $(5x+7)(5x-7)$
15. $-(6x+5)(6x-5)$
16. $(4x+9y)(4x-9y)$
17. $x=5, x=6$
18. $x=-7, x=3$
19. $x=7$
20. $x=-8, x=8$
21. $x=12$
22. $x=5/2, x=-5/2$
23. $x=-13$
24. $x=-10, x=-6$

9.7 Factoring Polynomials Completely

Learning Objectives

- Factor out a common binomial.
- Factor by grouping.
- Factor a quadratic trinomial where $a \neq 1$.
- Solve real world problems using polynomial equations.

Introduction

We say that a polynomial is **factored completely** when we factor as much as we can and we can't factor any more. Here are some suggestions that you should follow to make sure that you factor completely.

- Factor all common monomials first.
- Identify special products such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, factor using the methods we learned in the previous sections.
- Look at each factor and see if any of these can be factored further.

Here are some examples

Example 1

Factor the following polynomials completely.

a) $6x^2-30x+24$

b) $2x^2-8$

c) x^3+6x^2+9x

Solution

a) $6x^2-30x+24$

Factor the common monomial. In this case 6 can be factored from each term.

$$6(x^2-5x+6)$$

There are no special products. We factor x^2-5x+6 as a product of two binomials $(x \pm \_\_\_\_)(x \pm \_\_\_\_)$.

The two numbers that multiply to 6 and add to -5 are -2 and -3. Let's substitute them into the two parenthesis. The 6 is outside because it is factored out.

$$6(x^2-5x+6) = 6(x-2)(x-3)$$

If we look at each factor we see that we can't factor anything else.

The answer is $6(x-2)(x-3)$

b) $2x^2-8$

Factor common monomials $2x^2-8 = 2(x^2-4)$.

We recognize x^2-4 as a difference of squares. We factor as $2(x^2-4) = 2(x+2)(x-2)$.

If we look at each factor we see that we can't factor anything else.

The answer is $2(x+2)(x-2)$.

c) x^3+6x^2+9x

Factor common monomials $x^3+6x^2+9x = x(x^2+6x+9)$.

We recognize as a perfect square and factor as $x(x+3)^2$.

If we look at each factor we see that we can't factor anything else.

The answer is $x(x+3)^2$.

Example 2

Factor the following polynomials completely.

a) $-2x^4+162$

b) x^5-8x^3+16x

Solution

a) $-2x^4+162$

Factor the common monomial. In this case, factor -2 rather than 2. It is always easier to factor the negative number so that the leading term is positive.

$$-2x^4+162 = -2(x^4-81)$$

We recognize expression in parenthesis as a difference of squares. We factor and get this result.

$$-2(x^2-9)(x^2+9)$$

If we look at each factor, we see that the first parenthesis is a difference of squares. We factor and get this answers.

$$-2(x+3)(x-3)(x^2+9)$$

If we look at each factor, we see that we can factor no more.

The answer is $-2(x+3)(x-3)(x^2+9)$

b) x^5-8x^3+16x

Factor out the common monomial $x^5-8x^3+14x = x(x^4-8x^2+16)$.

We recognize $x^4 - 8x^2 + 16$ as a perfect square and we factor it as $x(x^2 - 4)^2$.

We look at each term and recognize that the term in parenthesis is a difference of squares.

We factor and get: $x[(x+2)^2(x-2)]^2 = x(x+2)^2(x-2)^2$.

We use square brackets "[" and "]" in this expression because x is multiplied by the expression $(x+2)^2(x-2)$. When we have "nested" grouping symbols we use brackets "[" and "]" to show the levels of nesting.

If we look at each factor now we see that we can't factor anything else.

The answer is: $x(x+2)^2(x-2)^2$.

Factor out a Common Binomial

The first step in the factoring process is often factoring the common monomials from a polynomial. Sometimes polynomials have common terms that are binomials. For example, consider the following expression.

$$x(3x+2) - 5(3x+2)$$

You can see that the term $(3x+2)$ appears in both term of the polynomial. This common term can be factored by writing it in front of a parenthesis. Inside the parenthesis, we write all the terms that are left over when we divide them by the common factor.

$$(3x+2)(x-5)$$

This expression is now completely factored.

Let's look at some more examples.

Example 3

Factor the common binomials.

a) $3x(x-1) + 4(x-1)$

b) $x(4x+5) + (4x+5)$

Solution

a) $3x(x-1) + 4(x-1)$ has a common binomial of $(x-1)$.

When we factor the common binomial, we get $(x-1)(3x+4)$.

b) $x(4x+5) + (4x+5)$ has a common binomial of $(4x+5)$.

When we factor the common binomial, we get $(4x+5)(x+1)$.

Factor by Grouping

It may be possible to factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factor by grouping**.

The next example illustrates how this process works.

Example 4

Factor $2x + 2y + ax + ay$.

Solution

There isn't a common factor for all four terms in this example. However, there is a factor of 2 that is common to the first two terms and there is a factor of a that is common to the last two terms. Factor 2 from the first two terms and factor a from the last two terms.

$$2x + 2y + ax + ay = 2(x + y) + a(x + y)$$

Now we notice that the binomial $(x + y)$ is common to both terms. We factor the common binomial and get.

$$(x + y)(2 + a)$$

Our polynomial is now factored completely.

Example 5

Factor $3x^2 + 6x + 4x + 8$.

Solution

We factor $3x$ from the first two terms and factor 4 from the last two terms.

$$3x(x + 2) + 4(x + 2)$$

Now factor $(x + 2)$ from both terms.

$$(x + 2)(3x + 4).$$

Now the polynomial is factored completely.

Factor Quadratic Trinomials Where a   1

Factoring by grouping is a very useful method for factoring quadratic trinomials where $a \neq 1$. A quadratic polynomial such as this one.

$$ax^2 + bx + c$$

This does not factor as $(x \pm m)(x \pm n)$, so it is not as simple as looking for two numbers that multiply to give c and add to give b. In this case, we must take into account the coefficient that appears in the first term.

To factor a quadratic polynomial where $a \neq 1$, we follow the following steps.

1. We find the product ac.
2. We look for two numbers that multiply to give ac and add to give b.
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

Let's apply this method to the following examples.

Example 6

Factor the following quadratic trinomials by grouping.

a) $3x^2 + 8x + 4$

b) $6x^2 - 11x + 4$

c) $5x^2 - 6x + 1$

Solution

Let's follow the steps outlined above.

a) $3x^2 + 8x + 4$

Step 1 $ac = 3 \cdot 4 = 12$

Step 2 The number 12 can be written as a product of two numbers in any of these ways:

$12 = 1 \cdot 12$	and	$1 + 12 = 13$	
$12 = 2 \cdot 6$	and	$2 + 6 = 8$	This is the correct choice.
$12 = 3 \cdot 4$	and	$3 + 4 = 7$	

Step 3 Re-write the middle term as: $8x = 2x + 6x$, so the problem becomes the following.

$$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$$

Step 4: Factor an x from the first two terms and 2 from the last two terms.

$$x(3x+2) + 2(3x+2)$$

Now factor the common binomial $(3x+2)$.

$$(3x+2)(x+2)$$

Our answer is $(3x+2)(x+2)$.

To check if this is correct we multiply $(3x+2)(x+2)$.

$$\begin{array}{rrrrr} & & 3x & + & 2 \\ & & x & + & 2 \\ \hline & & 6x & + & 4 \\ 3x^2 & + & 2x & & \\ \hline 3x^2 & + & 8x & + & 4 \end{array}$$

The answer checks out.

b) $6x^2 - 11x + 4$

Step 1 $ac = 6 \cdot 4 = 24$

Step 2 The number 24 can be written as a product of two numbers in any of these ways.

$24 = 1 \cdot 24$	and	$1 + 24 = 25$		
$24 = -1 \cdot (-24)$	and	$-1 + (-24) = -25$		
$24 = 2 \cdot 12$	and	$2 + 12 = 14$		
$24 = -2 \cdot (-12)$	and	$-2 + (-12) = -14$		
$24 = 3 \cdot 8$	and	$3 + 8 = 11$		
$24 = -3 \cdot (-8)$	and	$-3 + (-8) = -11$	$\leftarrow$	This is the correct choice.
$24 = 4 \cdot 6$	and	$4 + 6 = 10$		
$24 = -4 \cdot (-6)$	and	$-4 + (-6) = -10$		

Step 3 Re-write the middle term as $-11x = -3x - 8x$, so the problem becomes

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

Step 4 Factor by grouping. Factor a $3x$ from the first two terms and factor -4 from the last two terms.

$$3x(2x - 1) - 4(2x - 1)$$

Now factor the common binomial $(2x - 1)$.

$$(2x - 1)(3x - 4)$$

Our answer is $(2x - 1)(3x - 4)$.

c) $5x^2 - 6x + 1$

Step 1 $ac = 5 \cdot 1 = 5$

Step 2 The number 5 can be written as a product of two numbers in any of these ways.

$5 = 1 \cdot 5$	and	$1 + 5 = 6$	
$5 = -1 \cdot (-5)$	and	$-1 + (-5) = -6$	$\leftarrow$ This is the correct choice

Step 3 Rewrite the middle term as $-6x = -x - 5x$. The problem becomes

$$5x^2 - 6x + 1 = 5x^2 - x - 5x + 1$$

Step 4 Factor by grouping: factor an x from the first two terms and a factor of -1 from the last two terms

$$x(5x - 1) - 1(5x - 1)$$

Now factor the common binomial $(5x - 1)$.

$$(5x - 1)(x - 1).$$

Our answer is $(5x - 1)(x - 1)$.

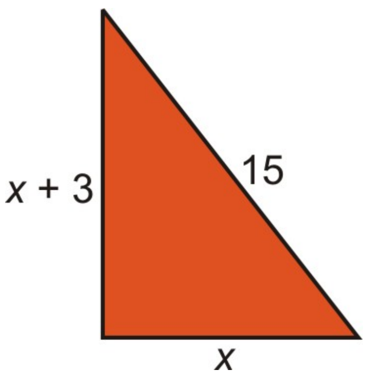
Solve Real-World Problems Using Polynomial Equations

Now that we know most of the factoring strategies for quadratic polynomials we can see how these methods apply to solving real world problems.

Example 7 Pythagorean Theorem

One leg of a right triangle is 3 feet *longer than the other leg. The hypotenuse is* 15 feet. *Find the dimensions of the right triangle.*

Solution



Let $x =$ the length of one leg of the triangle, then the other leg will measure $x+3$.

Let's draw a diagram.

Use the Pythagorean Theorem $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2 ora^2 + b^2 = c^2$.

Here a and b are the lengths of the legs and c is the length of the hypotenuse.

Let's substitute the values from the diagram.

$$a^2 + b^2 = c^2$$
$$x^2 + (x+3)^2 = 15^2$$

In order to solve, we need to get the polynomial in standard form. We must first distribute, collect like terms and **re- write** in the form polynomial = 0.

$$x^2 + x^2 + 6x + 9 = 225$$
$$2x^2 + 6x + 9 = 225$$
$$2x^2 + 6x - 216 = 0$$

Factor the common monomial $2(x + 3x - 108) = 0$.

To factor the trinomial inside the parenthesis we need to numbers that multiply to -108 and add to 3. It would take a long time to go through all the options so let's try some of the bigger factors.

$-108 = -12\cdot$	and	$-12 + 9 = -3$	
$-108 = 12 \cdot (-9)$	and	$12 + (-9) = 3$	$\leftarrow$ This is the correct choice.

We factor as: $2(x-9)(x+12) = 0$.

Set each term equal to zero and solve

$$x - 9 = 0 \qquad \text{or} \qquad x + 12 = 0$$
$$x = 9 \qquad\qquad x = -12$$

It makes no sense to have a negative answer for the length of a side of the triangle, so the answer must be the following.

Answer $x = 9$ for one leg, and $x + 3 = 12$ for the other leg.

Check $9^2 + 12^2 = 81 + 144 = 225 = 15^2$ so the answer checks.

Example 8 Number Problems

The product of two positive numbers is 60. *Find the two numbers if one of the numbers is* 4 *more than the other.*

Solution

Let $x =$ one of the numbers and $x + 4$ equals the other number.

The product of these two numbers equals 60. We can write the equation.

$$x(x + 4) = 60$$

In order to solve we must write the polynomial in standard form. Distribute, collect like terms and re-write in the form polynomial = 0.

$$x^2 + 4x = 60$$
$$x^2 + 4x - 60 = 0$$

Factor by finding two numbers that multiply to -60 and add to 4. List some numbers that multiply to -60:

$$-60 = -4 \cdot 15 \quad \text{and} \quad -4 + 15 = 11$$
$$-60 = 4 \cdot (-15) \quad \text{and} \quad 4 + (-15) = -11$$
$$-60 = -5 \cdot 12 \quad \text{and} \quad -5 + 12 = 7$$
$$-60 = 5 \cdot (-12) \quad \text{and} \quad 5 + (-12) = -7$$
$$-60 = -6 \cdot 10 \quad \text{and} \quad -6 + 10 = 4 \quad \leftarrow \text{This is the correct choice}$$
$$-60 = 6 \cdot (-10) \quad \text{and} \quad 6 + (-10) = -4$$

The expression factors as $(x + 10)(x - 6) = 0$.

Set each term equal to zero and solve.

$$x + 10 = 0 \qquad \text{or} \qquad x - 6 = 0$$
$$x = -10 \qquad\qquad x = 6$$

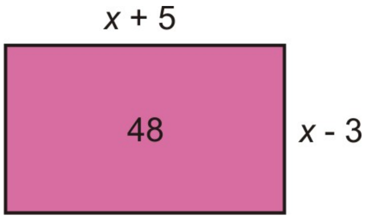
Since we are looking for positive numbers, the answer must be the following.

Answer $x = 6$ for one number, and $x + 4 = 10$ for the other number.

Check $6 \cdot 10 = 60$ so the answer checks.

Example 9 Area of a rectangle

A rectangle has sides of $x + 5$ *and* $x - 3$. *What value of* x *gives and area of* 48?



Solution:

Make a sketch of this situation.

Area of the rectangle = length × width

$$(x + 5)(x - 3) = 48$$

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite** in the form polynomial = 0.

$$x^2 + 2x - 15 = 48$$
$$x^2 + 2x - 63 = 0$$

Factor by finding two numbers that multiply to −63 and add to 2. List some numbers that multiply to −63.

$-63 = -7 \cdot 9$	and	$-7 + 9 = 2$	← This is the correct choice
$-63 = 7 \cdot (-9)$	and	$7 + (-9) = -2$	

The expression factors as $(x + 9)(x - 7) = 0$.

Set each term equal to zero and solve.

$$x + 9 = 0 \qquad \text{or} \qquad x - 7 = 0$$
$$x = -9 \qquad\qquad x = 7$$

Since we are looking for positive numbers the answer must be $x = 7$.

Answer The width is $x - 3 = 4$ and the length is $x + 5 = 12$.

Check $4 \cdot 12 = 48$ so the answer checks out.

Review Questions

Factor completely.

1. $2x^2 + 16x + 30$

2. $-x^3 + 17x^2 - 70x$
3. $2x^2 - 512$
4. $12x^3 + 12x^2 + 3x$

Factor by grouping.

5. $6x^2 - 9x + 10x - 15$
6. $5x^2 - 35x + x - 7$
7. $9x^2 - 9x - x + 1$
8. $4x^2 + 32x - 5x - 40$

Factor the following quadratic binomials by grouping.

9. $4x^2 + 25x - 21$
10. $6x^2 + 7x + 1$
11. $4x^2 + 8x - 5$
12. $3x^2 + 16x + 21$

Solve the following application problems:

13. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 13 feet. Find the dimensions of the right triangle.
14. A rectangle has sides of $x + 2$ and $x - 1$. What value of x gives and area of 108?
15. The product of two positive numbers is 120. Find the two numbers if one numbers is 7 more than the other.
16. Framing Warehouse offers a picture framing service. The cost for framing a picture is made up of two parts The cost of glass is \$1 persquarefoot. The cost of the frame is \$2 perlinearfoot . If the frame is a square, what size picture can you get framed for \$20?

Review Answers

1. $2(x + 3)(x + 5)$
2. $-x(x - 7)(x - 10)$
3. $2(x - 4)(x + 4)(x^2 + 16)$
4. $3x(2x + 1)^2$
5. $(2x - 3)(3x + 5)$
6. $(x - 7)(5x + 1)$
7. $(9x - 1)(x - 1)$
8. $(x + 8)(4x - 5)$
9. $(4x - 3)(x + 7)$
10. $(6x + 1)(x + 1)$
11. $(2x - 1)(2x + 5)$
12. $(x + 3)(3x + 7)$
13. $\text{Leg1} = 5, \text{Leg2} = 12$
14. $x = 10$
15. Numbers are 8 and 15.
16. You can frame a 2 foot $\times$ 2 foot picture.

Chapter 10

Quadratic Equations and Quadratic Functions

10.1 Graphs of Quadratic Functions

Learning Objectives

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Introduction

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples.

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.

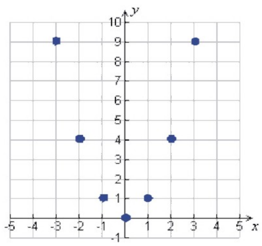
Graph Quadratic Functions

Let's see what a parabola looks like by graphing the simplest quadratic function, $y = x^2$.

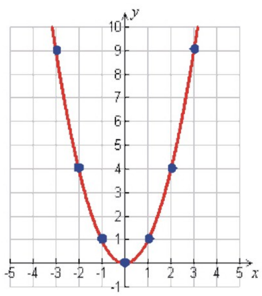
We will graph this function by making a table of values. Since the graph will be curved we need to make sure that we pick enough points to get an accurate graph.

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

We plot these points on a coordinate graph.



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example 1

Graph the following parabolas.

a) $y = 2x^2$

b) $y = -x^2$

c) $y = x^2 - 2x + 3$

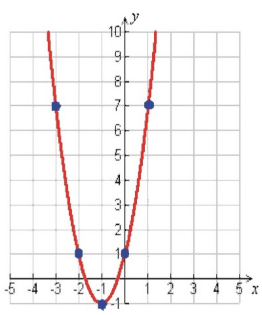
Solution

a) $y = 2x^2 + 4x + 1$

Make a table of values.

x	$y = 2x^2 + 4x + 1$
-3	$2(-3)^2 + 4(-3) + 1 = 7$
-2	$2(-2)^2 + 4(-2) + 1 = 1$
-1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Notice that the last two points have large $y-$ values. We will not graph them since that will make our $y-$ scale too big. Now plot the remaining points and join them with a smooth curve.

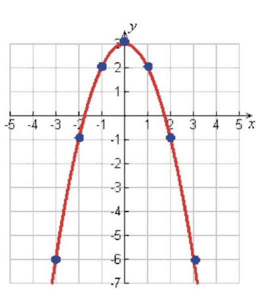


b) $y = -x^2 + 3$

Make a table of values.

x	$y = -x^2 + 3$
-3	$-(-3)^2 + 3 = -6$
-2	$-(-2)^2 + 3 = -1$
-1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Plot the points and join them with a smooth curve.



Notice that it makes an "upside down" parabola. Our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola turns up or down.

If the coefficient of x^2;is positive, then the parabola turns up.

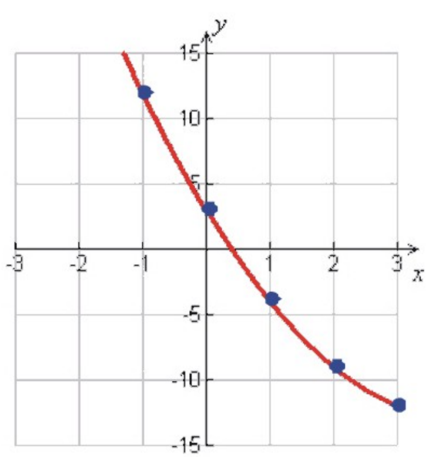
If the coefficient of x^2;is negative, then the parabola turns down.

c) $y = x^2 - 8x + 3$

Make a table of values.

x	$y = x^2 - 8x + 3$
-3	$(-3)^2 - 8(-3) + 3 = 36$
-2	$(-2)^2 - 8(-2) + 3 = 23$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

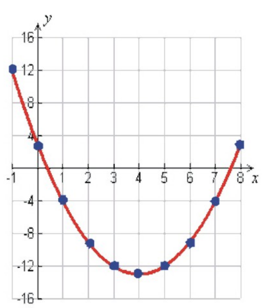
Let's not graph the first two points in the table since the values are very big. Plot the points and join them with a smooth curve.



This does not look like the graph of a parabola. What is happening here? If it is not clear what the graph looks like choose more points to graph until you can see a familiar curve. For negative values of x it looks like the values of y are getting bigger and bigger. Let's pick more positive values of x beyond $x = 3$.

x	$y = x^2 - 8x + 3$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



We now see the familiar parabolic shape. Graphing by making a table of values can be very tedious, especially in problems like this example. In the next few sections, we will learn some techniques that will simplify this process greatly, but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.** Remember a coefficient is just a number (i.e. a constant term) that goes before a variable or it can be alone. You should know that if you have a quadratic function, its graph is always a parabola. While the graph of a quadratic is always the same basic shape, we have different situations where the graph could be upside down. It could be shifted to different locations or it could be "fatter" or "skinnier". These situations are determined by the values of the coefficients. Let's see how changing the coefficient changes the orientation, location or shape of the parabola.

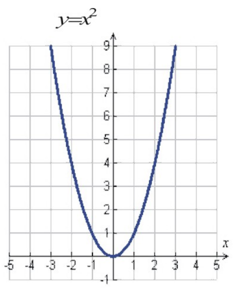
Orientation

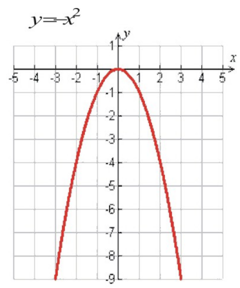
Does the parabola open up or down?

The answer to that question is pretty simple:

- If a is positive, the parabola opens up.
- If a is negative, the parabola opens down.

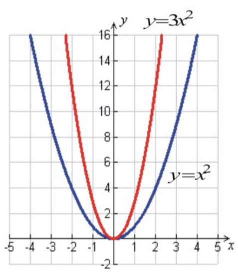
The following plot shows the graphs of $y = x^2$ and $y = -x^2$. You see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ is right-side-up and the graph of $y = -x^2$ is upside-down.

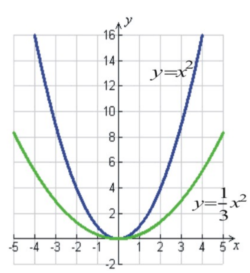




Dilation

Changing the value of the coefficient a makes the graph "fatter" or "skinnier". Let's look at how graphs compare for different positive values of a. The plot on the left shows the graphs of $y = -x^2$ and $y = 3x^2$. The plot on the right shows the graphs of $y = -x^2$ and $y = (1/3)x^2$.





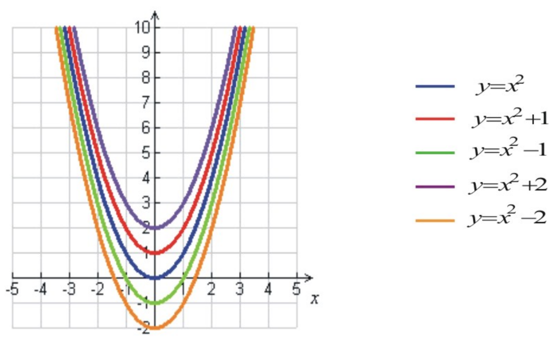
Notice that the larger the value of a is, the skinnier the graph is. For example, in the first plot, the graph of $y = 3x^2$ is skinnier than the graph of $y = x^2$. Also, the smaller a is (i.e. the closer to 0), the fatter the graph is. For example, in the second plot, the graph of $y = (1/3)x^2$ is fatter than the graph of $y = x^2$. This might seem counter-intuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$(-3)^2/3 = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$(-2)^2/3 = 4/3$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$(-1)^2/3 = 1/3$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$(0)^2/3 = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$(1)^2/3 = 1/3$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$(2)^2/3 = 4/3$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$(3)^2/3 = 3$

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result, the parabola will be skinnier because it grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = (1/3)x^2$ are smaller than the values of $y = x^2$. This is because each value of y gets divided by 3. As a result, the parabola will be fatter because it grows at one third the rate of $y = x^2$.

Vertical Shift

Changing the value of the coefficient c (called the constant term) has the effect of moving the parabola up and down. The following plot shows the graphs of $y = x^2, y = x^2 + 1, y = x^2 - 1, y = x^2 + 2, y = x^2 - 2$.



We see that if c is positive, the graph moves up by c units. If c is negative, the graph moves down by c units. In one of the later sections we will also talk about **horizontal shift (i.e. moving to the right or to the left).** Before we can do that we need to learn how to rewrite the quadratic equations in different forms.

Graph Quadratic Functions in Intercept Form

As you saw, in order to get a good graph of a parabola, we sometimes need to pick a lot of points in our table of values. Now, we will talk about different properties of a parabola that will make the graphing process less tedious. Let's look at the graph of $y = x^2 - 6x + 8$.

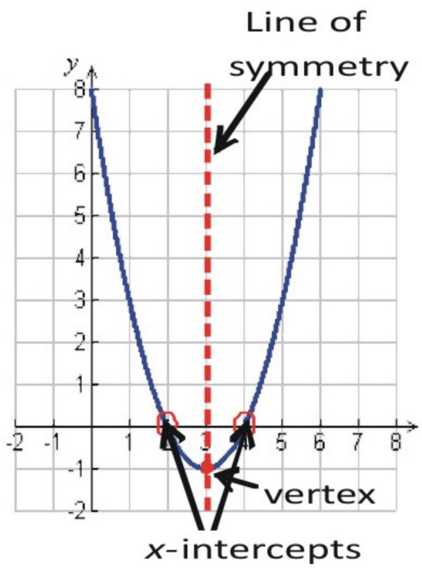
There are several things that we notice.

1. The parabola crosses the x axis at two points: $x = 2$ and $x = 4$.

- These points are called the $x-$ **intercepts** of the parabola.

2. The lowest point of the parabola occurs at point $(3, -1)$.

- This point is called the **vertex** of the parabola.
- The vertex is the lowest point in a parabola that turns upward, and it is the highest point in a parabola that turns downward.
- The vertex is exactly halfway between the two $x-$ intercepts. This will always be the case and you can find the vertex using that rule.



3. A parabola is **symmetric.** If you draw a vertical line through the vertex, you can see that the two halves of the parabola are mirror images of each other. The vertical line is called the **line of symmetry.**

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. If we can factor the quadratic expression, we can rewrite the function in **intercept form**

$$y = a(x-m)(x-n)$$

This form is very useful because it makes it easy for us to find the $x-$ intercepts and the vertex of the parabola. The $x-$ intercepts are the values of x where the graph crosses the $x-$ axis. In other words, they are the values of x when $y = 0$. To find the $x-$ intercepts from the quadratic function, we set $y = 0$ and solve.

$$0 = a(x-m)(x-n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations.

$$\begin{aligned} x - m &= 0 & & & x - n &= 0 \\ & & \text{or} & & & \\ x &= m & & & x &= n \end{aligned}$$

So the $x-$ intercepts are at points $(m,0)$ and $(n,0)$.

Once we find the $x-$ intercepts, it is simple to find the vertex. The $x-$ coordinate of the vertex is halfway between the two x intercepts, so we can find it by taking the average of the two values $(m+n)/2$.

The $y-$ value can be found by substituting the value of x back into the equation of the function.

Let's do some examples that find the $x-$ intercepts and the vertex:

Example 2

Find the $x-$ intercepts and the vertex of the following quadratic function.

(a) $y = x^2 - 8x + 15$

(b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation.

Remember, to factor the trinomial we need two numbers whose product is 15 and whose sum is -8. These numbers are -5 and -3.

The function in intercept form is $y = (x-5)(x-3)$

We find the $x-$ intercepts by setting $y = 0$.

We have

$$0 = (x-5)(x-3)$$

$$x - 5 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 5 \qquad\qquad x = 3$$

The $x-$ intercepts are $(5,0)$ **and** $(3,0)$.

The vertex is halfway between the two $x-$ intercepts. We find the x value by taking the average of the two $x-$ intercepts, $x = (5+3)/2 = 4$.

We find the y value by substituting the x value we just found back into the original equation.

$$y = x^2 - 8x + 15 \Rightarrow y = (4)^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

The vertex is $(4,-1)$.

b) $y = 3x^2 + 6x - 24$

Rewrite the function in intercept form.

Factor the common term of 3 first $y = 3(x^2 + 2x - 8)$.

Then factor completely $y = 3(x+4)(x-2)$

Set $y = 0$ and solve: $0 = 3(x+4)(x-2)$.

$$x + 4 = 0 \Rightarrow x = -4$$

or

$$x - 2 = 0 \Rightarrow x = 2$$

The $x-$ intercepts are: $(-4,0)$ **and** $(2,0)$

For the vertex,

$x = \frac{-4+2}{2} = -1$ and $y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$

The vertex is: $(-1,-27)$.

When graphing, it is very useful to know the vertex and $x-$intercepts. Knowing the vertex, tells us where the middle of the parabola is. When making a table of values we pick the vertex as a point in the table.

Then we choose a few smaller and larger values of x. In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example 3

Find the $x-$intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the $x-$intercepts and the vertex.

Factor the right-hand-side of the function to put the equation in intercept form.

$$y = (x-2)(x+2)$$

Set $y = 0$ and solve.

$$0 = (x-2)(x+2)$$

$x - 2 = 0$		$x + 2 = 0$
	or	
$x = 2$		$x = -2$

$x-$intercepts are: $(2, 0)$ and $(-2, 0)$

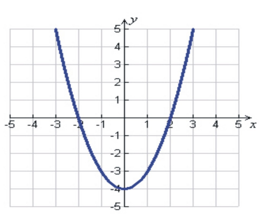
Find the vertex.

$$x = \frac{2-2}{2} = 0 \qquad y = (0)^2 - 4 = -4$$

The vertex is $(0, -4)$

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the $x-$intercepts in the table.

x	$y = x^2 - 4$
-3	$y = (-3)^2 - 4 = 5$
$x-$intercept -2	$y = (-2)^2 - 4 = 0$
-1	$y = (-1)^2 - 4 = -3$
vertex 0	$y = (0)^2 - 4 = -4$
1	$y = (1)^2 - 4 = -3$
$x-$intercept 2	$y = (2)^2 - 4 = 0$
3	$y = (3)^2 - 4 = 5$



b) $y = -x^2 + 14x - 48$

Let's find the $x-$intercepts and the vertex.

Factor the right hand side of the function to put the equation in intercept form.

$$y = -(x^2 - 14x + 48) = -(x-6)(x-8)$$

Set $y = 0$ and solve.

$$0 = -(x-6)(x-8)$$

$$x - 6 = 0 \qquad \text{or} \qquad x - 8 = 0$$

$$x = 6 \qquad \qquad x = 8$$

The $x-$ intercepts are: $(6,0)$ and $(8,0)$

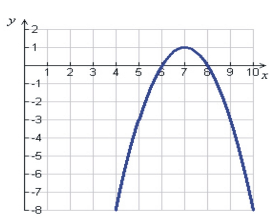
Find the vertex

$$x = \frac{6+8}{2} = 7 \qquad y = (7)^2 + 14(7) - 48 = 1$$

The vertex is $(7,1)$.

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 1$. Include the $x-$intercepts in the table. Then graph the parabola.

x	$y = -x^2 + 14x - 48$
4	$y = -(4)^2 + 14(4) - 48 = -8$
5	$y = -(5)^2 + 14(5) - 48 = -3$
$x -$ intercept 6	$y = -(6)^2 + 14(6) - 48 = 0$
vertex 7	$y = -(7)^2 + 14(7) - 48 = 1$
$x -$ intercept 8	$y = -(8)^2 + 14(8) - 48 = 0$
9	$y = -(9)^2 + 14(9) - 48 = -3$
10	$y = -(10)^2 + 14(10) - 48 = -8$



Analyze Graphs of Real-World Quadratic Functions.

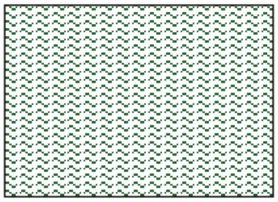
As we mentioned at the beginning of this section, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example 4 Area

Andrew has 100 feet *of fence to enclose a rectangular tomato patch. He wants to find the dimensions of the rectangle that encloses most area.*

Solution

We can find an equation for the area of the rectangle by looking at a sketch of the situation.



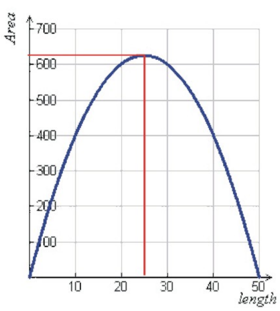
Let x be the length of the rectangle.

$50 - x$ is the width of the rectangle (Remember there are two widths so its not $100 - x$).

Let y be the area of the rectangle.

$$\text{Area} = \text{length} \times \text{width} \Rightarrow y = x(50 - x)$$

The following graph shows how the area of the rectangle depends on the length of the rectangle



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. Notice that the width is also 25, which makes the shape a square with side length 25.

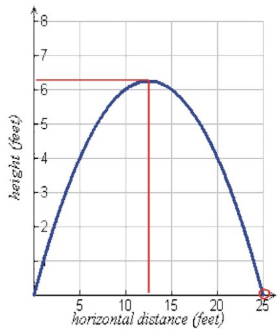
This is an example of an optimization problem.

Example 5 Projectile motion

Anne is playing golf. On the 4*th tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation,* $y = x - 0.04x^2$. *This relates the height of the ball y to the horizontal distance as the ball travels down the fairway. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance, x from the tee, does the ball attain its maximum height? What is the maximum height?*

Solution

Let's graph the equation of the path of the ball: $y = x - 0.04x^2$.



$y = x(1 - 0.04x)$ has solutions of $x = 0$ and $x = 25$

From the graph, we see that the ball hits the ground 25 feet from the tee.

We see that the maximum height is attained at 12.5 feet from the tee and the maximum height the ball reaches is 6.25 feet.

Review Questions

Rewrite the following functions in intercept form. Find the $x-$intercepts and the vertex.

1. $y = x^2 - 2x - 8$
2. $y = -x^2 + 10x - 21$
3. $y = 2x^2 + 6x + 4$

Does the graph of the parabola turn up or down?

4. $y = -2x^2 - 2x - 3$
5. $y = 3x^2$
6. $y = 16 - 4x^2$

The vertex of which parabola is higher?

7. $y = x^2$ or $y = 4x^2$
8. $y = -2x^2$ or $y = -2x^2 - 2$
9. $y = 3x^2 - 3$ or $y = 3x^2 - 6$

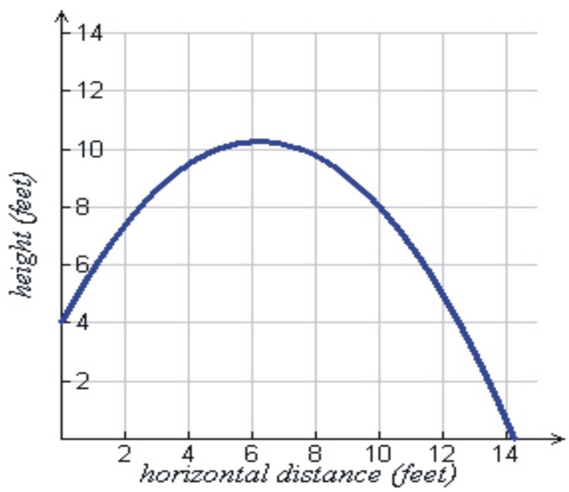
Which parabola is wider?

10. $y = x^2$ or $y = 4x^2$
11. $y = 2x^2 + 4$ or $y = \frac{1}{2}x^2 + 4$
12. $y = -2x^2 - 2$ or $y = -x^2 - 2$

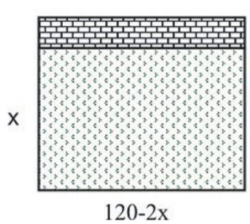
Graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

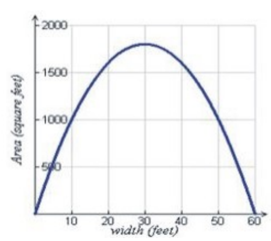
13. $y = 4x^2 - 4$
14. $y = -x^2 + x + 12$

15. $y = 2x^2 + 10x + 8$
16. $y = \frac{1}{2}x^2 - 2x$
17. $y = x - 2x^2$
18. $y = 4x^2 - 8x + 4$
19. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet. How far from Nadia does the ball hit the ground? At what distance, x from Nadia, does the ball attain its maximum height? What is the maximum height?



20. Peter wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $a = 120x - x^2$. From the graph find what dimensions of the rectangle would give him the greatest area.



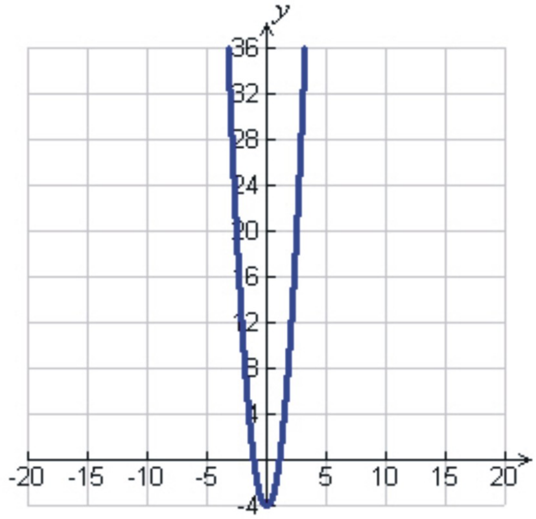


Review Answers

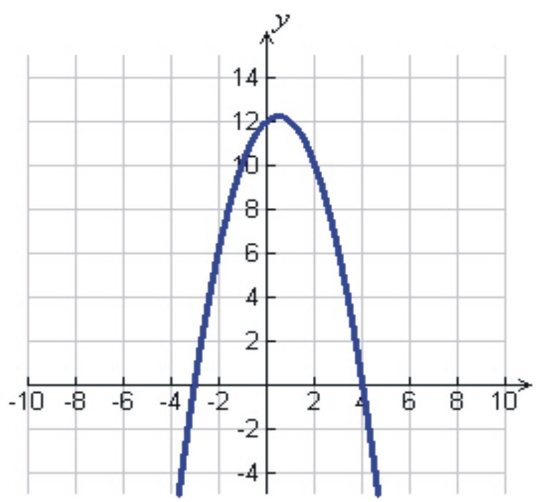
1. $x = -2, x = 4$ Vertex $(1, -9)$
2. $x = 3, x = 7$ Vertex $(5, 4)$
3. $x = -2, x = -1$ Vertex $(-3.5, 7.5)$
4. Down
5. Up
6. Down
7. $y = x^2 + 4$
8. $y = -2x^2$
9. $y = 3x^2 - 3$
10. $y = x^2$
11. $y = (1/2)x^2 + 4$

12. $y = -x^2 - 2$

13.



14.



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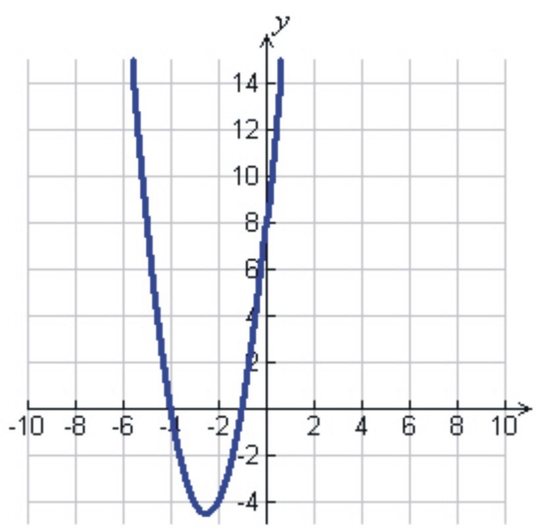
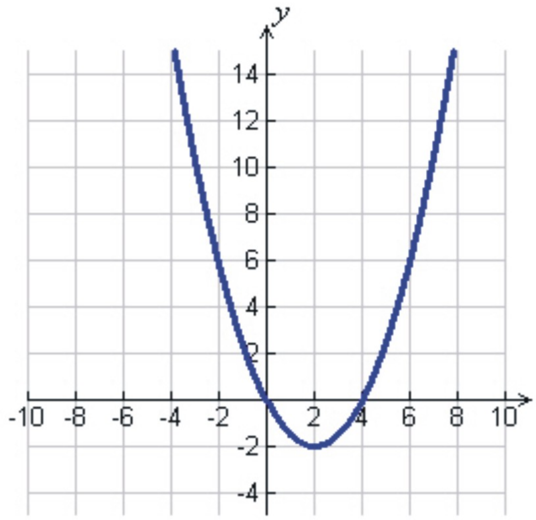
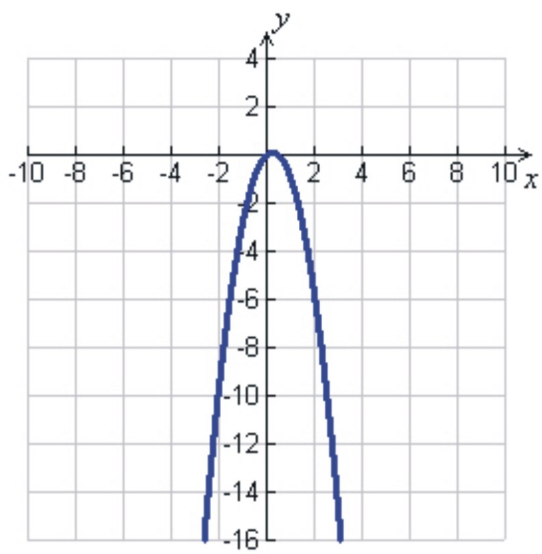


16.



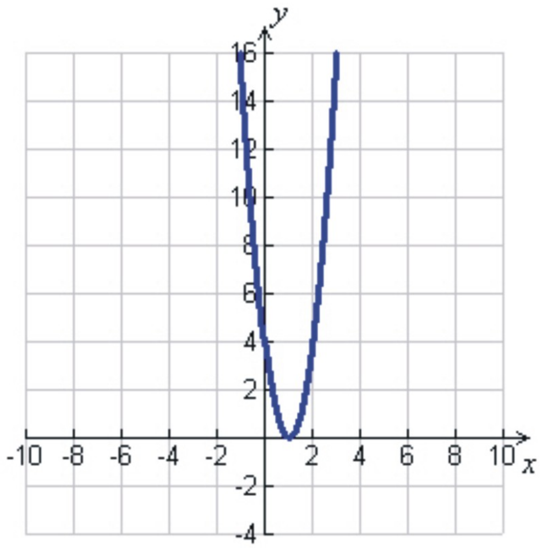
y
14
12
10
8
6
4
2
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4

17.



y
4
2
-10 -8 -6 -4 -2 2 4 6 8 10
x
-2
-4
-6
-8
-10
-12
-14
-16

18.



19. 14.25 feet, 6.25 feet, 10.25 feet
20. width = 30 feet, length = 60 feet

10.2 Quadratic Equations by Graphing

Learning Objectives

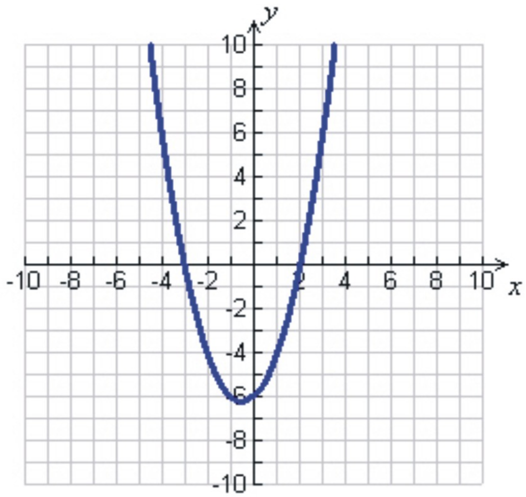
- Identify the number of solutions of quadratic equations.
- Solve quadratic equations by graphing.
- Find or approximate zeros of quadratic functions.
- Analyze quadratic functions using a graphing calculators.
- Solve real-world problems by graphing quadratic functions.

Introduction

In the last, section you learned how to graph quadratic equations. You saw that finding the $x-$ intercepts of a parabola is important because it tells us where the graph crosses the $x-$axis. and it also lets us find the vertex of the parabola. When we are asked to find the **solutions** of the quadratic equation in the form $ax^2 + bx + c = 0$, we are basically asked to find the $x-$ intercepts of the quadratic function.

Finding the $x-$intercepts of a parabola is also called finding the **roots** or **zeros** of the function.

Identify the Number of Solutions of Quadratic Equations



The graph of a quadratic equation is very useful in helping us identify how many solutions and what types of solutions a function has. There are three different situations that occur when graphing a quadratic function.

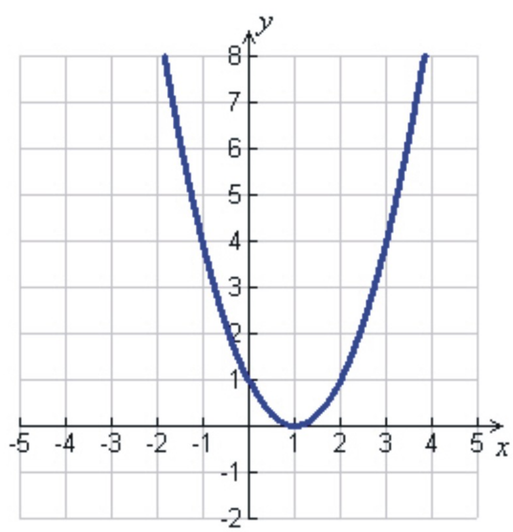
Case 1 The parabola crosses the $x-$axis at two points.

An example of this is $y = x^2 + x - 6$.

We can find the solutions to equation $x^2 + x - 6 = 0$ by setting $y = 0$. We solve the equation by factoring $(x + 3)(x - 2) = 0$ so $x = -3$ or $x = 2$.

Another way to find the solutions is to graph the function and read the $x-$intercepts from the graph. We see that the parabola crosses the $x-$axis at $x = -3$ and $x = 2$.

When the graph of a quadratic function crosses the x-axis at two points, we get **two distinct solutions** to the quadratic equation.



Case 2 The parabola touches the $x-$axis at one point.

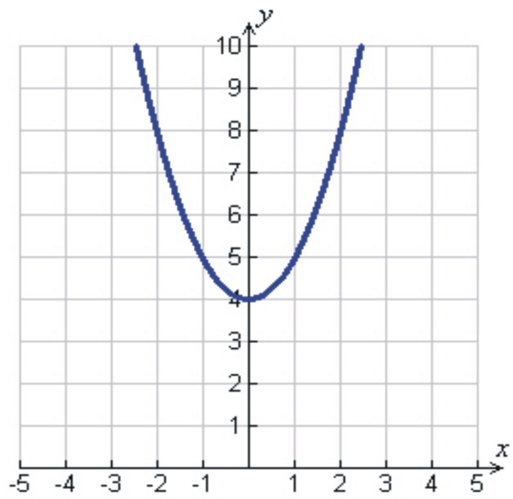
An example of this is $y = x^2 - 2x + 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x - 1)^2$ so $x = 1$.

Since the quadratic function is a perfect square, we obtain only one solution for the equation.

Here is what the graph of this function looks like. We see that the graph touches the $x-$axis at point $x = 1$.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**



Case 3 The parabola does not cross or touch the $x-$axis.

An example of this is $y = x^2 + 4$. If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor and the equation $x^2 = -4$ has no real solutions. When we look at the graph of this function, we see that the parabola does not cross or touch the $x-$axis.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing.

So far we have found the solutions to graphing equations using factoring. However, there are very few functions in real life that factor easily. As you just saw, graphing the function gives a lot of information about the solutions. We can find exact or approximate solutions to quadratic equations by graphing the function associated with it.

Example 1

Find the solutions to the following quadratic equations by graphing.

a) $-x^2 + 3 = 0$

b) $2x^2 + 5x - 7 = 0$

c) $-x^2 + x - 3 = 0$

Solution

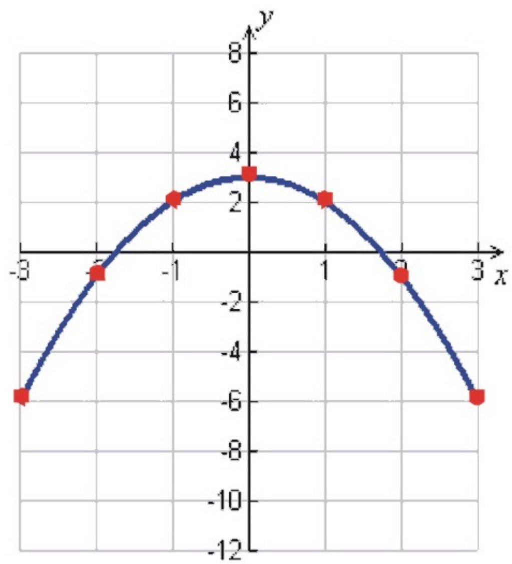
Let's graph each equation. Unfortunately none of these functions can be rewritten in intercept form because we cannot factor the right hand side. This means that you cannot find the $x-$ intercept and vertex before graphing since you have not learned methods other than factoring to do that.

a) To find the solution to $-x^2+3=0$, we need to find the $x-$ intercepts of $y=-x^2+3$.

Let's make a table of values so we can graph the function.

x	$y=-x^2+3$
-3	$y=-(-3)^2+3=-6$
-2	$y=-(-2)^2+3=-1$
-1	$y=-(-1)^2+3=2$
0	$y=-(0)^2+3=3$
1	$y=-(-1)^2+3=2$
2	$y=-(2)^2+3=-1$
3	$y=-(3)^2+3=-6$

We plot the points and get the following graph:



From the graph we can read that the $x-$ intercepts are approximately $x=1.7$ **and** $x=-1.7$.

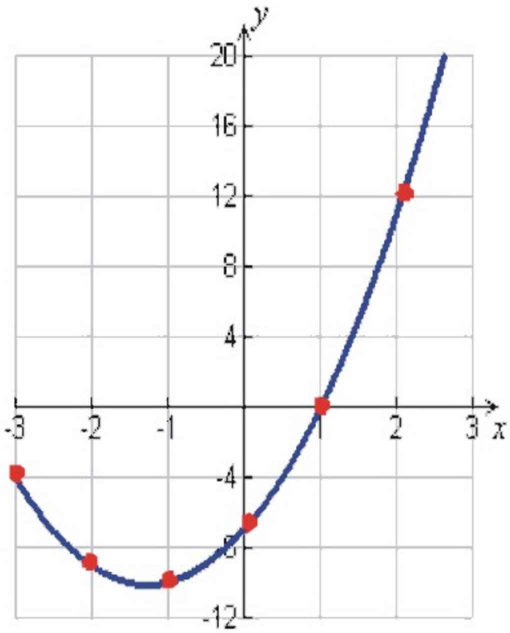
These are the solutions to the equation $-x^2+3=0$.

b) To solve the equation $2x^2+5x-7=0$ we need to find the $x-$ intercepts of $y=2x^2+5-7$.

Let's make a table of values so we can graph the function.

x	$y=2x^2+5x-7$
-3	$y=2(-3)^2+5(-3)-7=-4$
-2	$y=2(-2)^2+5(-2)-7=-9$
-1	$y=2(-1)^2+5(-1)-7=-10$
0	$y=2(0)^0+5(0)-7=-7$
1	$y=2(1)^2+5(1)-7=0$
2	$y=2(2)^2+5(2)-7=11$
3	$y=2(3)^2+5(3)-7=26$

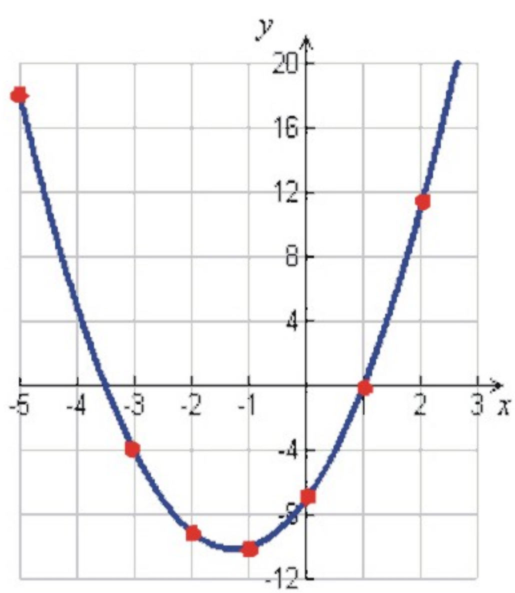
We plot the points and get the following graph:



Since we can only see one $x-$ intercept on this graph, we need to pick more points smaller than $x = -3$ and re-draw the graph.

x	$y = 2x^2 + 5x - 7$
-5	$y = 2(-5)^2 + 5(-5) - 7 = 18$
-4	$y = 2(-4)^2 + 5(-4) - 7 = 5$

Here is the graph again with both $x-$intercepts showing:



From the graph we can read that the $x-$intercepts are $x = 1$ **and** $x = -3.5$.

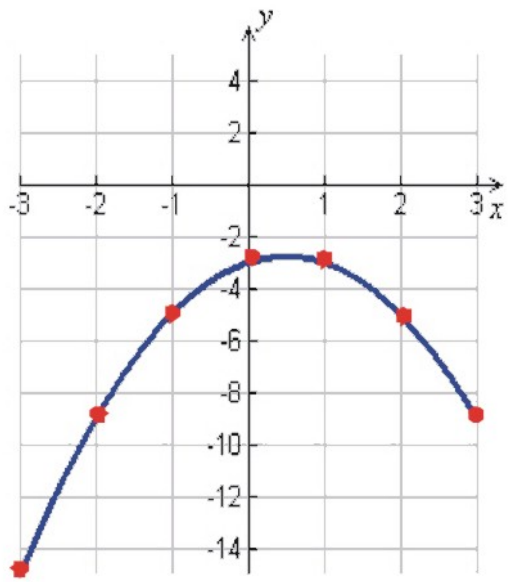
These are the solutions to equation $2x^2 + 5x - 7 = 0$.

c) To solve the equation $-x^2 + x - 3 = 0$ we need to find the $x-$intercepts of $y = -x^2 + x - 3$.

Let's make a table of values so we can graph the function.

x	$y = -x^2 + x - 3.$
-3	$y = -(-3)^2 + (-3) - 3 = -15$
-2	$y = -(-2)^2 + (-2) - 3 = -9$
-1	$y = -(-1)^2 + (-1) - 3 = -5$
0	$y = -(0)^2 + (0) - 3 = -3$
1	$y = -(1)^2 + (1) - 3 = -3$
2	$y = -(-2)^2 + (2) - 3 = -5$
3	$y = -(3)^2 + (3) - 3 = -9$

We plot the points and get the following graph:



This graph has no $x-$intercepts, so the equation $-x^2 + x - 3 = 0$ has **no real solutions.**

Find or Approximate Zeros of Quadratic Functions

From the graph of a quadratic function $y = ax^2 + bx + c$, we can find the **roots** or **zeros** of the function. The zeros are also the $x-$intercepts of the graph, and they solve the equation $ax^2 + bx + c = 0$. When the zeros of the function are integer values, it is easy to obtain exact values from reading the graph. When the zeros are not integers we must approximate their value.

Let's do more examples of finding zeros of quadratic functions.

Example 2 Find the zeros of the following quadratic functions.

a) $y = -x^2 + 4x - 4$

b) $y = 3x^2 - 5x$

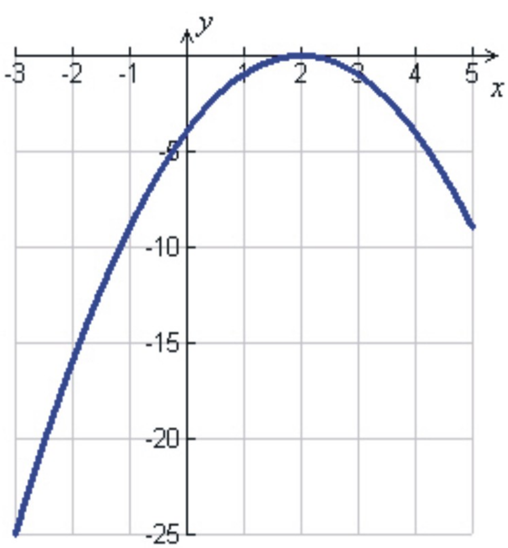
Solution

a) Graph the function $y = -x^2 + 4x - 4$ and read the values of the $x-$intercepts from the graph.

Let's make a table of values.

x	$y = -x^2 + 4x - 4$
-3	$y = -(-3)^2 + 4(-3) - 4 = -25$
-2	$y = -(-2)^2 + 4(-2) - 4 = -16$
-1	$y = -(-1)^2 + 4(-1) - 4 = -9$
0	$y = -(0)^2 + 4(0) - 4 = -4$
1	$y = -(1)^2 + 4(1) - 4 = -1$
2	$y = -(2)^2 + 4(2) - 4 = 0$
3	$y = -(3)^2 + 4(3) - 4 = -1$
4	$y = -(4)^2 + 4(4) - 4 = -4$
5	$y = -(5)^2 + 4(5) - 4 = -9$

Here is the graph of this function.



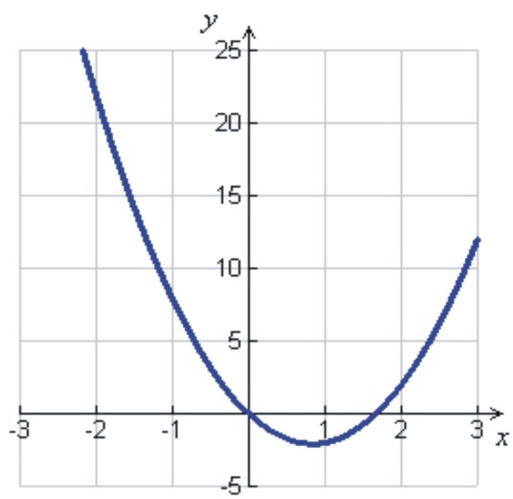
The function has a **double root** at $x = 2$.

b) Graph the function $y = 3x^2 - 5x$ and read the $x-$intercepts from the graph.

Let's make a table of values.

x	$y = 3x^2 - 5x$
-3	$y = 3(-3)^2 - 5(-3) = 42$
-2	$y = 3(-2)^2 - 5(-2) = 22$
-1	$y = 3(-1)^2 - 5(-1) = 8$
0	$y = 3(0)^2 - 5(0) = 0$
1	$y = 3(1)^2 - 5(1) = -2$
2	$y = 3(2)^2 - 5(2) = 2$
3	$y = 3(3)^2 - 5(3) = 12$

Here is the graph of this function.



The function has two roots: $x = 0$ **and** $x \approx 1.7$.

Analyze Quadratic Functions Using a Graphing Calculator

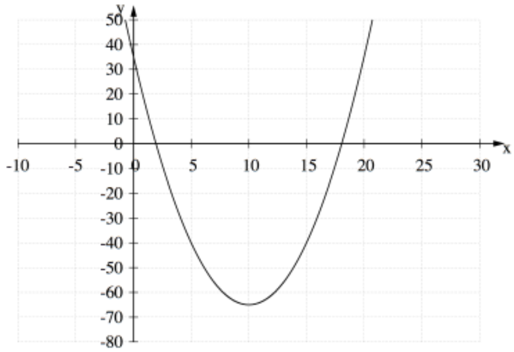
A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots of the function or the vertex of the function.

Example 3

Let's use the graphing calculator to analyze the graph of $y = x^2 - 20x + 35$.

1. **Graph** the function.

Press the **[Y=]** button and enter "$X^2 - 20X + 35$" next to $[Y_1 =]$.(Note, X is one of the buttons on the calculator)



Press the [**GRAPH**] button. This is the plot you should see. If this is not what you see change the window size. For the graph to the right, we used window size of XMIN = −10, XMAX = 30 and YMIN = −80, YMAX = 50. To change window size, press the [**WINDOW**] button.

2. Find the **roots.**

There are at least three ways to find the roots

Use [**TRACE**] to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen.You can improve your estimate by zooming in.

OR

Use [**TABLE**] and scroll through the values until you find values of Y equal to zero. You can change the accuracy of the solution by setting the step size with the [**TBLSET**] function.

OR

Use [**2nd**] [**TRACE**] (i.e. 'calc' button) and use option 'zero'.

Move cursor to the left of one of the roots and press [**ENTER**].

Move cursor to the right of the same root and press [**ENTER**].

Move cursor close to the root and press [**ENTER**].

The screen will show the value of the root. For the left side root, we obtained $x = 1.9$.

Repeat the procedure for the other root. For the right side root, we obtained $x = 18$.

3. Find the **vertex**

There are three ways to find the vertex.

Use [**TRACE**] to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use [**TABLE**] and scroll through the values until you find values the lowest or highest values of Y.

You can change the accuracy of the solution by setting the step size with the [**TBLSET**] function.

OR

Use [**2nd**] [**TRACE**] and use option 'maximum' if the vertex is a maximum or option 'minimum' if the vertex is a minimum.

Move cursor to the left of the vertex and press [**ENTER**].

Move cursor to the right of the vertex and press [**ENTER**].

Move cursor close to the vertex and press [**ENTER**].

The screen will show the x and y values of the vertex.

For this example, we obtained $x = 10$ and $x = -65$.

Solve Real-World Problems by Graphing Quadratic Functions

We will now use the methods we learned so far to solve some examples of real-world problems using quadratic functions.

Example 4 Projectile motion

Andrew is an avid archer. He launches an arrow that takes a parabolic path. Here is the equation of the height of the ball with respect to time.

$$y = -4.9t^2 + 48t$$

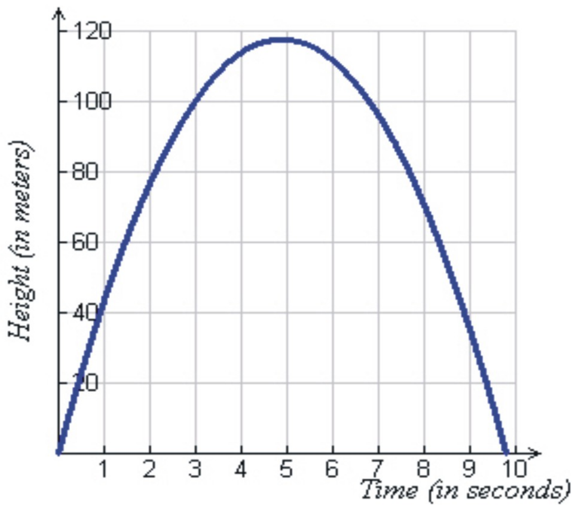
Here y is the height in meters and t is the time in seconds. Find how long it takes the arrow to come back to the ground.

Solution

Let's graph the equation by making a table of values.

t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.5$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here is the graph of the function.



The roots of the function are approximately $x = 0$ sec and $x = 9.8$ sec. The first root says that at time 0 seconds the height of the arrow is 0 meters. The second root says that it takes approximately 9.8 seconds for the arrow to return back to the ground.

Review Questions

Find the solutions of the following equations by graphing.

1. $x^2 + 3x + 6 = 0$
2. $-2x^2 + x + 4 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 9 = 0$
5. $10x^2 - 3x^2 = 0$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

Find the roots of the following quadratic functions by graphing.

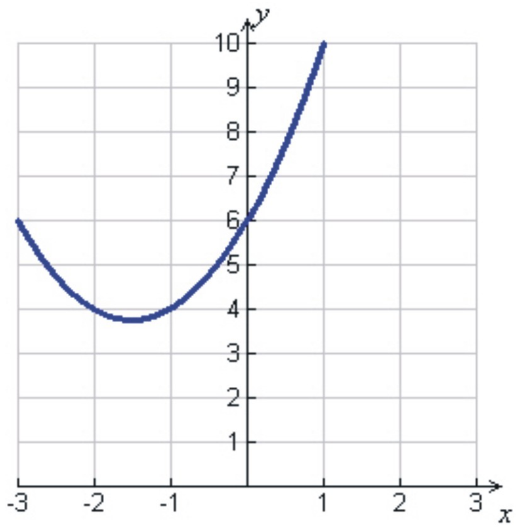
7. $y = -3x^2 + 4x - 1$
8. $y = 9 - 4x^2$
9. $y = x^2 + 7x + 2$
10. $y = -x^2 - 10x - 25$
11. $y = 2x^2 - 3x$
12. $y = x^2 - 2x + 5$ Using your graphing calculator
    (a) Find the roots of the quadratic polynomials.
    (b) Find the vertex of the quadratic polynomials.

13. $y = x^2 + 12x + 5$
14. $y = x^2 + 3x + 6$
15. $y = -x^2 - 3x + 9$
16. Peter throws a ball and it takes a parabolic path. Here is the equation of the height of the ball with respect to time: $y = -16t^2 + 60t$
    Here y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.

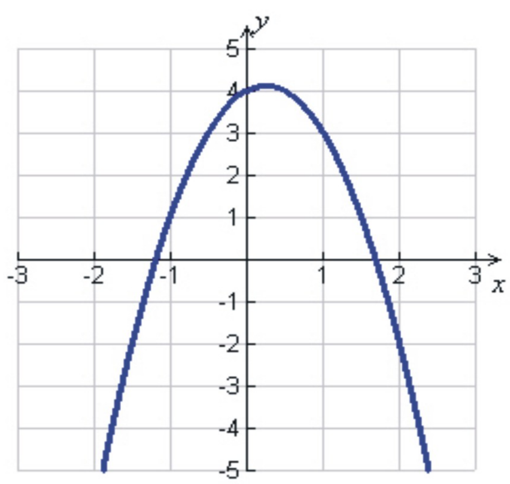
17. Use your graphing calculator to solve Ex. 5. You should get the same answers as we did graphing by hand but a lot quicker!

Review Answers

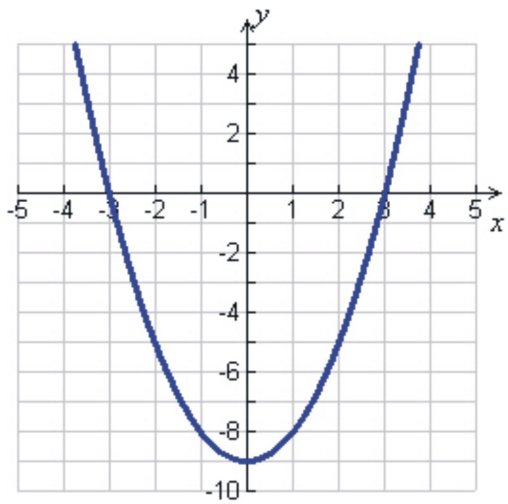
1. No real solutions



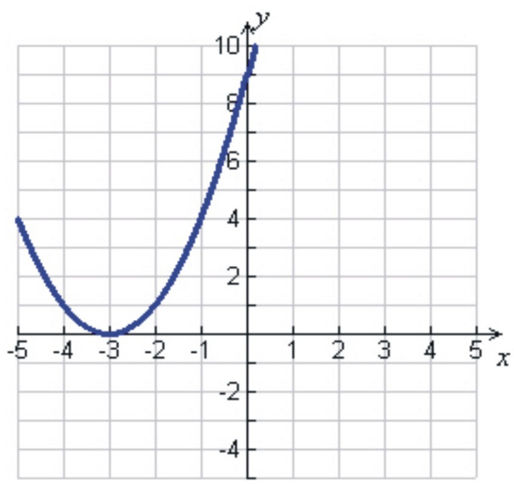
2. $x = -1.2, x = 1.87$



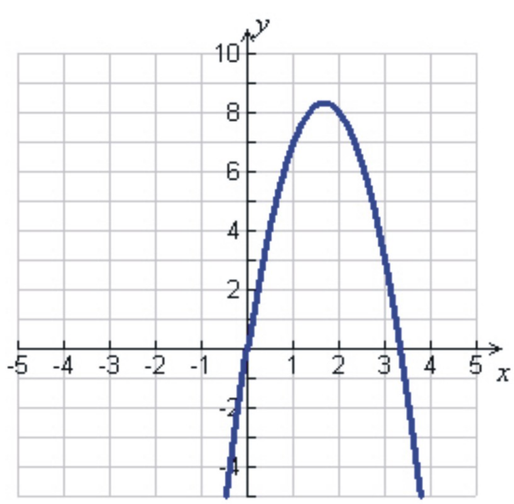
3. $x = -3, x = 3$



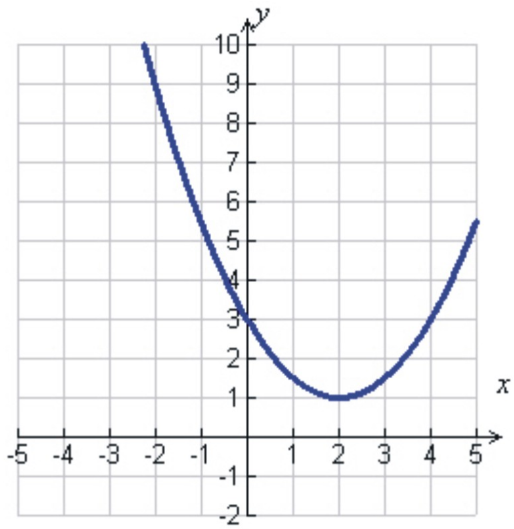
4. $x = -3$ double root



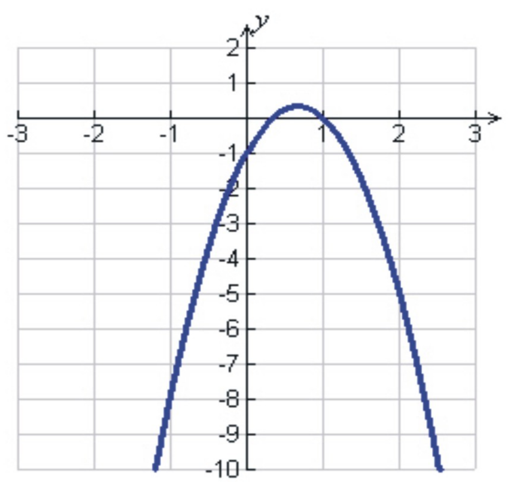
5. $x = 0, x = 3.23$



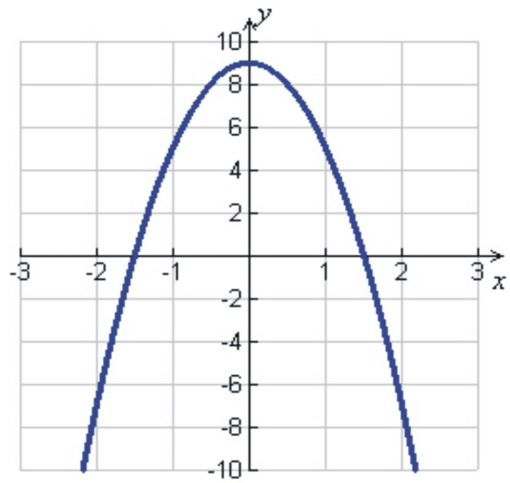
6. No real solutions.



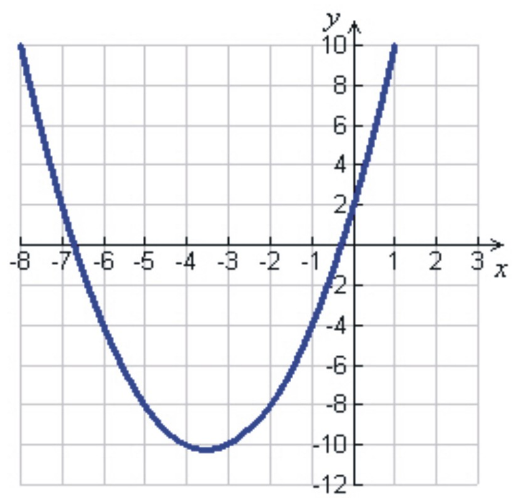
7. $x = 0.3, x = 1$



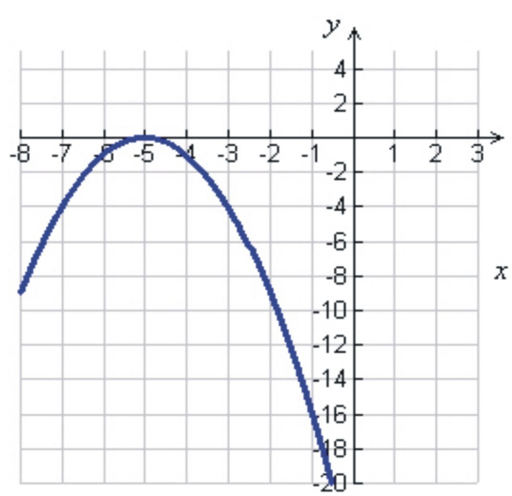
8. $x = -1.5, x = 1.5$



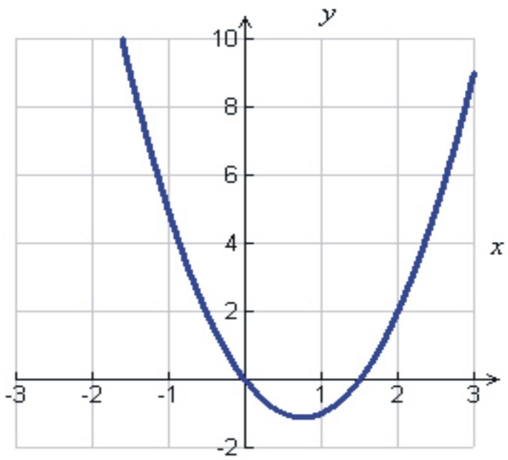
9. $x = -6.7, x = 0.3$



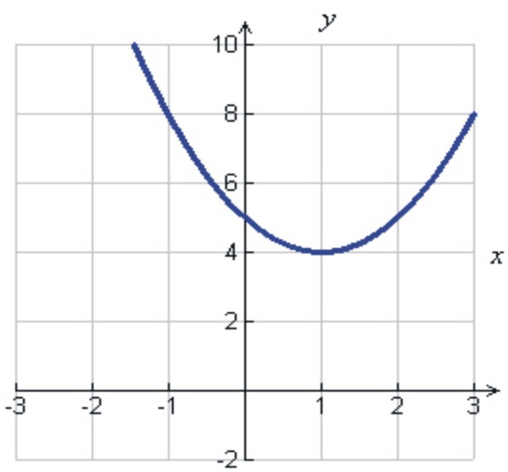
10. $x = -5$ double root



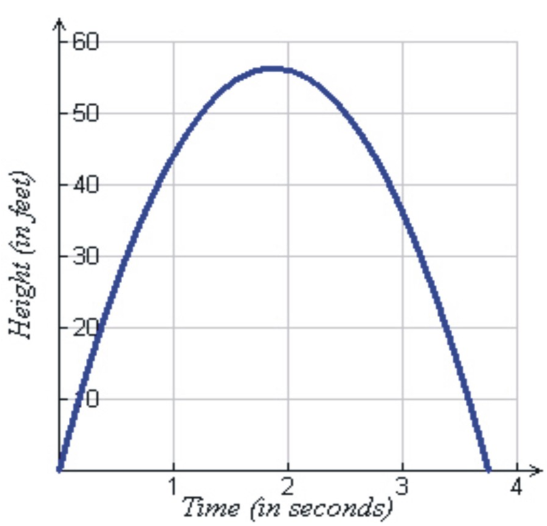
11. $x = 0, x = 1.5$



12. No real solutions.



13. .
14. .
15. .
16. time = 3.75 second



17. .

10.3 Quadratic Equations by Square Roots

Learning objectives

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Introduction

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. Unfortunately, in practice, most quadratic polynomials are not factorable. In this section you will continue learning new methods that can be used in solving quadratic equations. In particular, we will examine equations in which we can take the square root of both sides of the equation in order to arrive at the result.

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equation of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = c$

Once the x^2 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \qquad \text{and} \qquad x = -\sqrt{c}$$

Often this is written as $x = \pm\sqrt{c}$.

Example 1

Solve the following quadratic equations

a) $x^2 - 4 = 0$

b) $x^2 - 25 = 0$

Solution

a) $x^2 - 4 = 0$

Isolate the $x^2 x^2 = 4$

Take the square root of both sides $x = \sqrt{4}$ and $x = -\sqrt{4}$

Answer $x = 2$ and $x = -2$

b) $x^2 - 25 = 0$

Isolate the x^2 $\qquad x^2 = 25$

Take the square root of both sides $\qquad x = \sqrt{25}$ and $x = -\sqrt{25}$

Answer $x = 5$ and $x = -5$

Another type of equation where we can find the solution using the square root is

$$ax^2 - c = 0$$

We can solve this equation by isolating the x^2 term

$$ax^2 = c$$
$$x^2 = \frac{c}{a}$$

Now we can take the square root of both sides of the equation.

$$x = \sqrt{\frac{c}{a}} \qquad \text{and} \qquad x = -\sqrt{\frac{c}{a}}$$

Often this is written as $x = \pm\sqrt{\frac{c}{a}}$.

Example 2

Solve the following quadratic equations.

a) $9x^2 - 16 = 0$

b) $81x^2 - 1 = 0$

Solution

a) $9x^2 - 16 = 0$

Isolate the x^2.
$$9x^2 = 16$$
$$x^2 = \frac{16}{9}$$

Take the square root of both sides. $x = \sqrt{\frac{16}{9}}$ and $x = -\sqrt{169}$

Answer: $x = \frac{4}{3}$ and $x = -\frac{4}{3}$

b) $81x^2 - 1 = 0$

Isolate the x^2
$$81x^2 = 1$$
$$x^2 = \frac{1}{81}$$

Take the square root of both sides $x = \sqrt{\frac{1}{81}}$ and $x = -\sqrt{\frac{1}{81}}$

Answer $x = \frac{1}{9}$ and $x = -\frac{1}{9}$

As you have seen previously, some quadratic equations have no real solutions.

Example 3

Solve the following quadratic equations.

a) $x^2 + 1 = 0$

b) $4x^2 + 9 = 0$

Solution

a) $x^2 + 1 = 0$

$$\text{Isolate the } x^2 \qquad x^2 = -1$$

Take the square root of both sides: $x = \sqrt{-1}$ and $x = -\sqrt{-1}$

Answer Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation.

b) $4x^2 + 9 = 0$

$$\text{Isolate the } x^2 \qquad 4x^2 = -9$$
$$x^2 = -\frac{9}{4}$$

Take the square root of both sides $x = \sqrt{-\frac{9}{4}}$ and $x = -\sqrt{-\frac{9}{4}}$

Answer There are **no real solutions.**

We can also use the square root function in some quadratic equations where one side of the equation is a perfect square. This is true if an equation is of the form

$$(x-2)^2 = 9$$

Both sides of the equation are perfect squares. We take the square root of both sides.

$x - 2 = 3$ and $x - 2 = -3$

Solve both equations

Answer $x = 5$ and $x = -1$

Example 4

Solve the following quadratic equations.

a) $(x-1)^2 = 4$

b) $(x+3)^2 = 1$

Solution

a) $(x-1)^2 = 4$

$$\text{Take the square root of both sides.} \qquad x - 1 = 2 \text{ and } x - 1 = -2$$
$$\text{Solve each equation.} \qquad x = 3 \text{ and } x = -1$$

Answer $x = 3$ and $x = -1$

b) $(x+3)^2 = 1$

$$\text{Take the square root of both sides.} \qquad x + 3 = 1 \text{ and } x + 3 = -1$$
$$\text{Solve each equation.} \qquad x = -2 \text{ and } x = -4$$

It might be necessary to factor the right hand side of the equation as a perfect square before applying the method outlined above.

Example 5

Solve the following quadratic equations.

a) $x^2 + 8x + 16 = 25$

b) $4x^2 - 40x + 25 = 9$

Solution

a) $x^2 + 8x + 16 = 25$

Factor the right hand side.	$x^2 + 8x + 16 = (x+4)^2$	so $(x+4)^2 = 25$
Take the square root of both sides.	$x + 4 = 5$ and $x + 4 = -5$	
Solve each equation.	$x = 1$ and $x = -9$	

Answer $x = 1$ and $x = -9$

b) $4x^2 - 20x + 25 = 9$

Factor the right hand side.	$4x^2 - 20x + 25 = (2x-5)^2$	so $(2x-5)^2 = 9$
Take the square root of both sides.	$2x - 5 = 3$ and $2x - 5 = -3$	
Solve each equation.	$2x = 8$ and $2x = 2$	

Answer $x = 4$ and $x = 1$

Approximate Solutions of Quadratic Equations

We use the methods we learned so far in this section to find approximate solutions to quadratic equations. We can get approximate solutions when taking the square root does not give an exact answer.

Example 6

Solve the following quadratic equations.

a) $x^2 - 3 = 0$

b) $2x^2 - 9 = 0$

Solution

a)

Isolate the x^2.	$x^2 = 3$
Take the square root of both sides.	$x = \sqrt{3}$ and $x = -\sqrt{3}$

Answer $x \approx 1.73$ and $x \approx -1.73$

b)

Isolate the x^2.	$2x^2 = 9$ so $x^2 = \frac{9}{2}$
Take the square root of both sides.	$x = \sqrt{\frac{9}{2}}$ and $x = -\sqrt{\frac{9}{2}}$

Answer $x \approx 2.12$ and $x \approx -2.12$

Example 7

Solve the following quadratic equations.

a) $(2x+5)^2 = 10$

b) $x^2 - 2x + 1 = 5$

Solution.

a)

Take the square root of both sides.	$2x + 5 = \sqrt{10}$ and $2x + 5 = -\sqrt{10}$
Solve both equations.	$x = \frac{-5 + \sqrt{10}}{2}$ and $x = \frac{-5 - \sqrt{10}}{2}$

Answer $x \approx -0.92$ and $x \approx -4.08$

b)

Factor the right hand side.	$(x-1)^2 = 5$
Take the square root of both sides.	$x - 1 = \sqrt{5}$ and $x - 1 = -\sqrt{5}$
Solve each equation.	$x = 1 + \sqrt{5}$ and $x = 1 - \sqrt{5}$

Answer $x \approx 3.24$ and $x \approx -1.24$

Solve Real-World Problems Using Quadratic Functions and Square Roots

There are many real-world problems that require the use of quadratic equations in order to arrive at the solution. In this section, we will examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity and the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by

$$g = -9.8 \text{ m/s}^2 \qquad \text{or} \qquad g = -32 \text{ ft/s}^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we will be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the Moon.

The equation that shows the height of an object in free fall is given by

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object t is time, and g is the force of gravity. There are two choices for the equation you can use.

$y = -4.9t^2 + y_0$	If you wish to have the height in meters.
$y = -16t^2 + y_0$	If you wish to have the height in feet.

Example 8 Free fall

How long does it take a ball to fall from a roof to the ground 25 feet *below?*

Solution

$$\begin{aligned}
\text{Since we are given the height in feet, use equation} \quad & y = -16t^2 + y_0 \\
\text{The initial height is } y_0 = 25 \text{ feet, so} \quad & y = -16t^2 + 25 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \quad & 0 = -16t^2 + 25 \\
\text{Solve for t} \quad & 16t^2 = 25 \\
& t^2 = \frac{25}{16} \\
& t = \frac{5}{4} \text{ or } t = -\frac{5}{4}
\end{aligned}$$

We can discard the solution $t = -\frac{5}{4}$ since only positive values for time makes sense in this case,

Answer It takes the ball 1.25 seconds to fall to the ground.

Example 9 Free fall

A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds *later. How high is the cliff in meters?*

Solution

$$\begin{aligned}
\text{Since we want the height in meters, use equation} \quad & y = -4.9t^2 + y_0 \\
\text{The time of flight is } t = 7.2 \text{ seconds} \quad & y = -4.9(7.2)^2 + y_0 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \quad & 0 = -4.9(7.2)^2 + y_0 \\
\text{Simplify} \quad & 0 = -254 + y_0 \text{ so } y_0 = 254
\end{aligned}$$

Answer The cliff is 254 meters high.

Example 10

Victor drops an apple out of a window on the 10^{th} *floor which is* 120 feet *above ground. One second later Juan drops an orange out of a* 6^{th} *floor window which is* 72 feet *above the ground. Which fruit reaches the ground first? What is the time difference between the fruits' arrival to the ground?*

Solution Let's find the time of flight for each piece of fruit.

For the Apple we have the following.

$$\begin{aligned}
\text{Since we have the height in feet, use equation} \quad & y = -16t^2 + y_0 \\
\text{The initial height } y_0 = 120 \text{ feet.} \quad & y = -16t^2 + 120 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \quad & 0 = -16t^2 + 120 \\
\text{Solve for t} \quad & 16t^2 = 120 \\
& t^2 = \frac{120}{16} = 7.5 \\
& t = 2.74 \text{ or } t = -2.74 \text{ seconds.}
\end{aligned}$$

For the orange we have the following.

$$\begin{aligned}
\text{The initial height } y_0 = 72 \text{ feet.} \quad & 0 = -16t^2 + 72 \\
\text{Solve for t.} \quad & 16t^2 = 72 \\
& t^2 = \frac{72}{16} = 4.5 \\
& t = 2.12 \text{ or } t = -2.12 \text{ seconds}
\end{aligned}$$

But, don't forget that the orange was thrown out one second later, so add one second to the time of the orange. It hit the ground 3.12 seconds after Victor dropped the apple.

Answer The apple hits the ground first. It hits the ground 0.38 seconds before the orange. (Hopefully nobody was on the ground at the time of this experiment—don't try this one at home, kids!).

Review Questions

Solve the following quadratic equations.

1. $x^2 - 1 = 0$
2. $x^2 - 100 = 0$
3. $x^2 + 16 = 0$
4. $9x^2 - 1 = 0$
5. $4x^2 - 49 = 0$
6. $64x^2 - 9 = 0$
7. $x^2 - 81 = 0$
8. $25x^2 - 36 = 0$
9. $x^2 + 9 = 0$
10. $x^2 - 16 = 0$
11. $x^2 - 36 = 0$
12. $16x^2 - 49 = 0$
13. $(x - 2)^2 = 1$
14. $(x + 5)^2 = 16$
15. $(2x - 1)^2 - 4 = 0$
16. $(3x + 4)^2 = 9$
17. $(x - 3)^2 + 25 = 0$
18. $x^2 - 6 = 0$
19. $x^2 - 20 = 0$
20. $3x^2 + 14 = 0$
21. $(x - 6)^2 = 5$
22. $(4x + 1)^2 - 8 = 0$
23. $x^2 - 10x + 25 = 9$
24. $x^2 + 18x + 81 = 1$
25. $4x^2 - 12x + 9 = 16$
26. $(x + 10)^2 = 2$
27. $x^2 + 14x + 49 = 3$
28. $2(x + 3)^2 = 8$
29. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
30. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
31. Nisha drops a rock from the roof of a building 50 feet high. Ashaan drops a quarter from the top story window, 40 feet high, exactly half a second after Nisha drops the rock. Which hits the ground first?

Review Answers

1. $x = 1, x = -1$

2. $x = 10, x = -10$
3. No real solution.
4. $x = 1/3, x = -1/3$
5. $x = 7/2, x = -7/2$
6. $x = 3/8, x = -3/8$
7. $x = 9, x = -9$
8. $x = 6/5, x = -6/5$
9. No real solution.
10. $x = 4, x = -4$
11. $x = 6, x = -6$
12. $x = 7/4, x = -7/4$
13. $x = 3, x = 1$
14. $x = -1, x = -9$
15. $x = 3/2, x = -1/2$
16. $x = -1/3, x = -7/3$
17. No real solution.
18. $x \approx 2.45, x \approx -2.45$
19. $x \approx 4.47, x \approx -4.47$
20. No real solution.
21. $x \approx 8.24, x \approx 3.76$
22. $x \approx 0.46, x \approx -0.96$
23. $x = 8, x = 2$
24. $x = -8, x = -10$
25. $x = 7/2, x = -1/2$
26. $x \approx -8.59, x \approx -11.41$
27. $x \approx -5.27, x \approx -8.73$
28. $x = -1, x = -5$
29. $t = 5$ seconds
30. $y_0 = 137.6$ meters
31. .

10.4 Solving Quadratic Equations by Completing the Square

Learning objectives

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

Introduction

You saw in the last section that if you have a quadratic equation of the form

$(x - 2)^2 = 5$

We can easily solve it by taking the square root of each side.

$x - 2 = \sqrt{5}$ and $x - 2 = -\sqrt{5}$

Then simplify and solve.

$x = 2 + \sqrt{5} \approx 4.24$ and $x = 2 - \sqrt{5} \approx -0.24$

Unfortunately, quadratic equations are not usually written in this nice form. In this section, you will learn the method of **completing the squares** in which you take any quadratic equation and rewrite it in a form so that you can take the square root of both sides.

Complete the Square of a Quadratic Expression

The purpose of the method of completing the squares is to rewrite a quadratic expression so that it contains a perfect square trinomial that can be factored as the square of a binomial. Remember that the square of a binomial expands. Here is an example of this.

$$(x + a)^2 = x^2 + 2ax + a^2$$
$$(x - a)^2 = x^2 - 2ax + a^2$$

In order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example 1

Complete the square for the quadratic expression $x^2 + 4x$.

Solution To complete the square, we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always two times the product of the square roots of the other two terms, we rewrite our expression as

$$x^2 + 2(2)(x)$$

We see that the constant we are seeking must be 2^2.

$$x^2 + 2(2)(x) + 2^2$$

Answer By adding 4, this can be factored as: $(x + 2)^2$

BUT, we just changed the value of this expression $x^2 + 4x \neq (x + 2)^2$. Later we will show how to account for this problem. You need to add and subtract the constant term.

Also, this was a relatively easy example because a, the coefficient of the x^2 term was 1. If $a \neq 1$, we must factor a from the whole expression before completing the square.

Example 2

Complete the square for the quadratic expression $4x^2 + 32x$

Solution

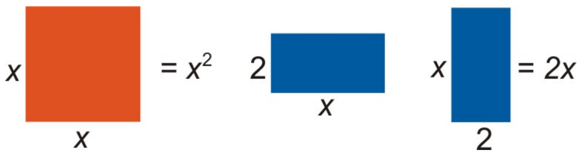
Factor the coefficient of the x^2 term.	$4(x^2 + 8x)$
Now complete the square of the expression in parentheses then rewrite the expression.	$4(x^2 + 2(4)(x))$
We complete the square by adding the constant 4^2.	$4(x^2 + 2(4)(x) + 4^2$
Factor the perfect square trinomial inside the parenthesis.	$4(x + 4)^2$

Our answer is $4(x + 4)^2$.

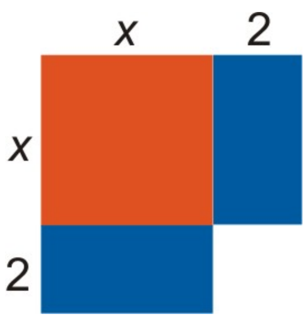
The expression **"completing the square"** comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in Example 1.

$$x^2 + 4x$$

We can think of this expression as the sum of three areas. The first term represents the area of a square of side x. The second expression represents the areas of two rectangles with a length of 2 and a width of x:

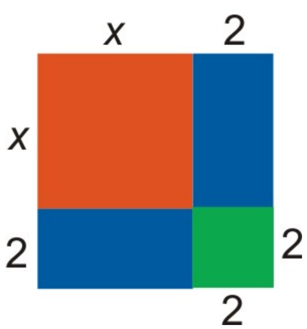


We can combine these shapes as follows



We obtain a square that is not quite complete.

In order to complete the square, we need a square of side 2.



We obtain a square of side $x + 2$.

The area of this square is: $(x + 2)^2$.

You can see that completing the square has a geometric interpretation.

Finally, here is the algebraic procedure for completing the square.

$$\begin{aligned}
x^2 + bx + c &= 0 \\
x^2 + bx &= -c \\
x^2 + bx + \left(\frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2 \\
\left(x + \frac{b}{2}\right)^2 &= -c + \left(\frac{b}{2}\right)^2
\end{aligned}$$

Solve Quadratic Equations by Completing the Square

Let's demonstrate the method of **completing the square** with an example.

Example 3

Solve the following quadratic equation $x^2 + 12x = 3$.

Solution

The method of completing the square is as follows.

1. Rewrite as $x^2 + 2(6)x = 3$

2. In order to have a perfect square trinomial on the right-hand-side we need to add the constant 6^2. Add this constant to both sides of the equation.

$$x^2 + 2(6)(x) + 6^2 = 3 + 6^2$$

3. Factor the perfect square trinomial and simplify the right hand side of the equation.

$$(x+6)^2 = 39$$

4. Take the square root of both sides.

$x + 6 = \sqrt{39}$	and	$x + 6 = -\sqrt{39}$
$x = -6 + \sqrt{39} \approx 0.24$	and	$x = -6 - \sqrt{39} \approx -12.24$

Answer $x = 0.24$ and $x = -12.24$

If the coefficient of the x^2 term is not one, we must divide that number from the whole expression before completing the square.

Example 4

Solve the following quadratic equation $3x^2 - 10x = -1$.

Solution:

1. Divide all terms by the coefficient of the x^2 term.

$$x^2 - \frac{10}{3}x = -\frac{1}{3}$$

2. Rewrite as

$$x^2 - 2\left(\frac{5}{3}\right)(x) = -\frac{1}{3}$$

3. In order to have a perfect square trinomial on the right hand side we need to add the constant $\left(\frac{5}{3}\right)^2$. Add this constant to both sides of the equation.

$$x^2 - 2\left(\frac{5}{3}\right)(x) + \left(\frac{5}{3}\right)^2 = -\frac{1}{3} + \left(\frac{5}{3}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x - \frac{5}{3}\right)^2 = \frac{1}{3} + \frac{25}{9}$$
$$\left(x - \frac{5}{3}\right)^2 = \frac{22}{9}$$

5. Take the square root of both sides.

$$x - \frac{5}{3} = \sqrt{\frac{22}{9}} \quad \text{and} \quad x - \frac{5}{3} = -\sqrt{\frac{22}{9}}$$
$$x = \frac{5}{3} + \sqrt{\frac{22}{9}} \approx 3.23 \quad \text{and} \quad x = \frac{5}{3} - \sqrt{\frac{22}{9}} \approx 0.1$$

Answer $x = 3.23$ and $x = 0.1$

Solve Quadratic Equations in Standard Form

An equation in standard form is written as $ax^2 + bx + c = 0$. We solve an equation in this form by the method of completing the square. First we move the constant term to the right hand side of the equation.

Example 5

Solve the following quadratic equation $x^2 15x + 12 = 0$.

Solution

The method of completing the square is as follows:

1. Move the constant to the other side of the equation.

$$x^2 + 15x = -12$$

2. Rewrite as

$$x^2 + 2\left(\frac{15}{2}\right)(x) = -12$$

3. Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation

$$x^2 + 2\left(\frac{15}{2}\right)(x) + \left(\frac{15}{2}\right)^2 = -12 + \left(\frac{15}{2}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x + \frac{15}{2}\right)^2 = -12 + \frac{225}{4}$$
$$\left(x + \frac{15}{2}\right)^2 = \frac{177}{4}$$

5. Take the square root of both sides.

$$x + \frac{15}{2} = \sqrt{\frac{177}{4}} \quad \text{and} \quad x + \frac{15}{2} = -\sqrt{\frac{177}{4}}$$
$$x + -\frac{15}{2} + \sqrt{\frac{177}{4}} \approx -0.85 \quad \text{and} \quad x + -\frac{15}{2} + \sqrt{\frac{177}{4}} \approx -14.15$$

Answer $x = -0.85$ and $x = -14.15$

Graph Quadratic Functions in Vertex Form

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form.

The vertex form of a quadratic function is $y - k = a(x - h)^2$.

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at point (h, k).

It is also simple to find the $x-$intercepts from the vertex from by setting $y = 0$ and taking the square root of both sides of the resulting equation.

The $y-$intercept can be found by setting $x = 0$ and simplifying.

Example 6

Find the vertex, the $x-$intercepts and the $y-$intercept of the following parabolas.

(a) $y - 2 = (x - 1)^2$

(b) $y + 8 = 2(x - 3)^2$

Solution

a) $y - 2 = (x - 1)^2$

Vertex is $(1, 2)$

To find $x-$intercepts,

$$\begin{array}{rlll} \text{Set } y = 0 & -2 = (x-1)^2 & & \\ \text{Take the square root of both sides} & \sqrt{-2} = x - 1 & \text{and} & -\sqrt{-2} = x - 1 \end{array}$$

The solutions are not real (because you cannot take the square root of a negative number), so there are **no** $x-$intercepts.

To find $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0 & y - 2 = (-1)^2 \\ \text{Simplify} & y - 2 = 1 \Rightarrow y = 3 \end{array}$$

b) $y + 8 = 2(x - 3)^2$

$$\begin{array}{ll} \text{Rewrite} & y - (-8) = 2(x-3)^2 \\ \text{Vertex is} & (3, -8) \end{array}$$

To find $x-$intercepts,

$$\begin{array}{rlll} \text{Set } y = 0: & 8 = 2(x-3)^2 & & \\ \text{Divide both sides by 2.} & 4 = (x-3)^2 & & \\ \text{Take the square root of both sides}: & 2 = x - 3 & \text{and} & -2 = x - 3 \\ \text{Simplify}: & x = 5 & \text{and} & x = 1 \end{array}$$

To find the $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0. & y + 8 = 2(-3)^2 \\ \text{Simplify}: & y + 8 = 18 \Rightarrow y = 10 \end{array}$$

To graph a parabola, we only need to know the following information.

- The coordinates of the vertex.
- The $x-$intercepts.
- The $y-$intercept.
- Whether the parabola turns up or down. Remember that if $a > 0$, the parabola turns up and if $a < 0$ then the parabola turns down.

Example 7

Graph the parabola given by the function $y + 1 = (x + 3)^2$.

Solution

Rewrite.	$y - (-1) = (x - (-3))^2$
Vertex is	$(-3, -1)$

To find the $x-$intercepts

Set $y = 0$	$1 = (x + 3)^2$		
Take the square root of both sides	$1 = x + 3$	and	$-1 = x + 3$
Simplify	$x = -2$	and	$x = -4$

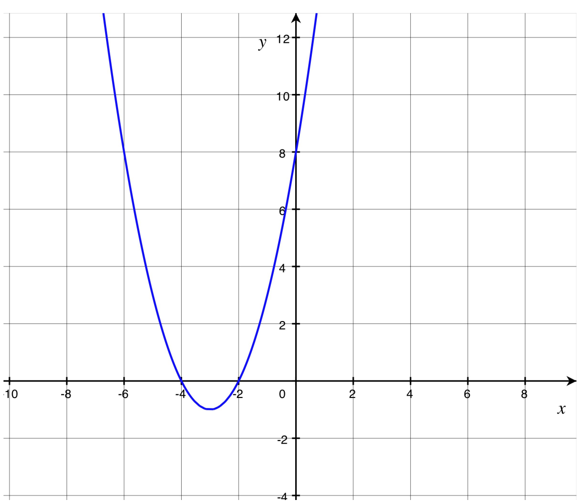
$x-$intercepts: $(-2, 0)$ and $(-4, 0)$

To find the $y-$intercept

Set $x = 0$	$y + 1(3)^2$	
Simplify	$y = 8$	$y -$ intercept : $(0, 8)$

Since $a > 0$, the parabola **turns up**.

Graph all the points and connect them with a smooth curve.



Example 8

Graph the parabola given by the function $y = -\frac{1}{2}(x - 2)^2$

Solution:

Re-write $y-(0)=-\frac{1}{2}(x-2)^2$

Vertex is $(2,0)$

To find the $x-$intercepts,

Set y = 0. $0=-\frac{1}{2}(x-2)^2$

Multiply both sides by -2. $0=(x-2)^2$

Take the square root of both sides. $0=x-2$

Simplify. $x=2$

$x-$intercept $(2,0)$

Note: there is only one $x-$intercept, indicating that the vertex is located at this point $(2,0)$.

To find the $y-$intercept

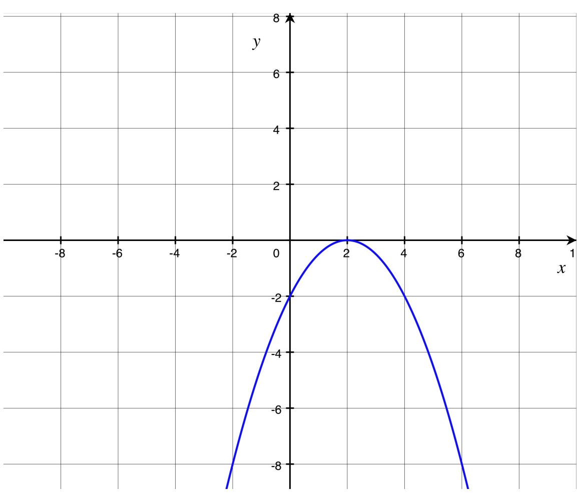
Set x = 0 $y=-\frac{1}{2}(-2)^2$

Simplify $y=-\frac{1}{2}(4)\Rightarrow y=-2$

$y-$intercept $(0,-2)$

Since $a<0$, the parabola **turns down.**

Graph all the points and connect them with a smooth curve.



Solve Real-World Problems Using Quadratic Functions by Completing the Square

Projectile motion with vertical velocity

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height with respect to time is given by

$$y = \frac{1}{2}\,\mathrm{gt}^2 + y_0$$

The term y_0 represents the initial height of the object and the coefficient of gravity on earth is given by $g = -9.8\ \mathrm{m/s}^2$ or $g = -32\ \mathrm{ft/s}^2$.

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation v_{0y}. Its value is positive if the object is thrown up in the air, and, it is negative if the object is thrown down. The equation for the height of the object in this case is given by the equation

$$y = \frac{1}{2}\,\mathrm{gt}^2 + v_{0y}t + y_0$$

There are two choices for the equation to use in these problems.

$y = -4.9t^2 + v_{0y}t + y_0$	If you wish to have the height in meters.
$y = -16t^2 + v_{0y}t + y_0$	If you wish to have the height in feet.

Example 9

An arrow is shot straight up from a height of 2 meters *with a velocity of* 50 m/s.

a) How high will an arrow be four seconds after being shot? After eight seconds?

b) At what time will the arrow hit the ground again?

c) What is the maximum height that the arrow will reach and at what time will that happen?

Solution

Since we are given the velocity in meters per second, use the equation $y = -4.9t^2 + v_{oy}t + y_0$

We know $v_{oy} = 50$ m/s and $y_0 = 2$ meters so, $y = -4.9t^2 + 50t + 2$

a) To find how high the arrow will be 4 seconds after being shot we substitute 4 for t

$$\begin{aligned} y &= -4.9(4)^2 + 50(4) + 2 \\ &= -4.9(16) + 200 + 2 = 123.6 \text{ meters} \end{aligned}$$

—we substitute— t = 8

$$\begin{aligned} y &= -4.9(8)^2 + 50(8) + 2 \\ &= -4.9(64) + 400 + 5 = 88.4 \text{ meters} \end{aligned}$$

b) The height of the arrow on the ground is $y = 0$, so $0 = -4.9t^2 + 50t + 2$

Solve for t by completing the square	$-4.9t^2 + 50t = -2$
Factor the	$-4.9 - 4.9(t^2 - 10.2t) = -2$
Divide both sides by	$-4.9t^2 - 10.2t = 0.41$
Add 5.1^2 to both sides	$t^2 - 2(5.1)t + (5.1)^2 = 0.41 + (5.1)^2$
Factor	$(t - 5.1)^2 = 26.43$
Solve	$t - 5.1 \approx 5.14$ and $t - 5.1 \approx -5.14$
	$t \approx 10.2$ sec and $t \approx -0.04$ sec

c) If we graph the height of the arrow with respect to time, we would get an upside down parabola ($a < 0$). The maximum height and the time when this occurs is really the vertex of this parabola (t, h).

We rewrite the equation in vertex form.

$$y = -4.9t^2 + 50t + 2$$
$$y - 2 = -4.9t^2 + 50t$$
$$y - 2 = -4.9(t^2 - 10.2t)$$

Complete the square inside the parenthesis.

$$y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$$
$$y - 129.45 = -4.9(t - 5.1)^2$$

The vertex is at $(5.1, 129.45)$. In other words, **when $t = 5.1$ seconds, the height is $y = 129$ meters.**

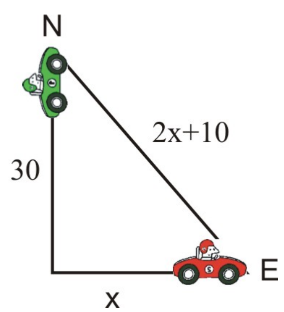
Another type of application problem that can be solved using quadratic equations is one where two objects are moving away in directions perpendicular from each other. Here is an example of this type of problem.

Example 10

Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, *the distance between the cars was* 10 miles *more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.*

Solution

Let $x =$ the distance traveled by the car heading east.



$2x + 10 =$ the distance between the two cars

Let's make a sketch

We can use the Pythagorean Theorem ($a^2 + b^2 = c^2$) to find an equation for x:

$$x^2 + 30^2 = (2x + 10)^2$$

Expand parentheses and simplify.

$$x^2 + 900 = 4x^2 + 40x + 100$$
$$800 = 3x^2 + 40x$$

Solve by completing the square.

$$\frac{800}{3} = x^2 + \frac{40}{3}x$$

$$\frac{800}{3} + \left(\frac{20}{3}\right)^4 = x^2 + 2\left(\frac{20}{3}\right)x + \left(\frac{20}{3}\right)^3$$

$$\frac{2800}{9} = \left(x + \frac{20}{3}\right)^2$$

$$x + \frac{20}{3} \approx 17.6 \text{ and } x + \frac{20}{3} \approx -17.6$$

$$x \approx 11 \text{ and } x \approx -24.3$$

Since only positive distances make sense here, the distance between the two cars is $2(11) + 10 = 32$ miles.

Answer The distance between the two cars is 32 miles.

Review Questions

Complete the square for each expression.

1. $x^2 + 5x$
2. $x^2 - 2x$
3. $x^2 + 3x$
4. $x^2 - 4x$
5. $3x^2 + 18x$
6. $2x^2 - 22x$
7. $8x^2 - 10x$
8. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9. $x^2 - 4x = 5$
10. $x^2 - 5x = 10$
11. $x^2 + 10x + 15 = 0$
12. $x^2 + 15x + 20 = 0$
13. $2x^2 - 18x = 0$
14. $4x^2 + 5x = -1$
15. $10x^2 - 30x - 8 = 0$
16. $5x^2 + 15x - 40 = 0$

Rewrite each quadratic function in vertex form.

17. $y = x^2 - 6x$
18. $y + 1 = -2x^2 - x$
19. $y = 9x^2 + 3x - 10$
20. $y = 32x^2 + 60x + 10$ For each parabola, find
    (a) The vertex
    (b) $x-$intercepts
    (c) $y-$intercept

(d) If it turns up or down.
(e) The graph the parabola.

21. $y - 4 = x^2 + 8x$
22. $y = -4x^2 + 20x - 24$
23. $y = 3x^2 + 15x$
24. $y + 6 = -x^2 + x$
25. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?
26. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

Review Answers

1. $x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$
2. $x^2 - 2x + 1 = (x - 1)^2$
3. $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$
4. $x^2 - 4x + 4 = (x - 2)^2$
5. $3(x^2 + 6x + 9) = 3(x + 3)^2$
6. $2\left(x^2 - 11x + \frac{121}{4}\right) = 2\left(x - \frac{11}{2}\right)^2$
7. $8\left(x^2 - \frac{5}{4}x + \frac{25}{64}\right) = 8\left(x - \frac{5}{8}\right)^2$
8. $5\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) = 5\left(x + \frac{6}{5}\right)^2$
9. $5, -1$
10. $6.53, -1.53$
11. $-8.16, -1.84$
12. $-13.52, -1.48$
13. $9.16, -.16$
14. $-1, -.25$
15. $-3.25, -.25$
16. $-4.7, 1.7$
17. $y + 9 = (x - 3)^2$
18. $y + \frac{7}{8} = -2\left(x + \frac{1}{4}\right)^2$
19. $y + 10.25 = 9\left(x + \frac{1}{6}\right)^2$
20. $y - \frac{305}{8} = -32\left(x - \frac{15}{16}\right)^2$
21. $y + 12 = (x + 4)^2$; vertex $(-4, -12)$; $x-$intercepts $(-7.46, 0), (-.54, 0)$; $y-$intercept $(0, 4)$; turns up.
22. $y - 1 = -4\left(x - \frac{5}{2}\right)^2$; vertex $(2.5, 1)$; $x-$intercepts $(2, 0), (3, 0)$ $y-$intercept $(0, -24)$; turns down.
23. $y + 18.75 = 3(x + 2.5)^2$; vertex $(-2.5, -18.75)$; $x-$intercepts $(0, 0), (-5, 0)$; $y-$intercept $(0, 0)$; turns up.
24. $y + \frac{23}{4} = -\left(x - \frac{1}{2}\right)^2$; vertex $(0.5, -5.75)$; $x-$intercepts none; $y-$intercept $(0, -6)$; turns down.
25. 0.85 seconds
26. Amanda 2.1 miles, Dolvin 5.1 miles

10.5 Solving Quadratic Equations by the Quadratic Formula

Learning objectives

- Solve quadratic equations using the quadratic formula.
- Identify and choose methods for solving quadratic equations.
- Solve real-world problems using functions by completing the square.

Introduction

In this section, you will solve quadratic equations using the **Quadratic Formula.** Most of you are already familiar with this formula from previous mathematics courses. It is probably the most used method for solving quadratic equations. For a quadratic equation in standard form

$$ax^2 + bx + c = 0$$

The solutions are found using the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will start by explaining where this formula comes from and then show how it is applied. This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation.	$ax^2 + bx + c = 0$
Subtract the constant term from both sides.	$ax^2 + bx = -c$
Divide by the coefficient of the x^2 term.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
Rewrite.	$x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$
Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides.	$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Factor the perfect square trinomial.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$
Simplify.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Take the square root of both sides.	$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ and $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$
Simplify.	$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{2a}}$ and $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{2a}}$
	$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

This can be written more compactly as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious. The quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

Applying the quadratic formula basically amounts to plugging the values of a, b and c into the quadratic formula.

Example 1

Solve the following quadratic equation using the quadratic formula.

a) $2x^2 + 3x + 1 = 0$

b) $x^2 - 6x + 5 = 0$

c) $-4x^2 + x + 1 = 0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 2, b = 3, c = 1.$ $x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(1)}}{2(2)}$

Simplify. $x = \frac{-3 \pm \sqrt{9 - 8}}{4} = \frac{-3 \pm \sqrt{1}}{4}$

Separate the two options. $x = \frac{-3 + 1}{4}$ and $x = \frac{-3 - 1}{4}$

Solve. $x = \frac{-2}{4} = -\frac{1}{2}$ and $x = \frac{-4}{4} = -1$

Answer $x = -\frac{1}{2}$ and $x = -1$

b)

Quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 1, b = -6, c = 5.$ $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

Simplify. $x = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm \sqrt{16}}{2}$

Separate the two options. $x = \frac{6 + 4}{2}$ and $x = \frac{6 - 4}{2}$

Solve $x = \frac{10}{2} = 5$ and $x = \frac{2}{2} = 1$

Answer $x = 5$ and $x = 1$

c)

Quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = -4, b = 1, c = 1.$ $x = \frac{-1 \pm \sqrt{(1)^2 - 4(-4)(1)}}{2(-4)}$

Simplify. $x = \frac{-1 \pm \sqrt{1 + 16}}{-8} = \frac{-1 \pm \sqrt{17}}{-8}$

Separate the two options. $x = \frac{-1 + \sqrt{17}}{-8}$ and $x = \frac{-1 - \sqrt{17}}{-8}$

Solve. $x \approx -.39$ and $x \approx .64$

Answer $x \approx -.39$ and $x \approx .64$

Often when we plug the values of the coefficients into the quadratic formula, we obtain a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced mathematics classes, you will learn how to work with ”complex” (or ”imaginary”) solutions to quadratic equations.

Example 2

Solve the following quadratic equation using the quadratic formula $x^2 + 2x + 7 = 0$

Solution:

a)

Quadratic formula. $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 1, b = 2, c = 7$. $x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(7)}}{2(1)}$

Simplify. $x = \frac{-2 \pm \sqrt{4 - 28}}{2} = \frac{-2 \pm \sqrt{-24}}{2}$

Answer There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, we must rewrite the equation before we apply the quadratic formula.

Example 3

Solve the following quadratic equation using the quadratic formula.

a) $x^2 - 6x = 10$

b) $8x^2 = 5x + 6$

Solution:

a)

Rewrite the equation in standard form. $x^2 - 6x - 10 = 0$

Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 1, b = -6, c = -10$. $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-10)}}{2(1)}$

Simplify. $x = \frac{6 \pm \sqrt{36 + 40}}{2} = \frac{6 \pm \sqrt{76}}{2}$

Separate the two options. $x = \frac{6 + \sqrt{76}}{2}$ and $x = \frac{6 - \sqrt{76}}{2}$

Solve. $x \approx 7.36$ and $x \approx -1.36$

Answer $x \approx 7.36$ and $x \approx -1.36$

b)

Rewrite the equation in standard form. $8x^2 + 5x + 6 = 0$

Quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = 8, b = 5, c = 6$. $x = \frac{-5 \pm \sqrt{(5)^2 - 4(8)(6)}}{2(8)}$

Simplify. $x = \frac{-5 \pm \sqrt{25 - 192}}{16} = \frac{-5 \pm \sqrt{-167}}{16}$

Answer no real solutions

Multimedia Link For more examples of solving quadratic equations using the quadratic formula, see Khan Academy Equation Part 2 (9:14) . This video is not necessarily different from the examples above, but it does help reinforce the procedure of using the quadratic formula to solve equations.

Finding the Vertex of a Parabola with the Quadratic Formula

Sometimes you get more information from a formula beyond what you were originally seeking. In this case, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

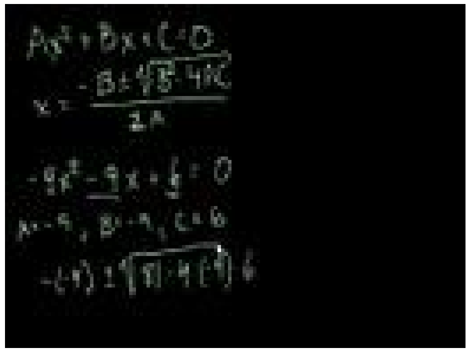
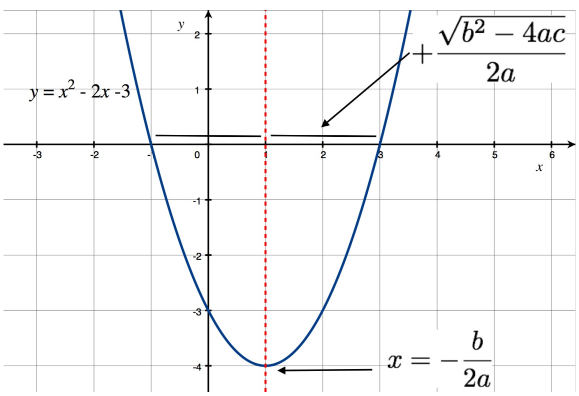


Figure 10.1: 2 more examples of solving equations using the quadratic equation (Watch on Youtube)

First, recall that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2 + bx + c = 0$. Those roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$



We can rewrite the fraction in the quadratic formula as

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $-\frac{b}{2a}$ because they move $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x = -\frac{b}{2a}$. The image to the right illustrates this for the equation $x^2 - 2x - 3 = 0$. The roots, -1 and 3 are both 2 units from the vertical line $x = 1$.

Identify and Choose Methods for Solving Quadratic Equations.

In mathematics, you will need to solve quadratic equations that describe application problems or that are part of more complicated problems. You learned four ways of solving a quadratic equation.

- Factoring.
- Taking the square root.
- Completing the square.
- Quadratic formula.

Usually you will not be told which method to use. You will have to make that decision yourself. However, here are some guidelines to which methods are better in different situations.

Factoring is always best if the quadratic expression is easily factorable. It is always worthwhile to check if you can factor because this is the fastest method. Many expressions are not factorable so this method is not used very often in practice.

Taking the square root is best used when there is no x term in the equation.

Completing the square can be used to solve any quadratic equation. This is usually not any better than using the quadratic formula (in terms of difficult computations), however it is a very important method for re-writing a quadratic function in vertex form. It is also be used to re-write the equations of circles, ellipses and hyperbolas in standard form (something you will do in algebra II, trigonometry, physics, calculus, and beyond...).

Quadratic formula is the method that is used most often for solving a quadratic equation. When solving directly by taking square root and factoring does not work, this is the method that most people prefer to use.

If you are using factoring or the quadratic formula make sure that the equation is in standard form.

Example 4

Solve each quadratic equation

a) $x^2 - 4x - 5 = 0$

b) $x^2 = 8$

c) $-4x^2 + x = 2$

d) $25x^2 - 9 = 0$

e) $3x^2 = 8x$

Solution

a) This expression if easily factorable so we can factor and apply the zero-product property:

Factor.	$(x-5)(x+1) = 0$
Apply zero-product property.	$x - 5 = 0$ and $x + 1 = 0$
Solve.	$x = 5$ and $x = -1$

Answer $x = 5$ and $x = -1$

b) Since the expression is missing the x term we can take the square root:

Take the square root of both sides.	$x = \sqrt{8}$ and $x = -\sqrt{8}$

Answer $x = 2.83$ and $x = -2.83$

c) Rewrite the equation in standard form.

It is not apparent right away if the expression is factorable, so we will use the quadratic formula.

Quadratic formula	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Plug in the values $a = -4, b = 1, c = -2$.	$x = \dfrac{-1 \pm \sqrt{1^2 - 4(-4)(-2)}}{2(-4)}$
Simplify.	$x = \dfrac{-1 \pm \sqrt{1 - 32}}{-8} = \dfrac{-1 \pm \sqrt{-31}}{-8}$

Answer no real solution

d) This problem can be solved easily either with factoring or taking the square root. Let's take the square root in this case.

Add 9 to both sides of the equation.	$25x^2 = 9$
Divide both sides by 25.	$x^2 = \frac{9}{25}$
Take the square root of both sides.	$x = \sqrt{\frac{9}{25}}$ and $x = -\sqrt{\frac{9}{25}}$
Simplify.	$x = \frac{3}{5}$ and $x = -\frac{3}{5}$

Answer $x = \frac{3}{5}$ and $x = -\frac{3}{5}$

e)

Rewrite the equation in standard form	$3x^2 - 8x = 0$
Factor out common x term.	$x(3x - 8) = 0$
Set both terms to zero.	$x = 0$ and $3x = 8$
Solve.	$x = 0$ and $x = \frac{8}{3} = 2.67$

Answer $x = 0$ and $x = 2.67$

Solve Real-World Problems Using Quadratic Functions by any Method

Here are some application problems that arise from number relationships and geometry applications.

Example 5

The product of two positive consecutive integers is 156. *Find the integers.*

Solution

For two consecutive integers, one integer is one more than the other one.

Define

Let $x =$ the smaller integer

$x + 1 =$ the next integer

Translate

The product of the two numbers is 156. We can write the equation:

$$x(x + 1) = 156$$

Solve

$$x^2 + x = 156$$
$$x^2 + x - 156 = 0$$

Apply the quadratic formula with $a = 1, b = 1, c = -156$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-156)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{625}}{2} = \frac{-1 \pm 25}{2}$$
$$x = \frac{-1 + 25}{2} \text{ and } x = \frac{-1 - 25}{2}$$
$$x = \frac{24}{2} = 12 \text{ and } x = \frac{-26}{2} = -13$$

Since we are looking for positive integers take, $x = 12$

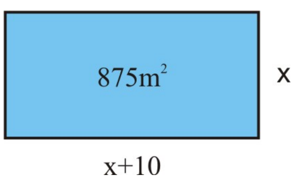
Answer 12 and 13

Check $12 \times 13 = 156$. The answer checks out.

Example 6

The length of a rectangular pool is 10 meters *more than its width. The area of the pool is* 875 square meters. *Find the dimensions of the pool.*

Solution:



Draw a sketch

Define

Let $x =$ the width of the pool

$x + 10 =$ the length of the pool

Translate

The area of a rectangle is A = length × width, so

$$x(x + 10) = 875$$

Solve

$$x^2 + 10x = 875$$
$$x^2 + 10x - 875 = 0$$

Apply the quadratic formula with $a = 1, b = 10$ and $c = -875$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-875)}}{2(1)}$$
$$x = \frac{-10 \pm \sqrt{100 + 3500}}{2}$$
$$x = \frac{-10 \pm \sqrt{3600}}{2} = \frac{-10 \pm 60}{2}$$
$$x = \frac{-10 + 60}{2} \text{ and } x = \frac{-10 - 60}{2}$$
$$x = \frac{50}{2} = 25 \text{ and } x = \frac{-70}{2} = -35$$

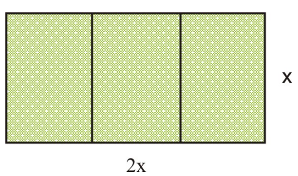
Since the dimensions of the pools should be positive, then $x = 25$ meters.

Answer The pool is 25 meters $\times$ 35 meters.

Check $25 \times 35 = 875 \text{ m}^2$. **The answer checks out.**

Example 7

Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is 200 ft^2. *How much fencing does Suzie need?*



Solution

Draw a Sketch

Define

Let $x =$ the width of the plot

$2x =$ the length of the plot

Translate

Area of a rectangle is A = length × width, so

$$x(2x) = 200$$

Solve

$$2x^2 = 200$$

Solve by taking the square root.

$$x^2 = 100$$
$$x = \sqrt{100} \text{ and } x = -\sqrt{100}$$
$$x = 10 \text{ and } x = -10$$

We take $x = 10$ since only positive dimensions make sense.

The plot of land is 10 feet $\times$ 20 feet.

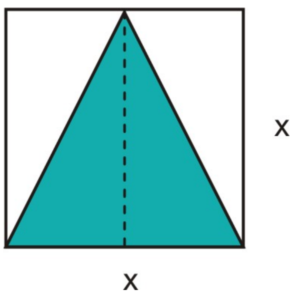
To fence the garden the way Suzie wants, we need 2 lengths and 4 widths $= 2(20) + 4(10) = 80$ feet of fence.

Answer: The fence is 80 feet.

Check $10 \times 20 = 200 \text{ ft}^2$ and $2(20) + 4(10) = 80$ feet. **The answer checks out.**

Example 8

An isosceles triangle is enclosed in a square so that its base coincides with one of the sides of the square and the tip of the triangle touches the opposite side of the square. If the area of the triangle is 20 in^2 *what is the area of the square?*



Solution:

Draw a sketch.

Define

Let $x =$ base of the triangle

$x =$ height of the triangle

Translate

Area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$, so

$$\frac{1}{2} \cdot x \cdot x = 20$$

Solve

$$\frac{1}{2}x^2 = 20$$

Solve by taking the square root.

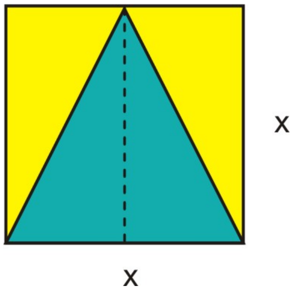
$$x^2 = 40$$
$$x = \sqrt{40} \text{ and } x = -\sqrt{40}$$
$$x \approx 6.32 \text{ and } x \approx -6.32$$

The side of the square is 6.32 inches.

The area of the square is $(6.32)^2 = 40 \text{ in}^2$, twice as big as the area of the triangle.

Answer: Area of the triangle is 40 in^2

Check: It makes sense that the area of the square will be twice that of the triangle. If you look at the figure you can see that you can fit two triangles inside the square.



The answer checks out.

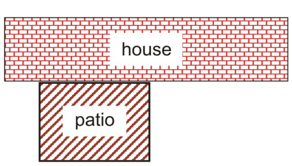
Review Questions

Solve the following quadratic equations using the quadratic formula.

1. $x^2 + 4x - 21 = 0$
2. $x^2 - 6x = 12$
3. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
4. $2x^2 + x - 3 = 0$
5. $-x^2 - 7x + 12 = 0$
6. $-3x^2 + 5x = 0$
7. $4x^2 = 0$
8. $x^2 + 2x + 6 = 0$

Solve the following quadratic equations using the method of your choice.

9. $x^2 - x = 6$
10. $x^2 - 12 = 0$
11. $-2x^2 + 5x - 3 = 0$
12. $x^2 + 7x - 18 = 0$
13. $3x^2 + 6x = -10$
14. $-4x^2 + 4000x = 0$
15. $-3x^2 + 12x + 1 = 0$
16. $x^2 + 6x + 9 = 0$
17. $81x^2 + 1 = 0$
18. $-4x^2 + 4x = 9$
19. $36x^2 - 21 = 0$
20. $x^2 - 2x - 3 = 0$
21. The product of two consecutive integers is 72. Find the two numbers.
22. The product of two consecutive odd integers is 1 less than 3 times their sum. Find the integers.
23. The length of a rectangle exceeds its width by 3 inches. The area of the rectangle is 70 square inches, find its dimensions.
24. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is 4 feet $\times 8$ feet and the cut off part is 1/3 of the total area of the plywood sheet. What is the length of the side of the square?



25. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 192 ft^2 and the length is 4 feet longer than the width. Find how much fencing Mike will need.

Review Answers

1. $x = -7, x = 3$
2. $x = -1.58, x = 7.58$
3. $x = -.28, x = .45$
4. $x = -1.5, x = 1$
5. $x = -8.42, x = 1.42$
6. $x = 1, x = 2/3$
7. $x = 0, x = 1/4$
8. No real solution
9. $x = -2, x = 3$
10. $x = -3.46, x = 3.46$
11. $x = 1, x = 1.5$
12. $x = -9, x = 2$
13. No real solution
14. $x = 0, x = 1000$
15. $x = -.08, x = 4.08$
16. $x = -3$
17. No real solution
18. No real solution
19. $x = -.76, x = .76$
20. $x = -1, x = 3$
21. 8 and 9
22. 5 and 7
23. 7 in and 10 in
24. side = 3.27 ft
25. 40 feet of fencing.

10.6 The Discriminant

Learning Objectives

- Find the discriminant of a quadratic equation.
- Interpret the discriminant of a quadratic equation.
- Solve real-world problems using quadratic functions and interpreting the discriminant.

Introduction

The quadratic equation is $ax^2 + bx + c = 0$.

It can be solved using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The expression inside the square root is called the **discriminant**, $D = b^2 - 4\text{ac}$. The discriminant can be used to analyze the types of solutions of quadratic equations without actually solving the equation. Here are some guidelines.

- If $b^2 - 4ac > 0$, we obtain two separate real solutions.
- If $b^2 - 4ac < 0$, we obtain non-real solutions.
- If $b^2 - 4ac = 0$, we obtain one real solution, a **double root.**

Find the Discriminant of a Quadratic Equation

To find the discriminant of a quadratic equation, we calculate $D = b^2 - 4ac$.

Example 1

Find the discriminant of each quadratic equation. Then tell how many solutions there will be to the quadratic equation without solving.

a) $x^2 - 5x + 3 = 0$

b) $4x^2 - 4x + 1 = 0$

c) $-2x^2 + x = 4$

Solution:

a) Substitute $a = 1, b = -5$ and $c = 3$ into the discriminant formula $D = (-5)^2 - 4(1)(3) = 13$.

There are two real solutions because $D > 0$.

b) Substitute $a = 4, b = -4$ and $c = 1$ into the discriminant formula $D = (-4)^2 - 4(4)(1) = 0$.

There is one real solution because $D = 0$.

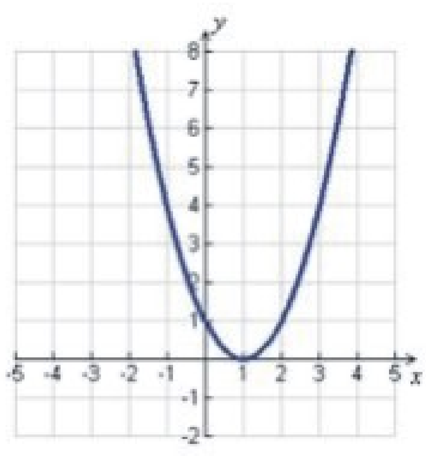
c) Rewrite the equation in standard form $-2x^2 + x - 4 = 0$.

Substitute $a = -2, b = 1$ and $c = -4$ into the discriminant formula: $D = (1)^2 - 4(-2)(-4) = -31$.

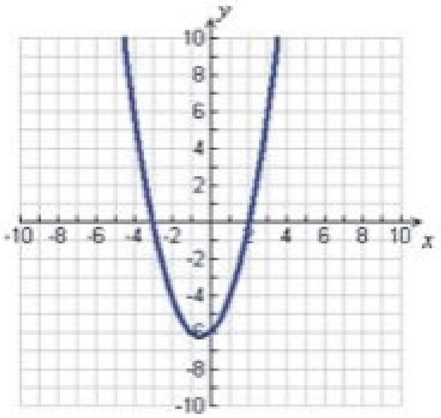
There are no real solutions because $D < 0$.

Interpret the Discriminant of a Quadratic Equation

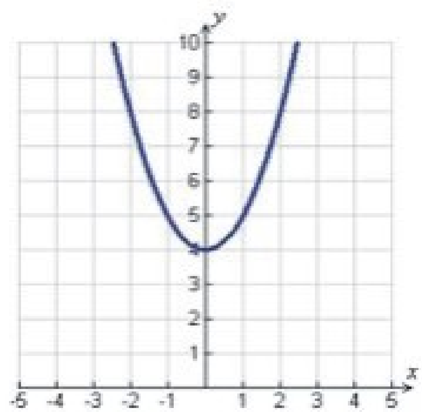
The sign of the discriminant tells us the nature of the solutions (or roots) of a quadratic equation. We can obtain two distinct real solutions if $D > 0$, no real solutions if $D < 0$ or one solution (called a "double root") if $D = 0$. Recall that the number of solutions of a quadratic equation tell us how many times a parabola crosses the $x-$axis.



D > 0, the parabola crosses the x-axis in two places.



D = 0, the parabola only touches the x-axis in one place.



D < 0, the parabola does not cross the x-axis.

Example 2

Determine the nature of solutions of each quadratic equation.

a) $4x^2 - 1 = 0$

b) $10x^2 - 3x = -4$

c) $x^2 - 10x + 25 = 0$

Solution

Use the value of the discriminant to determine the nature of the solutions to the quadratic equation.

a) Substitute $a = 4, b = 0$ and $c = -1$ into the discriminant formula $D = (0)^2 - 4(4)(-1) = 16$.

The discriminant is positive, so the equation has two distinct real solutions.

The solutions to the equation are: $\frac{0 \pm \sqrt{16}}{8} = \pm\frac{4}{8} = \pm\frac{1}{2}$.

b) Rewrite the equation in standard form $10x^2 - 3x + 4 = 0$.

Substitute $a = 10, b = -3$ and $c = 4$ into the discriminant formula $D = (-3)^2 - 4(10)(4) = -151$.

The discriminant is negative, so the equation has two non-real solutions.

c) Substitute $a = 1, b = -10$ and $c = 25$ into the discriminant formula $D = (-10)^2 - 4(1)(25) = 0$.

The discriminant is 0, so the equation has a double root.

The solution to the equation is $\frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$.

If the discriminant is a perfect square, then the solutions to the equation are rational numbers.

Example 3

Determine the nature of the solutions to each quadratic equation.

a) $2x^2 + x - 3 = 0$

b) $-x^2 - 5x + 14 = 0$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $a = 2, b = 1$ and $c = -3$ into the discriminant formula $D = (1)^2 - 4(2)(-3) = 25$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}$ so, $x = 1$ and $x = -\frac{3}{2}$.

b) Substitute $a = -1, b = -5$ and $c = 14$ into the discriminant formula: $D = (-5)^2 - 4(-1)(14) = 81$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{5 \pm \sqrt{81}}{-2} = \frac{5 \pm 9}{-2}$ so, $x = -7$ and $x = 2$.

If the discriminant is not a perfect square, then the solutions to the equation are irrational numbers.

Example 4

Determine the nature of the solutions to each quadratic equation.

a) $-3x^2 + 4x + 1 = 0$

b) $5x^2 - x - 1 = 0$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $a = -3, b = 2$ and $c = 1$ into the discriminant formula $D = (4)^2 - 4(-3)(1) = 28$.

The discriminant is a positive perfect square, so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{-2 \pm \sqrt{28}}{-6}$ so, $x \approx -0.55$ and $x \approx 1.22$.

b) Substitute $a = 5, b = -1$ and $c = -1$ into the discriminant formula $D = (-1)^2 - 4(5)(-1) = 21$.

The discriminant is a positive perfect square so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{1 \pm \sqrt{20}}{10}$ so, $x \approx 0.56$ and $x \approx -0.36$.

Solve Real-World Problems Using Quadratic Functions and Interpreting the Discriminant

You saw that calculating the discriminant shows what types of solutions a quadratic equation possesses. Knowing the types of solutions is very useful in applied problems. Consider the following situation.

Example 5

Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y = -\frac{32}{6400}x^2 + x$ *where* y *is the height and* x *is the horizontal distance the ball travels. We want to know if he kicked the ball hard enough to go over the goal post which is* 10 feet *high.*

Solution

Define

Let $y =$ height of the ball in feet

$x =$ distance from the ball to the goalpost.

Translate We want to know if it is possible for the height of the ball to equal 10 feet at some real distance from the goalpost.

$$10 = -\frac{32}{6400}x^2 + x$$

Solve

Write the equation in standard form.	$-\frac{32}{6400}x^2 + x - 10 = 0$
Simplify.	$-0.005x^2 + x - 10 = 0$
Find the discriminant.	$D = (1)^2 - 4(-0.005)(-10) = 0.8$

Since the discriminant is positive, we know that it is possible for the ball to go over the goal post, if Marcus kicks it from an acceptable distance x from the goal post. From what distance can he score a field goal? See the next example.

Example 6 (continuation)

What is the farthest distance that he can kick the ball from and still make it over the goal post?

Solution

We need to solve for the value of x by using the quadratic formula.

$$x = \frac{-1 \pm \sqrt{0.8}}{-0.01} \approx 10.6 \text{ or } 189.4$$

This means that Marcus has to be closer that 189.4 feet or further than 10.6 feet to make the goal. (Why are there two solutions to this equation? Think about the path of a ball after it is kicked).

Example 7

Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function

$$P = 0.003x^2 + 12x + 27760$$

In this equation x is the number of helmets produced. Their goal is to make a profit of \$40,000 *this year. Is this possible?*

Solution

We want to know if it is possible for the profit to equal \$40,000.

$$40000 = -0.003x^2 + 12x + 27760$$

Solve

Write the equation in standard form	$-0.003x^2 + 12x - 12240 = 0$
Find the discriminant.	$D = (12)^2 - 4(-0.003)(-12240) = -2.88$

Since the discriminant is negative, we know that there are no real solutions to this equation. Thus, it is not possible for Emma and Bradon to make a profit of \$40,000 this year no matter how many helmets they make.

Review Questions

Find the discriminant of each quadratic equation.

1. $2x^2 - 4x + 5 = 0$
2. $x^2 - 5x = 8$
3. $4x^2 - 12x + 9 = 0$
4. $x^2 + 3x + 2 = 0$
5. $x^2 - 16x = 32$
6. $-5x^2 + 5x - 6 = 0$

Determine the nature of the solutions of each quadratic equation.

7. $-x^2 + 3x - 6 = 0$
8. $5x^2 = 6x$
9. $41x^2 - 31x - 52 = 0$
10. $x^2 - 8x + 16 = 0$
11. $-x^2 + 3x - 10 = 0$
12. $x^2 - 64 = 0$

Without solving the equation, determine whether the solutions will be rational or irrational.

13. $x^2 = -4x + 20$
14. $x^2 + 2x - 3 = 0$
15. $3x^2 - 11x = 10$
16. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
17. $x^2 - 10x + 25 = 0$
18. $x^2 = 5x$
19. Marty is outside his apartment building. He needs to give Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of 55 feet/second. Will the phone reach her if she is 36 feet up? (Hint: The equation for the height is given by $y = -32t^2 + 55t + 4$.)
20. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $R = x(200 - 0.4x)$ where x is the number of tires sold. Can Bryson's business generate revenue of $\$20,000$ in the month of July?

Review Answers

1. $D = -24$
2. $D = 57$
3. $D = 0$
4. $D = 1$
5. $D = 384$
6. $D = -95$
7. $D = -15$ no real solutions
8. $D = 36$ two real solutions
9. $D = 9489$ two real solutions
10. $D = 0$ one real solutions
11. $D = -31$ no real solutions

12. $D = 256$ two real solutions
13. $D = 96$ two real irrational solutions
14. $D = 16$ two real rational solutions
15. $D = 241$ two real irrational solutions
16. $D = 8/3$ two real irrational solutions
17. $D = 0$ one real rational solution
18. $D = 25$ two real rational solutions
19. no
20. yes

10.7 Linear, Exponential and Quadratic Models

Learning Objectives

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function models.

Introduction

In this course you have learned about three types of functions, linear, quadratic and exponential.

Linear functions take the form $y = mx = b$.

Quadratic functions take the form $y = ax^2 + bx + c$.

Exponential functions take the form $y = a \cdot b^x$.

In real-world applications, the function that describes some physical situation is not given. Finding the function is an important part of solving problems. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. One job for the scientist is to figure out which function best fits the data. In this section, you will learn some methods that are used to identify which function describes the relationship between the dependent and independent variables in a problem.

Identify Functions Using Differences or Ratios.

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable.

We use differences to identify linear functions.

If the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear.*

Example 1

Determine if the function represented by the following table of values is linear.

x	y	difference of $y-$ values
-2	-4	$-1+4$ = **3**
-1	-1	$2+1$ = **3**
0	2	$5-2$ = **3**
1	5	$8-5$ = **3**
2	8	

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$values, we must make sure that we examine entries for which the $x-$values increase by the same amount.

For example, examine the values in the following table.

difference of $x-$ values	x	y	difference of $y-$ values
1 - 0 = **1**	0	5	$-1+4$ = **3**
3 - 1 = **2**	1	10	$2+1$ = **3**
4 - 3 = **1**	3	20	$5-2$ = **3**
6 - 4 = **2**	4	25	$8-5$ = **3**
	6	35	

At first glance, this function might not look linear because the difference in the $y-$values is not always the same.

However, we see that the difference in $y-$values is 5 when we increase the $x-$values by 1, and it is 10 when we increase the $x-$values by 2. This means that the difference in $y-$values is always 5 when we increase the $x-$values by 1. Therefore, the function is linear. The key to this observation is that **the ratio of the differences is constant**.

In mathematical notation, we can write the linear property as follows.

If $\frac{y_2-y_1}{x_2-x_1}$ is always the same for values of the dependent and independent variables, then the points are on a line. Notice that the expression we wrote is the definition of the slope of a line.

Differences can also be used to identify quadratic functions. For a quadratic function, when we increase the $x-$values by the same amount,

the difference between $y-$values will not be the same. However, the difference of the differences of the $y-$values will be the same.

Here are some examples of quadratic relationships represented by tables of values.

a)

x	$y = x^2$	difference of $y-$ values	difference of differences
0	0	1 – 0 = **1**	
1	1	4 – 1 = **3**	3 – 1 = **2**
2	4	9 – 4 = **5**	5 – 3 = **2**
3	9	16 – 9 = **7**	7 – 5 = **2**
4	16	25 – 16 = **9**	9 – 7 = **2**
5	25	36 – 25 = **11**	11 – 9 = **2**
6	36		

In this quadratic function, $y = x^2$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 2.

b)

x	$y = 2x^2\ 3x + 1$	difference of $y-$ values	difference of differences
0	0	0 – 1 = **-1**	
1	1	3 – 0 = **3**	3 + 1 = **4**
2	3	10 – 3 = **7**	7 – 3 = **4**
3	10	21 – 10 = **11**	11 – 7 = **4**
4	21	36 – 21 = **15**	15 – 11 = **4**
5	36	55 – 36 = **19**	19 – 15 = **4**
6	55		

In this quadratic function, $y = x^2 - 3x + 1$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

We use ratios to identify exponential functions.

If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential.*

Example 2

Determine if the function represented by the following table of values is exponential.

a)

x	y
0	4
1	12
2	36
3	108
4	324

ratio of y - values

$\frac{12}{4} = 3$

$\frac{36}{12} = 3$

$\frac{108}{36} = 3$

$\frac{324}{108} = 3$

If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by 3.

Since the ratio is always the same, **the function is exponential.**

b)

x	y
0	240
1	120
2	60
3	30
4	15

ratio of y - values

$\frac{120}{240} = \frac{1}{2}$

$\frac{60}{120} = \frac{1}{2}$

$\frac{30}{60} = \frac{1}{2}$

$\frac{15}{30} = \frac{1}{2}$

If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by $1/2$.

Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions.

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

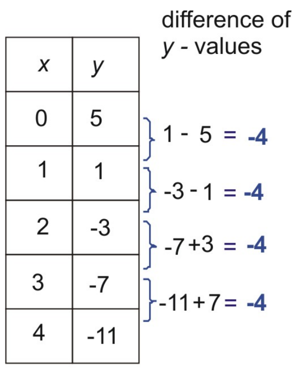
Example 3

Determine what type of function represents the values in the following table.

x	y
0	3
1	1
2	-3
3	-7
4	-11

Solution

Let's first check the difference of consecutive values of y.



If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always decreases by 4. Since the difference is always the same, **the function is linear.**

To find the equation for the function that represents these values, we start with the general form of a linear function.

$$y = mx + b$$

Here $''m''$ is the slope of the line and is defined as the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is

$$y = -4x + 5$$

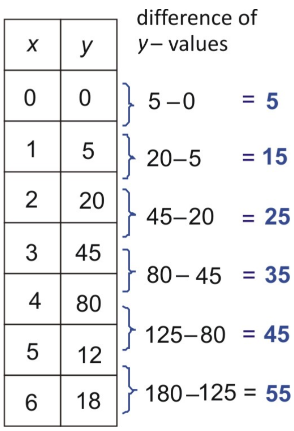
Example 4

Determine what type of function represents the values in the following table.

x	y
0	0
1	5
2	20
3	45
4	80
5	125
6	180

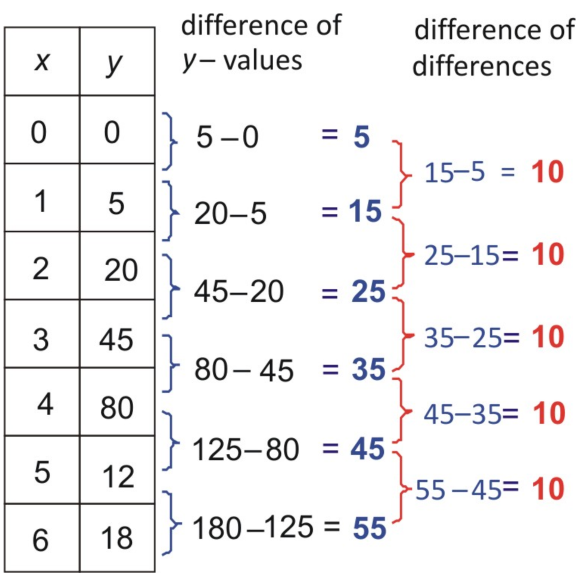
Solution

Let's first check the difference of consecutive values of y.



If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value does not remain constant. Since the difference is not the same, **the function is not linear.**

Now, let's check the difference of the differences in the values of y.



When we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant. The difference of the differences is always 10 when we increase the $x-$value by one.

The function describing these set of values is **quadratic.** To find the equation for the function that represents these values, we start with the general form of a quadratic function.

$$y = ax^2 + bx + c$$

We need to use the values in the table to find the values of the constants a, b and c.

The value of c represents the value of the function when $x = 0$, so $c = 0$.

Then	$y = ax^2 + bx$
Plug in the point $(1, 5)$.	$5 = a + b$
Plug in the point $(2, 20)$.	$20 = 4a + 2b \Rightarrow 10 = 2a + b$
To find a and b, we solve the system of equations	$5 = a + b$
	$10 = 2a + b$
Solve the first equation for b.	$5 = a + b \Rightarrow b = 5 - a$
Plug the first equation into the second.	$10 = 2a + 5 - a$
Solve for a and b.	$a = 5$ and $b = 0$

Therefore the equation of the quadratic function is

$$y = 5x^2$$

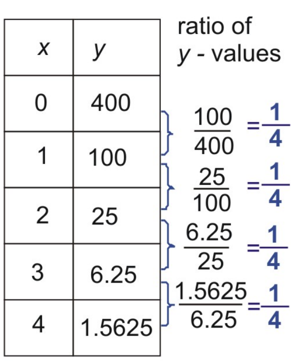
Example 5

Determine what type of function represents the values in the following table.

x	y
0	400
1	100
2	25
3	625
4	1.5625

Solution:

Let's check the ratio of consecutive values of y.



If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by $1/4$.

Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function.

$$y = a \cdot b^x$$

b is the ratio between the values of y each time that x is increased by one. The constant a is the value of the function when $x = 0$. Therefore, our answer is

$$y = 400\left(\frac{1}{4}\right)^x$$

Perform Exponential and Quadratic Regressions with a Graphing Calculator.

Earlier you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you will learn how to perform exponential and quadratic regression to find equations for functions that describe non-linear relationships between the variables in a problem.

Example 6

Find the quadratic function that is a best fit for the data in the following table. The following table shows how many miles per gallon a car gets at different speeds.

Table 10.1:

Speed (mi/h)	Miles Per Gallon
30	18
35	20
40	23
45	25
50	28
55	30
60	29
65	25
70	25

Using a graphing calculator.

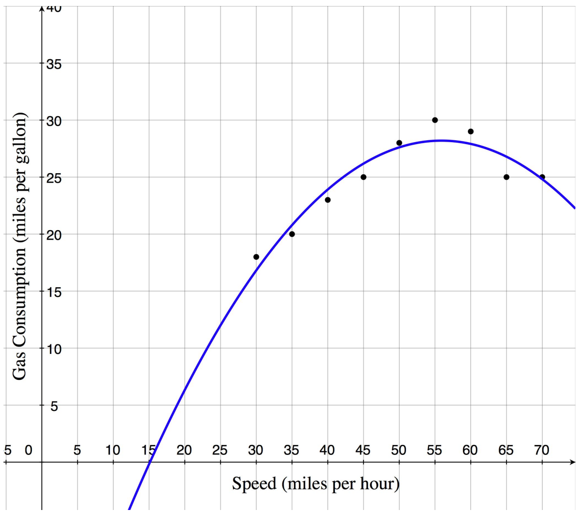
a) Draw the scatterplot of the data.

b) Find the quadratic function of best fit.

c) Draw the quadratic function of best fit on the scatterplot.

d) Find the speed that maximizes the miles per gallon.

e) Predict the miles per gallon of the car if you drive at a speed of 48 miles per gallon.



Solution

Step 1 Input the data

Press [**STAT**] and choose the [**EDIT**]option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Note: In order to clear a list, move the cursor to the top so that L_1 or L_2 is highlighted. Then press [**CLEAR**] button and then [**ENTER**].

Step 2 Draw the scatter plot.

First press [**Y=**] and clear any function on the screen by pressing [**CLEAR**] when the old function is highlighted.

Press [**STATPLOT**] [**STAT**] and [**Y=**] and choose option 1.

Choose the ON option, after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press [**GRAPH**] and make sure that the window is set so you see all the points in the scatterplot. In this case $30 \leq x \leq 80$ and $0 \leq y \leq 40$.

You can set the window size by pressing on the [**WINDOW**] key at top.

Step 3 Perform quadratic regression.

Press [**STAT**] and use right arrow to choose [**CALC**].

Choose Option 5 (QuadReg) and press [**ENTER**]. You will see "QuadReg" on the screen.

Type in L_1, L_2 after 'QuadReg' and Press [**ENTER**]. The calculator shows the quadratic function.

Function $y = -0.017x^2 + 1.9x - 25$

Step 4: Graph the function.

Press [**Y=**] and input the function you just found.

Press [**GRAPH**] and you will see the curve fit drawn over the data points.

To find the speed that maximizes the miles per gallons, use [**TRACE**] and move the cursor to the top of the parabola. You can also use [**CALC**] [**2nd**] [**TRACE**] and option 4 Maximum, for a more accurate answer. The speed that maximizes miles per gallons = 56 mi/h

Plug $x = 56$ into the equation you found: $y = -0.017(56)^2 = 1.9(56) - 25 = 28$ miles per gallon

Note: The image to the right shows our data points from the table and the function plotted on the same graph. One thing that is clear from this graph is that predictions made with this function will not make sense for all values of x. For example, if $x < 15$, this graph predicts that we will get negative mileage, something that is impossible. Thus, part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Example 7

The following data represents the amount of money an investor has in an account each year for 10 years.

Table 10.2:

year	value of account
1996	$5000
1997	$5400
1998	$5800
1999	$6300
2000	$6800
2001	$7300
2002	$7900
2003	$8600
2004	$9300
2005	$10000
2006	$11000

Using a graphing calculator

a) Draw a scatterplot of the value of the account as the dependent variable, and the number of years *since* 1996 as the independent

variable.

b) Find the exponential function that fits the data.

c) Draw the exponential function on the scatterplot.

d) What will be the value of the account in 2020?

Solution

Step 1 Input the data

Press [**STAT**] and choose the [**EDIT**] option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2 Draw the scatter plot.

First press [**Y=**] and clear any function on the screen.

Press [**GRAPH**] and choose Option 1.

Choose the ON option and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press [**GRAPH**] make sure that the window is set so you see all the points in the scatterplot. In this case: $0 \leq x \leq 10$ and $0 \leq y \leq 11000$.

Step 3 Perform exponential regression.

Press [**STAT**] and use right arrow to choose [**CALC**] .

Choose Option 0 and press [**ENTER**]. You will see “ExpReg” on the screen.

Press [**ENTER**] . The calculator shows the exponential function.

Function $y = 4975.7(1.08)^x$

Step 4: Graph the function.

Press [**Y=**] and input the function you just found. Press [**GRAPH**].

Substitute $x = 2020 - 1996 = 24$ into the function $y = 4975.7(1.08)^{24} = \$31551.81$.

Note: This is a curve fit. So the function above is the curve that comes closest to all the data points. It will not return y values that are exactly the same as in the data table, but they will be close. It is actually more accurate to use the curve fit values than the

data points.

Solve Real-World Problems by Comparing Function Models

Example 8

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.

Table 10.3:

Year	Number of Students (millions)
1990	26.6
1991	26.6
1992	27.1
1993	27.7
1994	28.1
1995	28.4
1996	28.1
1997	29.1
1998	29.3
2003	32.5

Solution

We will perform linear, quadratic and exponential regression on this data set and see which function represents the values in the table the best.

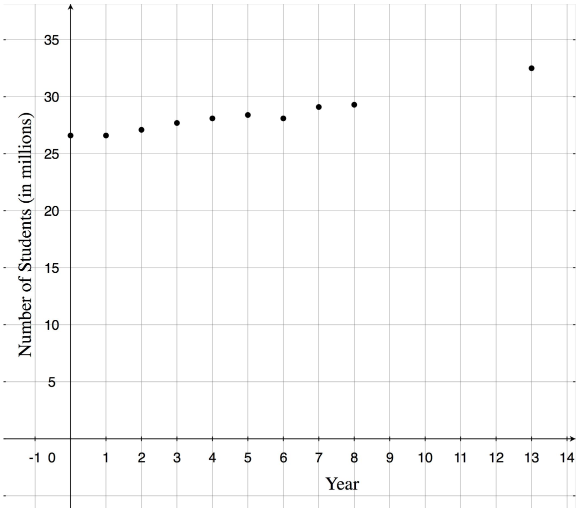
Step 1 Input the data.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2 Draw the scatter plot.

Set the window size: $0 \le x \le 10$ and $20 \le y \le 40$.

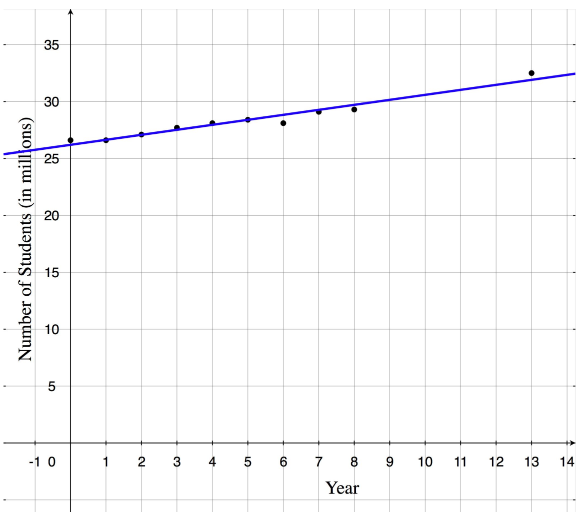
Here is the scatter plot.



Step 3 Perform Regression.

Linear Regression

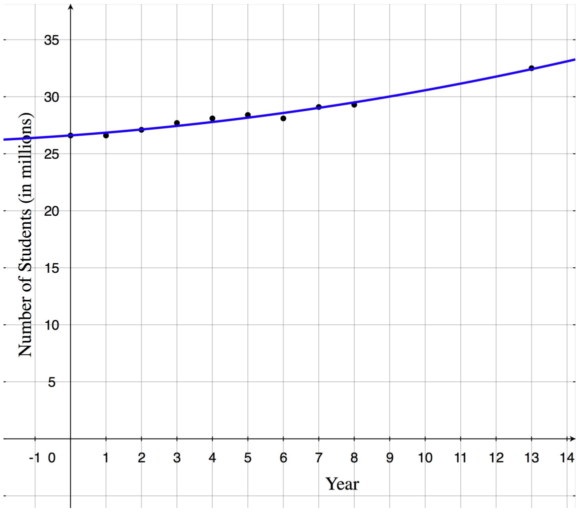
The function of the line of best fit is $y = 0.44x + 26.1$.



Here is the graph of the function on the scatter plot.

Quadratic Regression

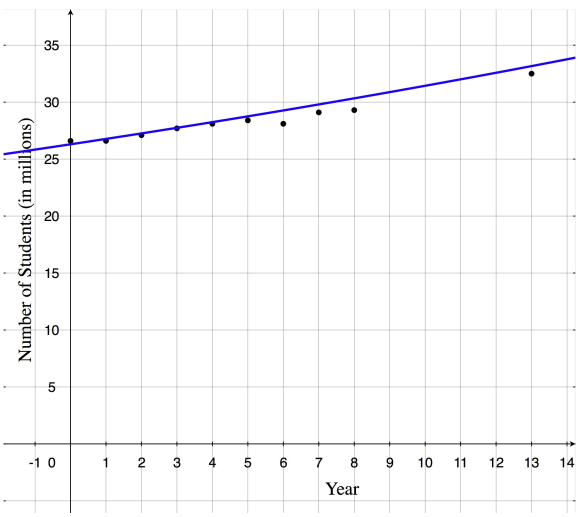
The quadratic function of best fit is $y = 0.064x^2 - .067x + 26.84$.



Here is the graph of the function on the scatter plot.

Exponential Regression

The exponential function of best fit is $y = 26.2(1.018)^x$.



Here is the graph of the function on the scatter plot.

From the graphs, it looks like the quadratic function is the best fit for this data set. We use this function to predict school enrollment in 2007.

Substitute $x = 2007 - 1990 = 17$

$$y = 0.064(17)^2 - .067(17) + 26.84 = \underline{44.2 \text{ million students}}$$

Review Questions

Determine whether the data in the following tables can be represented by a linear function.

1.

x	y
−4	10
−3	7
−2	4
−1	1
0	−2
1	−5

2.

x	y
−2	4
−1	3
0	2
1	3
2	6
3	11

3.

x	y
0	50
1	75
2	100
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function:

4.

x	y
−10	10
−5	2.5
0	0
5	2.5
10	10
15	22.5

5.

x	y
1	4
2	6
3	6
4	4
5	0
6	−6

6.

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

Determine whether the data in the following tables can be represented by an exponential function.

7.

x	y
0	200
1	300
2	1800
3	8300
4	25800
5	62700

8.

x	y
0	120
1	180
2	270
3	405
4	607.5
5	911.25

9.

x	y
0	4000
1	2400
2	1440
3	864
4	518.4
5	311.04

Determine what type of function represents the values in the following table and find the equation of the function.

10.

x	y
0	400
1	500
2	625
3	781.25
4	976.5625

11.

x	y
-9	-3
-7	-2
-5	-1
-3	0
-1	1
1	2

12.

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2
2	4
3	14

13. As a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, the table shows the height of the ball with respect to time.

Table 10.4:

Time (seconds)	Height (inches)
2	2
2.2	16
2.4	24
2.6	33
2.8	38
3.0	42
3.2	36
3.4	30
3.6	28
3.8	14
4.0	6

14.
15. Using a graphing calculator
16. (a) Draw the scatter plot of the data.
 (b) Find the quadratic function of best fit.
 (c) Draw the quadratic function of best fit on the scatter plot.
 (d) Find the maximum height the ball reaches on the bounce.
 (e) Predict how high the ball is at time $t = 2.5$ seconds.
17. A chemist has a 250 gram sample of a radioactive material. She records the amount of radioactive material remaining in the sample every day for a week and obtains the data in the following table.

Table 10.5:

Day	Weight (grams)
0	250
1	208
2	158
3	130
4	102
5	80
6	65
7	50

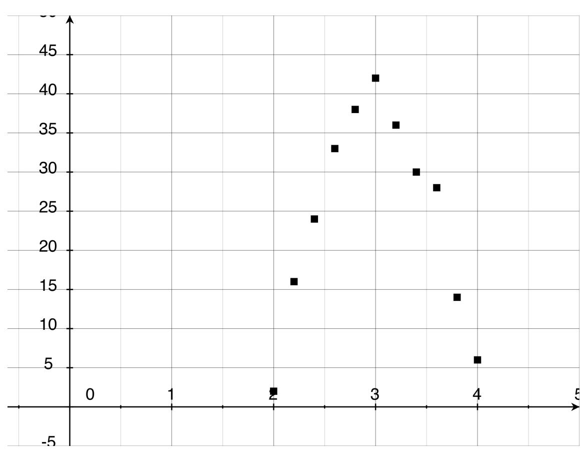
18.
19. Using a graphing calculator,
20. (a) Draw a scatterplot of the data.
 (b) Find the exponential function of best fit.
 (c) Draw the exponential function of best fit on the scatter plot.
 (d) Predict the amount of material after 10 days.
21. The following table shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau). Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the rate of teen pregnancies in the year 2010.

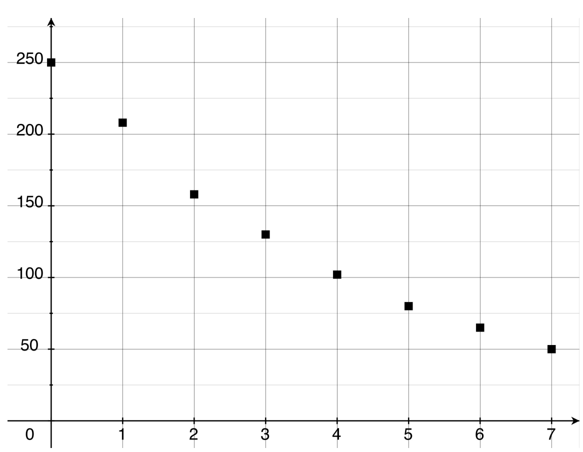
Table 10.6:

Year	Rate of Pregnancy (per 1000)
1990	116.9
1991	115.3
1992	111.0
1993	108.0
1994	104.6
1995	99.6
1996	95.6
1997	91.4
1998	88.7
1999	85.7
2000	83.6
2001	79.5
2002	75.4

22.

Review Answers

1. Linear common difference = −3
2. Not Linear
3. Linear common difference = 25
4. Quadratic difference of difference = 5
5. Quadratic difference of difference = −2
6. Not Quadratic
7. Not Exponential
8. Exponential common ratio = 1.5
9. Exponential common ratio = 0.6
10. Exponential $y = 400(1.25)^x$
11. Linear $y = (1/2)x + (3/2)$
12. Quadratic $y = 2x^2 - 4$
13.
14. (a) 

    (b) $y = -35.4x^2 + 213.3x - 282.4$;
    (c) Maximum height = 38.9 inches.
    (d) $t = 2.5$ sec, height = 29.6 inches.
15.
16. (a) 

    (b) $y = 255.25(0.79)^x$
    (c) After 10 days, there is 24.17 grams of material left.
17. linear function is best fit: $y = -3.54x + 117.8$ In year 2010, $x = 20$, rate of teen pregnancies = 47 per

10.8 Problem Solving Strategies: Choose a Function Model

Learning Objectives

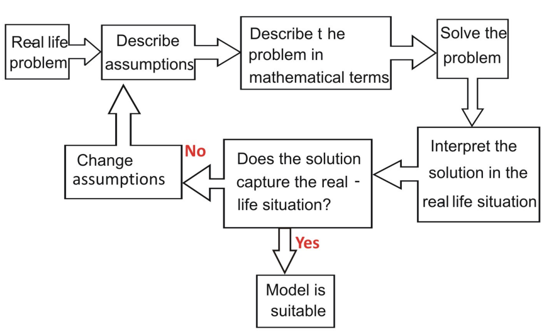
- Read and understand given problem situations
- Develop and use the strategy: Choose a Function
- Develop and use the strategy: Make a Model
- Plan and compare alternative approaches to solving problems
- Solve real-world problems using selected strategies as part of a plan

Introduction

As you learn more and more mathematical methods and skills, it is important to think about the purpose of mathematics and how it works as part of a bigger picture. Mathematics is used to solve problems which often arise from real-life situations. **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution. Modeling involves creating a set of mathematical equations that describes a situation, solving those equations and using them to understand the real-life problem. Often the model needs to be adjusted because it does not describe the situation as well as we wish.

A mathematical model can be used to gain understanding of a real-life situation by learning how the system works, which variables are important in the system and how they are related to each other. Models can also be used to predict and forecast what a system will do in the future or for different values of a parameter. Lastly, a model can be used to estimate quantities that are difficult to evaluate exactly.

Mathematical models are like other types of models. The goal is not to produce an exact copy of the "real" object but rather to give a representation of some aspect of the real thing. The modeling process can be summarized as follows.



Notice that the modeling process is very similar to the problem solving format we have been using throughout this book. In this section, we will focus mostly on the assumptions we make and the validity of the model. Functions are an integral part of the modeling process because they are used to describe the mathematical relationship in a system. One of the most difficult parts of the modeling process is determining which function best describes a situation. We often find that the function we chose is not appropriate. Then, we must choose a different one, or we findthat a function model is good for one set of parameters but we need to use another function for a different set of parameters. Often, for certain parameters, more than one function describes the situation well and using the simplest function is most practical.

Here we present some mathematical models arising from real-world applications.

Example 1 Stretching springs beyond the "elastic limit"

A spring is stretched as you attach more weight at the bottom of the spring. The following table shows the length of the spring in inches for different weights in ounces.

Weight (oz)	0	2	4	6	8	10	12	14	16	18	20
Length (in)	2	2.4	2.8	3.2	3.5	3.9	4.1	4.4	4.6	4.7	4.8

a) Find the length of the spring as a function of the weight attached to it.

b) Find the length of the spring when you attach 5 ounces.

c) Find the length of the spring when you attach 19 ounces.

Solution

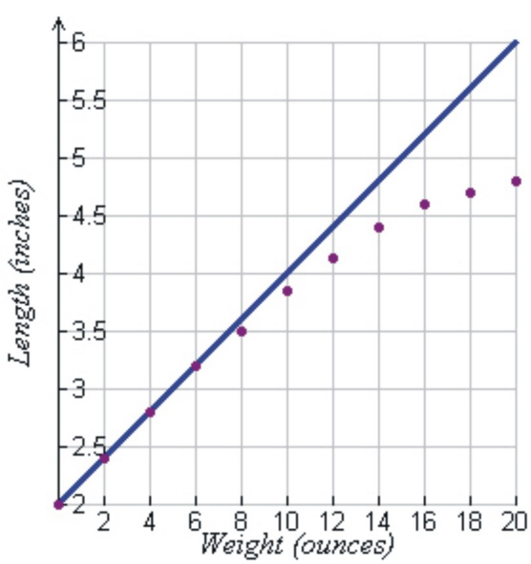
Step 1 Understand the problem

Define $x =$ weight in ounces on the spring

$y =$ length in inches of the spring

Step 2 Devise a plan

Springs usually have a linear relationship between the weight on the spring and the stretched length of the spring. If we make a scatter plot, we notice that for lighter weights the points do seem to fit on a straight line (see graph). Assume that the function relating the length of the spring to the weight is linear.



Step 3 Solve

Find the equation of the line using points describing lighter weights:

$(0, 2)$ and $(4, 2.8)$.

The slope is $m = \frac{.8}{4} = 0.2$

Using $y = mx + b$

a) We obtain the function $y = .2x + 2$.

b) To find the length of the spring when the weight is 5 ounces, we plug in $x = 5$.

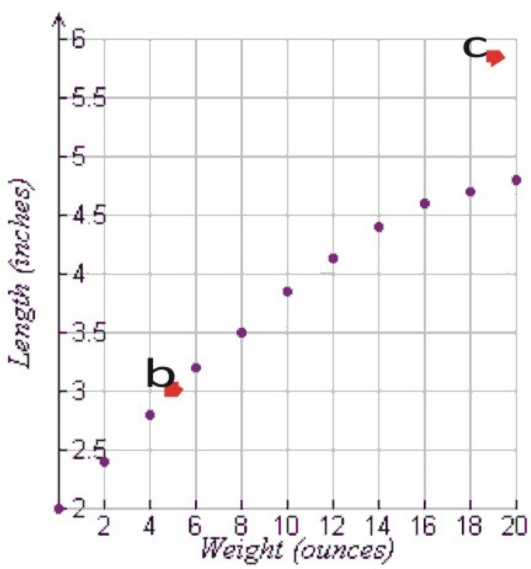
$$y = .2(5) + 2 = 3 \text{ inches}$$

c) To find the length of the spring when the weight is 19 ounces, we plug in $x = 19$.

$$y = .2(19) + 2 = 5.8 \text{ inches}$$

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that for small weights, the relationship between the length of the spring and the weight is a linear function.

For larger weights, the spring does not seem to stretch as much for each added ounces. We must change our assumption. There must be a non-linear relationship between the length and the weight.

Step 5 Solve with New Assumptions

Let's find the equation of the function by cubic regression with a graphing calculator.

a) We obtain the function $y = -.000145x^3 - .000221x^2 + .202x + 2.002$.

b) To find the length of the spring when the weight is 5 ounces, we plug in $x = 5$.

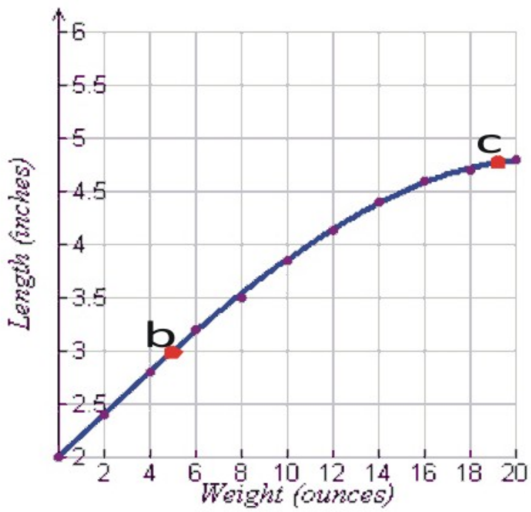
$$y = -.000145(5)^3 - .000221(5)^2 + .202(5) + 2.002 = 3 \text{ inches}$$

c) To find the length of the spring when the weight is 19 ounces, we plug in $x = 19$.

$$y = -.000145(19)^3 - .000221(19)^2 + .202(19) + 2.002 = 4.77 \text{ inches}$$

Step 6 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a cubic function represents the stretching of the spring more accurately than a linear function. However, for small weights the linear function is an equally good representation, and it is much easier to use in most cases. In fact, the linear approximation usually allows us to easily solve many problems that would be very difficult to solve by using the cubic function.

Example 2 Water flow

A thin cylinder is filled with water to a height of 50 centimeters. The cylinder has a hole at the bottom which is covered with a stopper. The stopper is released at time $t = 0$ seconds and allowed to empty. The following data shows the height of the water in the cylinder at different times.

Time (sec)	0	2	4	6	8	10	12	14	16	18	20	22	24
Height (cm)	50	42.5	35.7	29.5	23.8	18.8	14.3	10.5	7.2	4.6	2.5	1.1	0.2

a) Find the height (in centimeters) of water in the cylinder as a function of time in seconds.

b) Find the height of the water when $t = 5$ seconds.

c) Find the height of the water when $t = 13$ seconds.

Solution:

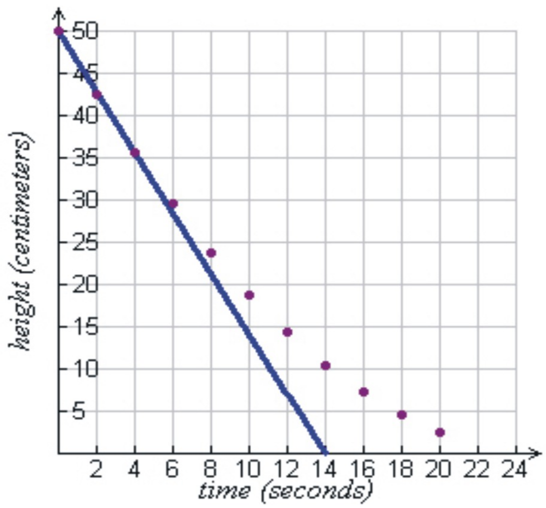
Step 1 Understand the problem

Define $x =$ the time in seconds

$y =$ height of the water in centimeters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of water on the vertical axis.



Notice that most of the points seem to fit on a straight line when the water level is high. Assume that a function relating the height of the water to the time is linear.

Step 3 Solve

Find the equation of the line using points describing lighter weights:

$(0,50)$ and $(4,35.7)$.

The slope is $m = \frac{-14.3}{4} = -3.58$

Using $y = mx + b$

a) We obtain the function: $y = -3.58x + 50$

b) The height of the water when $t = 5$ seconds is

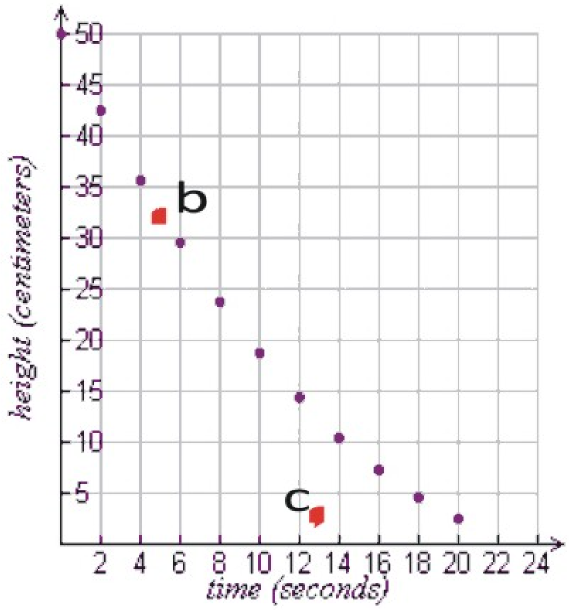
$$y = -3.58(5) + 50 = 32.1 \text{ centimeters}$$

c) The height of the water when $t = 13$ seconds is

$$y = -3.58(13) + 50 = 3.46 \text{ centimeters}$$

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that when the water level is high, the relationship between the height of the water and the time is a linear function. When the water level is low, we must change our assumption. There must be a non-linear relationship between the height and the time.

Step 5 Solve with new assumptions

Let's assume the relationship is quadratic and let's find the equation of the function by quadratic regression with a graphing calculator.

a) We obtain the function $y = .075x^2 - 3.87x + 50$

b) The height of the water when $t = 5$ seconds is

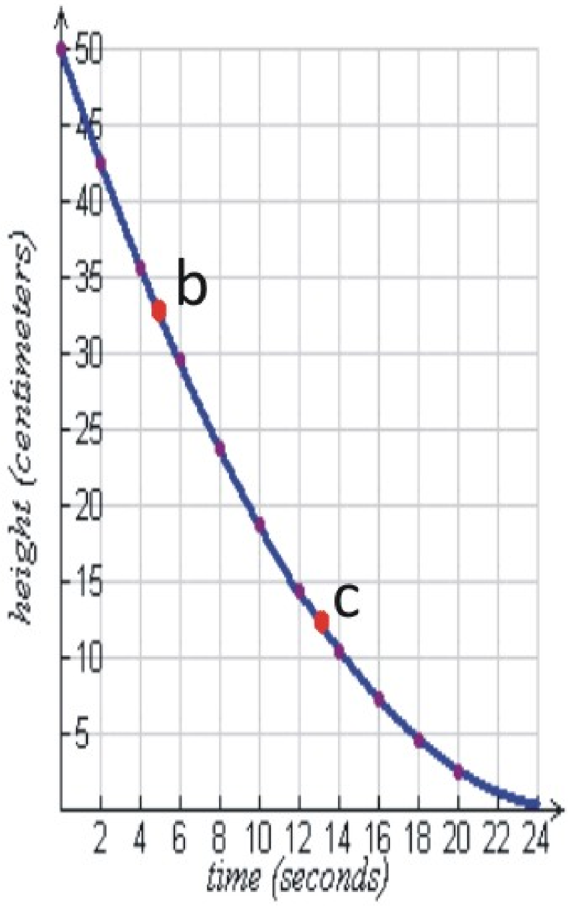
$$y = .075(5)^2 - 3.87(5) + 50 = 32.53 \text{ centimeters}$$

c) The height of the water when $t = 13$ seconds is

$$y = .075(13)^2 - 3.87(13) + 50 = 12.37 \text{ centimeters}$$

Step 6: Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a quadratic function represents the situation more accurately than a linear function. However, for high water levels the linear function is an equally good representation.

Example 3 Projectile motion

A golf ball is hit down a straight fairway. The following table shows the height of the ball with respect to time. The ball is hit at an angle of 70 degrees with the horizontal with a speed of 40 meters/sec.

Time (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Height (meters)	0	17.2	31.5	42.9	51.6	57.7	61.2	62.3	61.0	57.2	51.0	42.6	31.9	19.0	4.1

a) Find the height of the ball as a function of time.

b) Find the height of the ball when $t = 2.4$ seconds.

c) Find the height of the ball when $t = 6.2$ seconds.

Solution

Step 1 Understand the problem

Define $x =$ the time in seconds

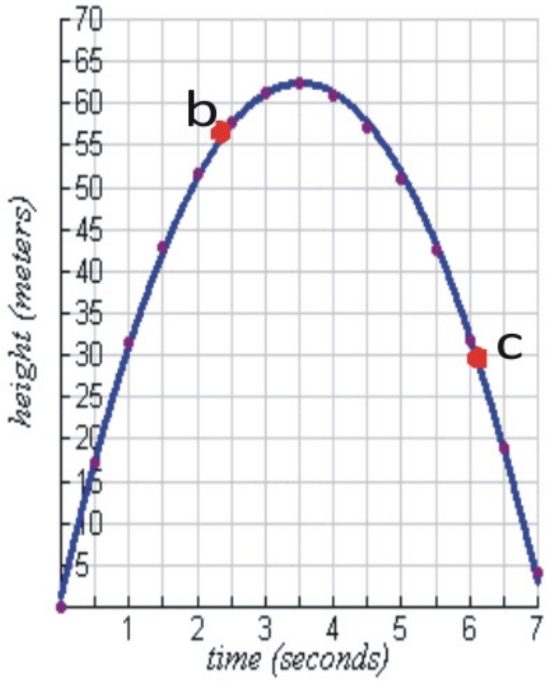
$y =$ height of the ball in meters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of the ball on the vertical axis. We know that a projectile follows a parabolic path, so we assume that the function relating height to time is quadratic.

Step 3 Solve

Let's find the equation of the function by quadratic regression with a graphing calculator.



a) We obtain the function $y = -4.92x^2 + 34.7x + 1.2$

b) The height of the ball when $t = 2.4$ seconds is:

$$y = -4.92(2.4)^2 + 34.7(2.4) + 1.2 = 56.1 \text{ meters}$$

c) The height of the ball when $t = 6.2$ seconds is:

$$y = -4.92(6.2)^2 + 34.7(6.2) + 1.2 = 27.2 \text{ meters}$$

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) follow the trend very closely. The quadratic function is a very good model for this problem

Example 4 Population growth

A scientist counts two thousand fish in a lake. The fish population increases at a rate of 1.5 fish per generation but the lake has space and food for only 2,000,000 fish. The following table gives the number of fish (in thousands) in each generation.

Generation	0	4	8	12	16	20	24	28
Number (thousands)	2	15	75	343	1139	1864	1990	1999

a) Find the number of fish as a function of generation.

b) Find the number of fish in generation 10.

c) Find the number of fish in generation 25.

Solution:

Step 1 Understand the problem

Define $x =$ the generation number $y =$ the number of fish in the lake

Step 2 Devise a plan

Let's make a scatterplot of our data with the generation number on the horizontal axis and the number of fish in the lake on the vertical axis. We know that a population can increase exponentially. So, we assume that we can use an exponential function to describe the relationship between the generation number and the number of fish.

Step 3 Solve

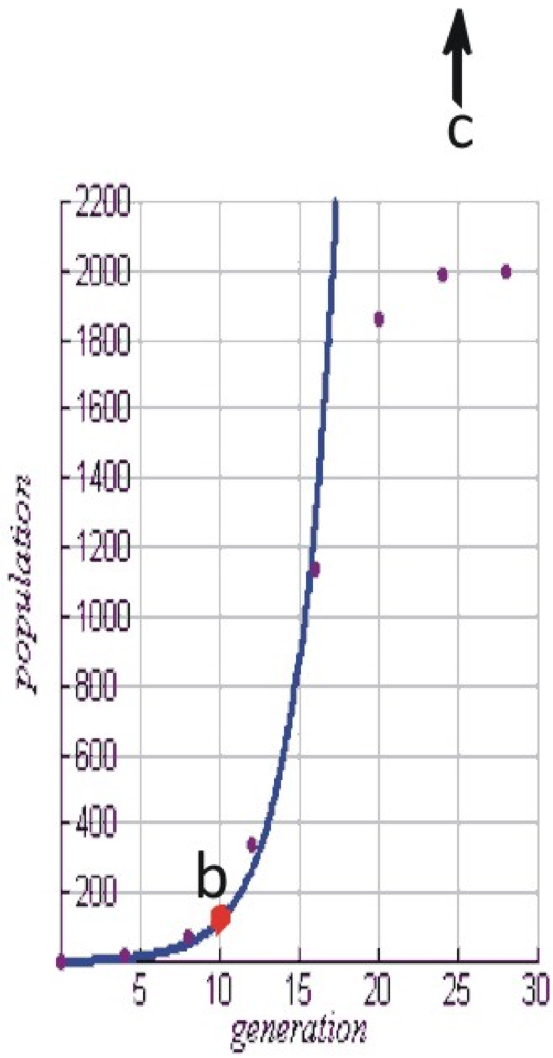
a) Since the population increases at a rate of 1.5 per generation, assume the function $y = 2(1.5)x$

b) The number of fish in generation 10 is: $y = 2(1.5)^{10} = 115$ thousand fish

c) The number of fish in generation 25 is: $y = 2(1.5)^{25} = 50502$ thousand fish

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) fits the data well but the answer to c) does not seem to follow the trend very closely. The result is not even on our graph!



When the population of fish is high, the fish compete for space and resources so they do not increase as fast. We must change our assumptions.

Step 5 Solve with new assumptions

When we try different regressions with the graphing calculator, we find that logistic regression fits the data the best.

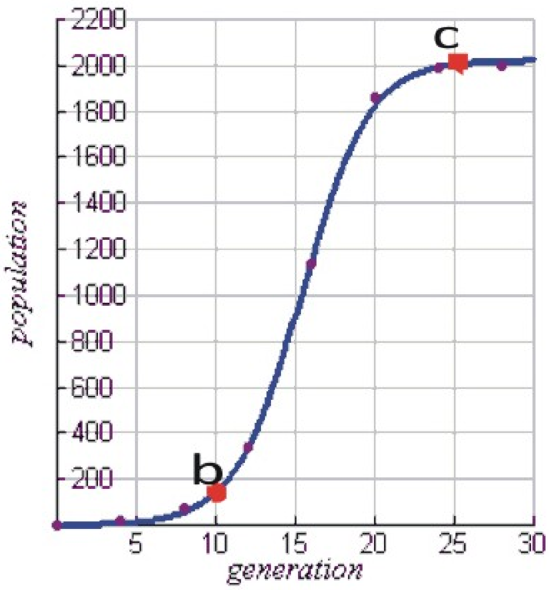
a) We obtain the function $y - \frac{2023.6}{1+1706.3(2.71)^{-484x}}$

b) The number of fish in generation 10 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(10)}} = 139.6$ thousand fish

c) The number of fish in generation 25 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(25)}} = 2005$ thousand fish

Step 6 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a logistic function represents the situation more accurately than an exponential function. However, for small populations the exponential function is an equally good representation, and it is much easier to use in most cases.

Review Questions

1. In Example 1, evaluate the length of the spring for weight = 3 ounces by
   (a) Using the linear function
   (b) Using the cubic function
   (c) Figuring out which function is best to use in this situation.
2. In Example 1, evaluate the length of the spring for weight = 15 ounces by
   (a) Using the linear function
   (b) Using the cubic function
   (c) Figuring out which function is best to use in this situation.
3. In Example 2, evaluate the height of the water in the cylinder when $t = 4.2$ seconds by
   (a) Using the linear function
   (b) Using the quadratic function
   (c) Figuring out which function is best to use in this situation.
4. In Example 2, evaluate the height of the water in the cylinder when $t = 19$ seconds by
   (a) Using the linear function
   (b) Using the quadratic function
   (c) Figuring out which function is best to use in this situation.
5. In Example 3, evaluate the height of the ball when $t = 5.2$ seconds. Find when the ball is at its highest point.
6. In Example 4, evaluate the number of fish in generation 8 by

(a) Using the exponential function
(b) Using the logistic function
(c) Figuring out which function is best to use in this situation.

7. In Example 4, evaluate the number of fish in generation 18 by

(a) Using the exponential function
(b) Using the logistic function
(c) Figuring out which function is best to use in this situation.

Review Answers

1.
2. (a) 2.6 inches
   (b) 2.6 inches
   (c) Both functions give the same result. The linear function is best because it is easier to use.
3.
4. (a) 5 inches
   (b) 4.5 inches
   (c) The two functions give different answers. The cubic function is better because it gives a more accurate answer.
5.
6. (a) 34.96 cm
   (b) 35.07 cm
   (c) The results from both functions are almost the same. The linear function is best because it is easier to use.
7.
8. (a) -18.02 cm
   (b) 3.5 cm
   (c) The two function give different results. The quadratic function is better because it gives a more accurate answer.
9.
10. (a) 48.6 meters
    (b) 3.7 seconds
11.
12. (a) $51,000$
    (b) $55,000$
    (c) The results from both functions are almost the same. The linear function is best because it is easier to use.
13.
14. (a) $2,956,000$
    (b) $1,571,000$
    (c) the two functions give different results; the logistic function is better because it gives a more accurate answer.

Chapter 11

Algebra and Geometry Connections; Working with Data

11.1 Graphs of Square Root Functions

Learning Objectives

- Graph and compare square root functions.
- Shift graphs of square root functions.
- Graph square root functions using a graphing calculator.
- Solve real-world problems using square root functions.

Introduction

In this chapter, you will be learning about a different kind of function called the **square root function**. You have seen that taking the square root is very useful in solving quadratic equations. For example, to solve the equation $x^2 = 25$ we take the square root of both sides $\sqrt{x^2} = \pm\sqrt{25}$ and obtain $x = \pm 5$. A square root function has the form $y = \sqrt{f(x)}$. In this type of function, the expression in terms of x is found inside the square root sign (also called the "radical" sign).

Graph and Compare Square Root Functions

The square root function is the first time where you will have to consider the domain of the function before graphing. The domain is very important because the function is undefined if the expression inside the square root sign is negative, and as a result there will be no graph in that region.

In order to is cover how the graphs of square root function behave, we should make a table of values and plot the points.

Example 1

Graph the function $y = \sqrt{x}$.

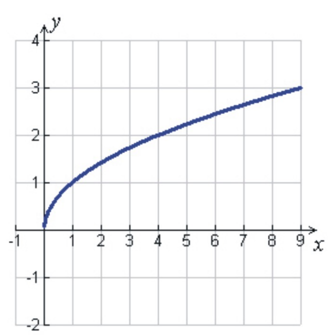
Solution

Before we make a table of values, we need to find the domain of this square root function. The domain is found by realizing that the function is only defined when the expression inside the square root is greater than or equal to zero. We find that the domain is all values of x such that $x \geq 0$.

This means that when we make our table of values, we should pick values of x that are greater than or equal to zero. It is very useful to include the value of zero as the first value in the table and include many values greater than zero. This will help us in determining what the shape of the curve will be.

x	$y = \sqrt{x}$
0	$y = \sqrt{0} = 0$
1	$y = \sqrt{1} = 1$
2	$y = \sqrt{2} = 1.4$
3	$y = \sqrt{3} = 1.7$
4	$y = \sqrt{4} = 2$
5	$y = \sqrt{5} = 2.2$
6	$y = \sqrt{6} = 22.4$
7	$y = \sqrt{7} = 2.6$
8	$y = \sqrt{8} = 2.8$
9	$y = \sqrt{9} = 3$

Here is what the graph of this table looks like.



The graphs of square root functions are always curved. The curve above looks like half of a parabola lying on its side. In fact the square root function we graphed above comes from the expression $y^2 = x$.

This is in the form of a parabola but with the x and y switched. We see that when we solve this expression for y we obtain two solutions $y = \sqrt{x}$ and $y = -\sqrt{x}$. The graph above shows the positive square root of this answer.

Example 2

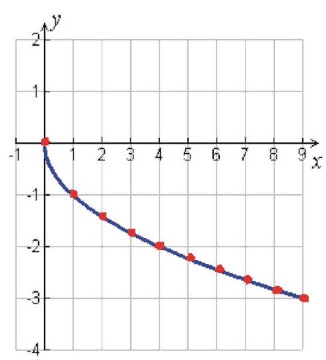
Graph the function $y = -\sqrt{x}$.

Solution

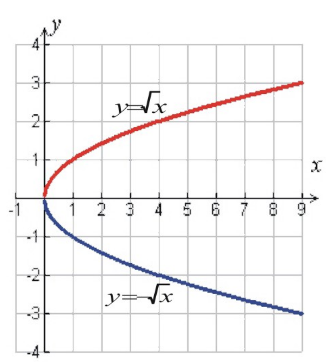
Once again, we must look at the domain of the function first. We see that the function is defined only for $x \geq 0$. Let's make a table of values and calculate a few values of the function.

x	$y = -\sqrt{x}$
0	$y = -\sqrt{0} = -0$
1	$y = -\sqrt{1} = -1$
2	$y = -\sqrt{2} = -1.4$
3	$y = -\sqrt{3} = -1.7$
4	$y = -\sqrt{4} = -2$
5	$y = -\sqrt{5} = -2.2$
6	$y = -\sqrt{6} = -22.4$
7	$y = -\sqrt{7} = -2.6$
8	$y = -\sqrt{8} = 2.8$
9	$y = -\sqrt{9} = -3$

Here is the graph from this table.



Notice that if we graph the two separate functions on the same coordinate axes, the combined graph is a parabola lying on its side.



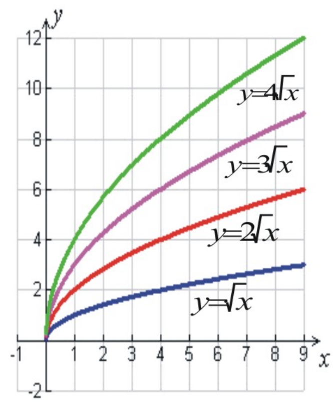
Now let's compare square root functions that are multiples of each other.

Example 3

Graph functions $y = \sqrt{x}$, $y = 2\sqrt{x}$, $y = 3\sqrt{x}$, $y = 4\sqrt{x}$ *on the same graph.*

Solution

Here we will show just the graph without the table of values.

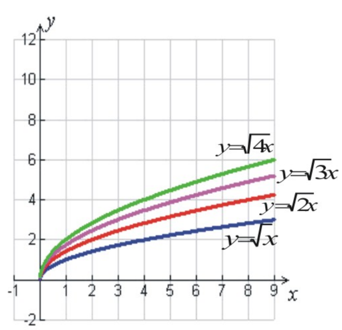


If we multiply the function by a constant bigger than one, the function increases faster the greater the constant is.

Example 4

Graph functions $y = \sqrt{x}$, $y = \sqrt{2x}$, $y = \sqrt{3x}$, $y = \sqrt{4x}$ *on the same graph.*

Solution

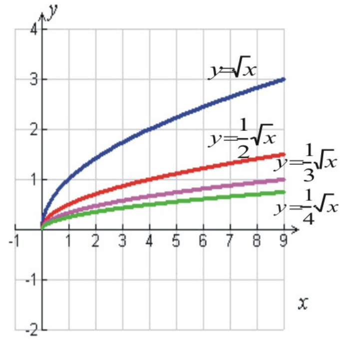


Notice that multiplying the expression inside the square root by a constant has the same effect as in the previous example but the function increases at a slower rate because the entire function is effectively multiplied by the square root of the constant. Also note that the graph of $\sqrt{4x}$ is the same as the graph of $2\sqrt{2x}$. This makes sense algebraically since $\sqrt{4} = 2$.

Example 5

Graph functions $y = \sqrt{x}$, $y = \frac{1}{2}\sqrt{x}$, $y = \frac{1}{3}\sqrt{x}$,$y = \frac{1}{4}\sqrt{x}$ *on the same graph.*

Solution



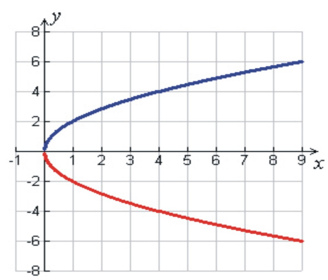
If we multiply the function by a constant between 0 and 1, the function increases at a slower rate for smaller constants.

Example 6

Graph functions $y = 2\sqrt{x}, y = -2\sqrt{x}$ *on the same graph.*

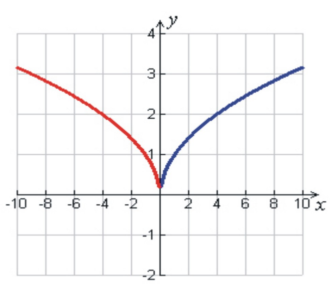
Solution

If we multiply the function by a negative function, the square root function is reflected about the $x-$axis.



Example 7

Graph functions $y = \sqrt{x}, y = \sqrt{-x}$ *on the same graph.*



Solution

Notice that for function $y = \sqrt{x}$ the domain is values of $x \geq 0$, and for function $y = \sqrt{-x}$ the domain is values of $x \leq 0$.

When we multiply the argument of the function by a negative constant the function is reflected about the $y-$axis.

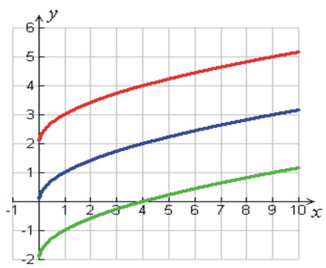
Shift Graphs of Square Root Functions

Now, let's see what happens to the square root function as we add positive and negative constants to the function.

Example 8

Graph the functions $y = \sqrt{x}$, $y = \sqrt{x} + 2$,$y = \sqrt{x} - 2$.

Solution

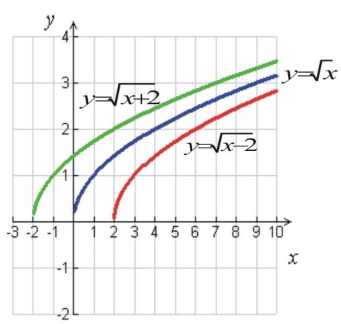


We see that the graph keeps the same shape, but moves up for positive constants and moves down for negative constants.

Example 9

Graph the functions $y = \sqrt{x}$, $y = \sqrt{x-2}$,$y = \sqrt{x+2}$.

Solution



When we add constants to the argument of the function, the function shifts to the left for a positive constant and to the right for a negative constant because the domain shifts. There can't be a negative number inside the square root.

Now let's graph a few more examples of square root functions.

Example 10

Graph the function $y = 2\sqrt{3x-1} + 2$.

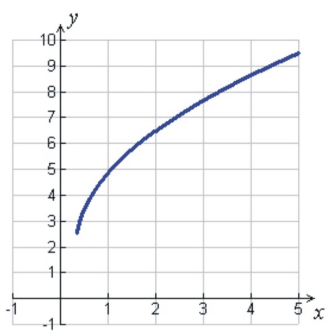
Solution

We first determine the domain of the function. The function is only defined if the expression inside the square root is positive $3x - 1 \geq 0$ or $x \geq 1/3$.

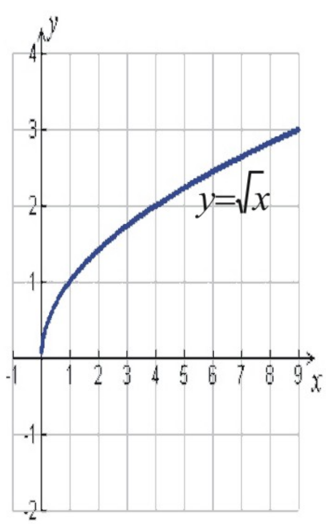
Make a table for values of x greater than or equal to 1/3.

x	$y = 2\sqrt{3x-1}+2$
$\frac{1}{3}$	$y = 2\sqrt{3.\frac{1}{3}-1}+2 = 2$
1	$y = 2\sqrt{3(1)-1}+2 = 4.8$
2	$y = 2\sqrt{3(2)-1}+2 = 6.5$
3	$y = 2\sqrt{3(3)-1}+2 = 7.7$
4	$y = 2\sqrt{3(4)-1}+2 = 8.6$
5	$y = 2\sqrt{3(5)-1}+2 = 9.5$

Here is the graph.

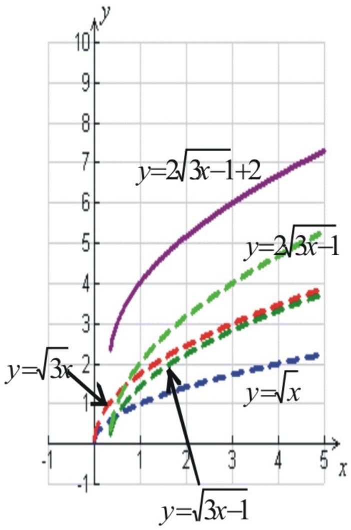


You can also think of this function as a combination of shifts and stretches of the basic square root function $y = \sqrt{x}$. We know that the graph of this function looks like the one below.



If we multiply the argument by 3 to obtain $y = \sqrt{3x}$, this stretches the curve vertically because the value of y increases faster by a factor of $y = \sqrt{3}$.

Next, when we subtract the value of 1 from the argument to obtain $y = \sqrt{3x-1}$ this shifts the entire graph to the left by one unit.



Multiplying the function by a factor of 2 to obtain $y = 2\sqrt{3x-1}$ stretches the curve vertically again, and y increases faster by a factor of 2.

Finally, we add the value of 2 to the function to obtain $y = \sqrt{3x-1} + 2$. This shifts the entire function vertically by 2 units.

This last method of graphing showed a way to graph functions without making a table of values. If we know what the basic function looks like, we can use shifts and stretches to **transform** the function and get to the desired result.

Graph Square Root Functions Using a Graphing Calculator

Next, we will demonstrate how to use the graphing calculator to plot square root functions.

Example 11

Graph the following functions using a graphing calculator.

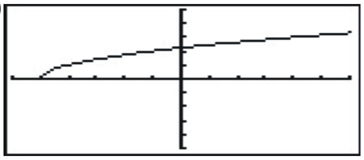
a) $y = \sqrt{x+5}$

b) $y = \sqrt{9-x^2}$

Solution:

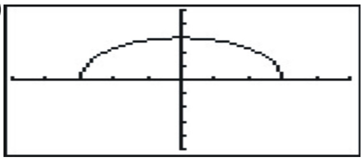
In all the cases we start by pressing the [**Y=button**] and entering the function on the function screen of the calculator:

We then press [**GRAPH**] to display the results. Make sure your window is set appropriately in order to view the function well. This is done by pressing the [**WINDOW**] button and choosing appropriate values for the Xmin, Xmax, Ymin and Ymax.

a) 

The window of this graph is $-6 \leq x \leq 5;\ -5 \leq y \leq 5$.

The domain of the function is $x \geq -5$

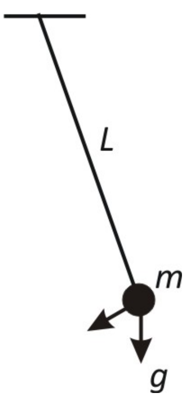
b) 

The window of this graph is $-5 \leq x \leq 5; -5 \leq y \leq 5$.

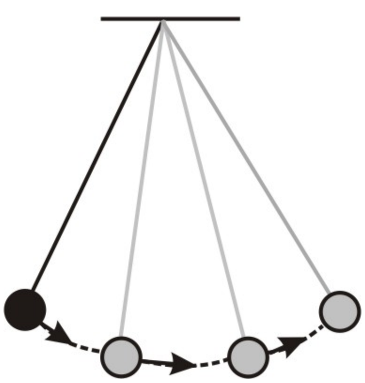
The domain of the function is $-3 \leq x \leq 3$

Solve Real-World Problems Using Square Root Functions

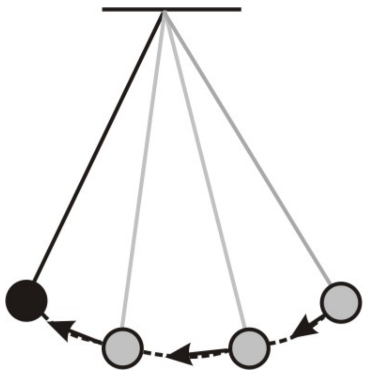
Pendulum



Mathematicians and physicists have studied the motion of a pendulum in great detail because this motion explains many other behaviors that occur in nature. This type of motion is called **simple harmonic motion** and it is very important because it describes anything that repeats periodically. Galileo was the first person to study the motion of a pendulum around the year 1600. He found that the time it takes a pendulum to complete a swing from a starting point back to the beginning does not depend on its mass or on its angle of swing (as long as the angle of the swing is small). Rather, it depends only on the length of the pendulum.



The time it takes a pendulum to swing from a starting point and back to the beginning is called the **period** of the pendulum.



Galileo found that the period of a pendulum is proportional to the square root of its length $T = a\sqrt{L}$. The proportionality constant depends on the acceleration of gravity $a = 2\pi/\sqrt{g}$. At sea level on Earth the acceleration of gravity is $g = 9.81 \text{ m/s}^2$ (meters per second squared). Using this value of gravity, we find $a = 2.0$ with units of $\text{s}/\sqrt{\text{m}}$ (seconds divided by the square root of meters). Up until the mid 20th century, all clocks used pendulums as their central time keeping component.

Example 12

Graph the period of a pendulum of a clock swinging in a house on Earth at sea level as we change the length of the pendulum. What does the length of the pendulum need to be for its period to be one second?

Solution

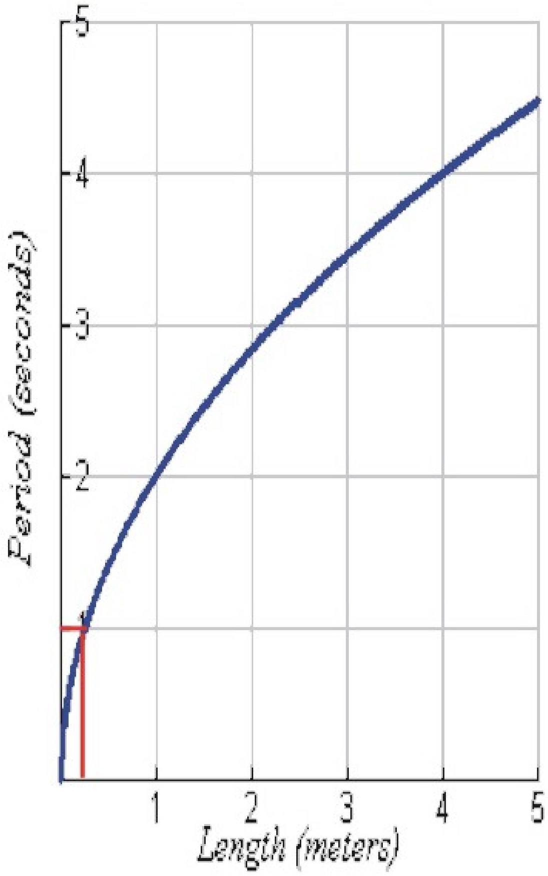
The function for the period of a pendulum at sea level is: $T = 2\sqrt{L}$.

We make a graph with the horizontal axis representing the length of the pendulum and with the vertical axis representing the period of the pendulum.

We start by making a table of values.

L	$T = 2\sqrt{L}$
0	$T = 2\sqrt{0} = 0$
1	$T = 2\sqrt{1} = 2$
2	$T = 2\sqrt{2} = 2.8$
3	$T = 2\sqrt{3} = 3.5$
4	$T = 2\sqrt{4} = 4$
5	$T = 2\sqrt{5} = 4.5$

Now let's graph the function.



We can see from the graph that a length of approximately 1/4 meters gives a period of one second. We can confirm this answer by using our function for the period and plugging in $T = 1$ second.

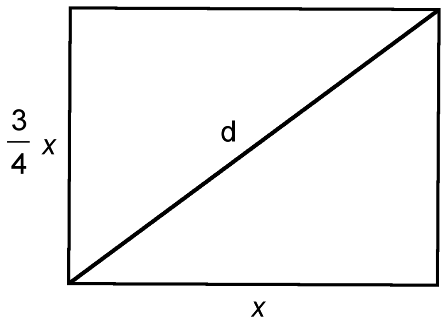
$$T = 2\sqrt{L} \Rightarrow 1 = 2\sqrt{L}$$

Square both sides of the equation:	$1 = 4L$
Solve for L:	$L = 1/4$ meters

Example 13

"Square" TV screens have an aspect ratio of 4:3. This means that for every four inches of length on the horizontal, there are three inches of length on the vertical. TV sizes represent the length of the diagonal of the television screen. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with an area of 180 in^2?

Solution



Let $d =$ length of the diagonal, $x =$ horizontal length

$$4 \cdot \text{vertical length} = 3 \cdot \text{horizontal length}$$

—Or,—

$$\text{vertical length} = \frac{3}{4}x.$$

The area of the screen is: $A = \text{length} \cdot \text{width}$ or $A = \frac{3}{4}x^2$

Find how the diagonal length and the horizontal length are related by using the Pythagorean theorem, $a^2 + b^2 = c^2$.

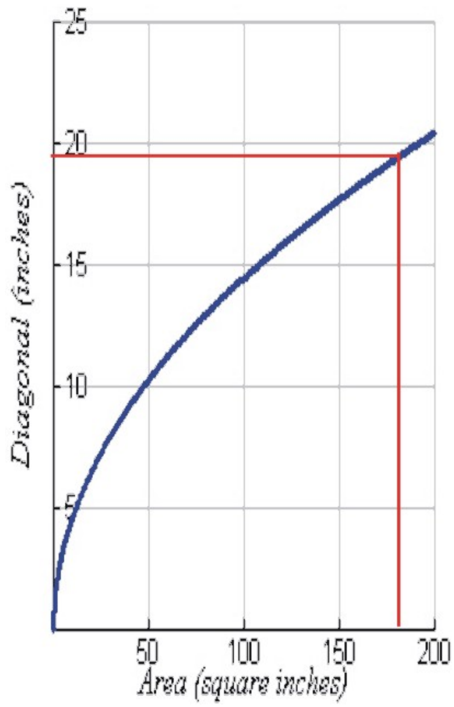
$$x^2 + \left(\frac{3}{4}x\right)^2 = d^2$$
$$x^2 + \frac{9}{16}x^2 = d^2$$
$$\frac{25}{16}x^2 = d^2 \Rightarrow x^2 = \frac{16}{25}d^2 \Rightarrow x^2 = \frac{4}{5}d$$

$$A = \frac{3}{4}\left(\frac{4}{5}d\right)^2 = \frac{3}{4} \cdot \frac{16}{25}d^2 = \frac{12}{25}d^2$$

We can also find the diagonal length as a function of the area $d^2 = \frac{25}{12}A$ or $d = \frac{5}{2\sqrt{3}}\sqrt{A}$.

Make a graph where the horizontal axis represents the area of the television screen and the vertical axis is the length of the diagonal. Let's make a table of values.

A	$d = \frac{5}{2\sqrt{3}}\sqrt{A}$
0	0
25	7.2
50	10.2
75	12.5
100	14.4
125	16.1
150	17.6
175	19
200	20.4



From the graph we can estimate that when the area of a TV screen is 180 in^2 the length of the diagonal is approximately 19.5 inches inches. We can confirm this by substituting $a = 180$ into the formula that relates the diagonal to the area.

$$d = \frac{5}{2\sqrt{3}}\sqrt{A} = \frac{5}{2\sqrt{3}}\sqrt{180} = 19.4 \text{ inches}$$

Review Questions

Graph the following functions on the same coordinate axes.

1. $y = \sqrt{x}$, $y = 2.5\sqrt{x}$ and $y = -2.5\sqrt{x}$
2. $y = \sqrt{x}$ $y = 0.3\sqrt{x}$,and $y = 0.6\sqrt{x}$
3. $y = \sqrt{x}$, $y = \sqrt{x-5}$ and $y = \sqrt{x+5}$
4. $y = \sqrt{x}$, $y = \sqrt{x} + 8$ and $y = \sqrt{x} - 8$

Graph the following functions.

5. $y = \sqrt{2x-1}$
6. $y = \sqrt{4x+4}$
7. $y = \sqrt{5-x}$
8. $y = 2\sqrt{x} + 5$
9. $y = 3 - \sqrt{x}$
10. $y = 4 + 2\sqrt{x}$
11. $y = 2\sqrt{2x+3} + 1$
12. $y = 4 + 2\sqrt{2-x}$
13. $y = \sqrt{x+1} - \sqrt{4x-5}$

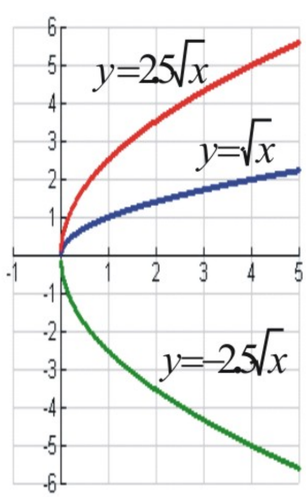
14. The acceleration of gravity can also given in feet per second squared. It is $g = 32$ ft/s^2 at sea level. Graph the period of a pendulum with respect to its length in feet. For what length in feet will the period of a pendulum be two seconds?
15. The acceleration of gravity on the Moon is 1.6 m/s^2. Graph the period of a pendulum on the Moon with respect to its length in meters. For what length, in meters, will the period of a pendulum be 10 seconds?
16. The acceleration of gravity on Mars is 3.69 m/s^2. Graph the period of a pendulum on the Mars with respect to its length in meters. For what length, in meters, will the period of a pendulum be three seconds?
17. The acceleration of gravity on the Earth depends on the latitude and altitude of a place. The value of g is slightly smaller for places closer to the Equator than places closer to the Poles, and the value of g is slightly smaller for places at higher altitudes that it is for places at lower altitudes. In Helsinki, the value of $g = 9.819$ m/s^2, in Los Angeles the value of $g = 9.796$ m/s^2 and in Mexico City the value of $g = 9.779$ m/s^2. Graph the period of a pendulum with respect to its length for all three cities on the same graph. Use the formula to find the length (in meters) of a pendulum with a period of 8 seconds for each of these cities.
18. The aspect ratio of a wide-screen TV is 2.39:1. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with area 150 in^2?

Graph the following functions using a graphing calculator.

19. $y = \sqrt{3x - 2}$
20. $y = 4 + \sqrt{2 - x}$
21. $y = \sqrt{x^2 - 9}$
22. $y = \sqrt{x} - \sqrt{x + 2}$

Review Answers

1.



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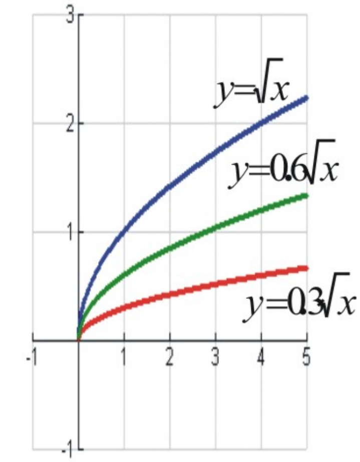
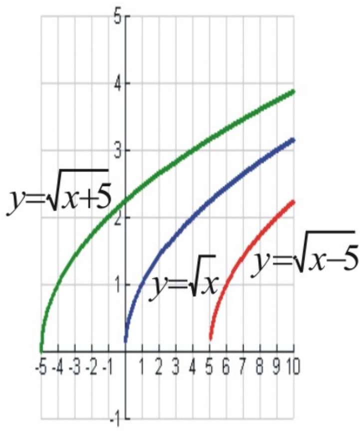


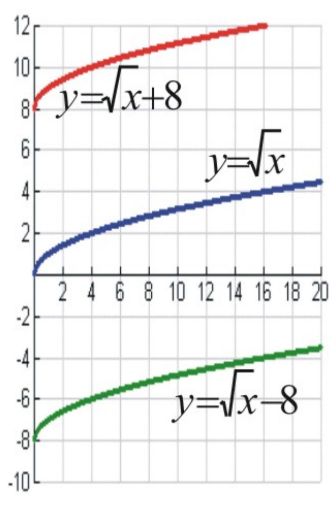
$y=\sqrt{x}$
$y=0.6\sqrt{x}$
$y=0.3\sqrt{x}$

3.



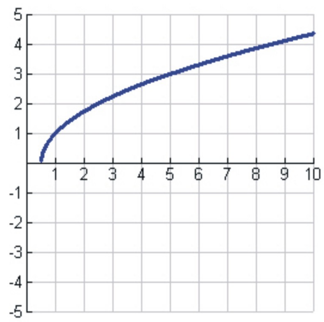
$y=\sqrt{x+5}$
$y=\sqrt{x}$
$y=\sqrt{x-5}$

4.



$y=\sqrt{x}+8$
$y=\sqrt{x}$
$y=\sqrt{x}-8$

5.



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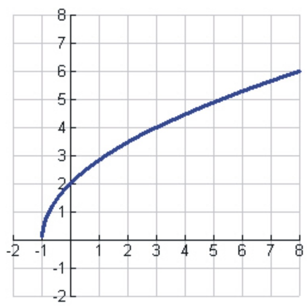

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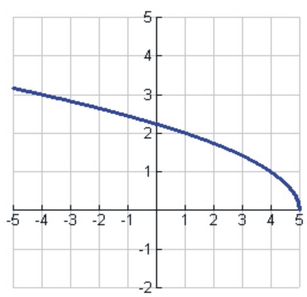

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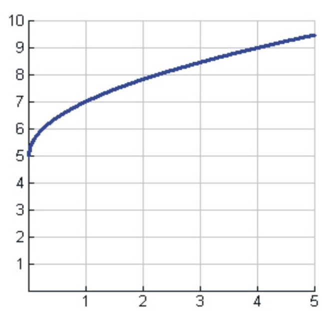

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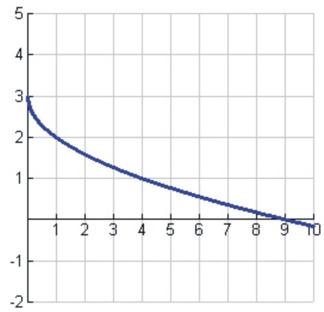
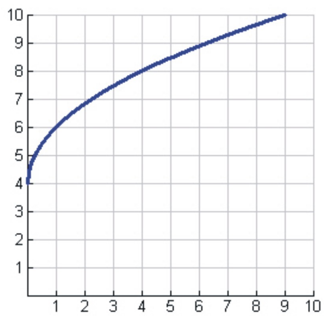


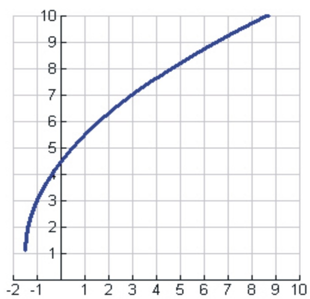
5
4
3
2
1
-1
-2
1 2 3 4 5 6 7 8 9 10

10.



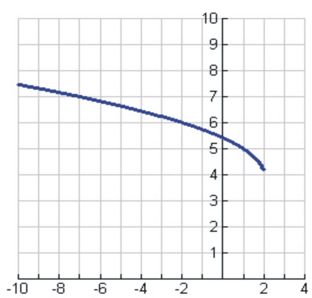
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10

11.



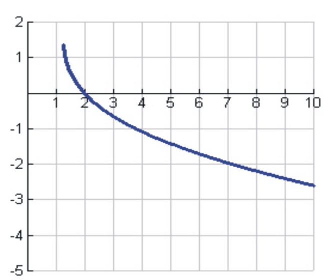
10
9
8
7
6
5
4
3
2
1
-2 -1 1 2 3 4 5 6 7 8 9 10

12.



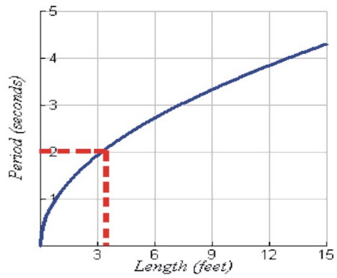
10
9
8
7
6
5
4
3
2
1
-10 -8 -6 -4 -2 2 4

13.

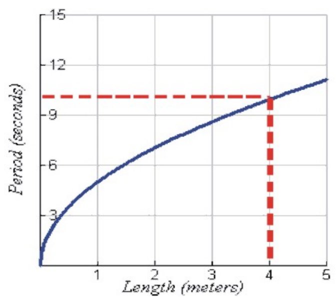


2
1
-1
-2
-3
-4
-5
1 2 3 4 5 6 7 8 9 10

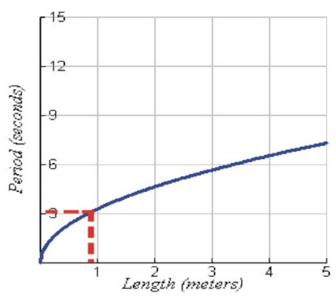
14. $L = 3.25$ feet



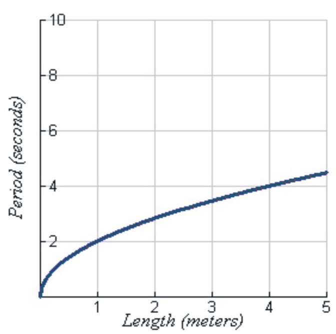
15. $L = 4.05$ meters



16. $L = 0.84$ meters

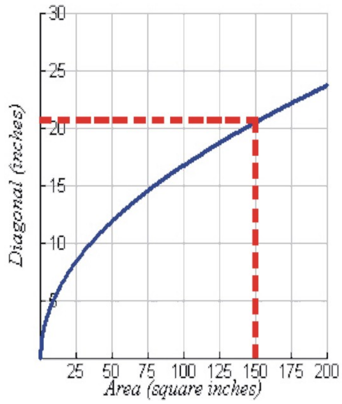


17. Note: The differences are so small that all of the lines appear to coincide

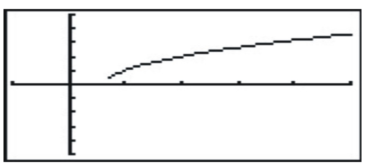


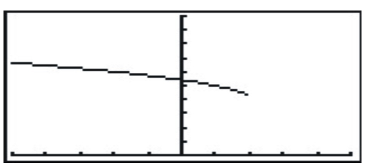
on this graph. If you zoom (way) in you can see slight differences.The period of an 8 meter pedulum in Helsinki is 1.8099 seconds, in Los Angeles it is 1.8142 seconds , and in Mexico City it is 1.8173 seconds .

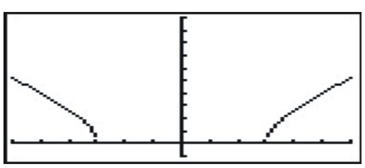
18. $D = 20.5$ inches

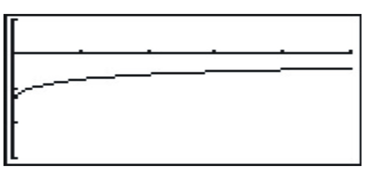


15.92 m Helsinki
15.88 m Los Angeles
15.85 m Mexico City

19.  Window $-1 \le x \le 5; -5 \le y \le 5$

20.  Window $-5 \le x \le 5; 0 \le y \le 10$

21.  Window $-6 \le x \le 6; -1 \le y \le 10$

22.  Window $0 \le x \le 5; -3 \le y \le 1$

11.2 Radical Expressions

Learning objectives

- Use the product and quotient properties of radicals.
- Rationalize the denominator.
- Add and subtract radical expressions.
- Multiply radical expressions.
- Solve real-world problems using square root functions.

Introduction

A radical reverses the operation of raising a number to a power. For example, to find the square of 4, we write $4^2 = 4 \cdot 4 = 16$. The reverse process is called finding the square root. The symbol for a square root is $\sqrt{\ }$ .This symbol is also called the **radical sign**. When we take the square root of a number, the result is a number which when squared gives the number under the square root sign. For example,

$$\sqrt{9} = 3 \qquad \text{since} \qquad 3^2 = 3 \cdot 3 = 9$$

Radicals often have an index in the top left corner. The index indicates which root of the number we are seeking. Square roots have an index of 2 but many times this index is not written.

$$\sqrt[2]{36} = 6 \qquad \text{since} \qquad 6^2 = 36$$

The cube root of a number gives a number which when raised to the third power gives the number under the radical sign.

$$\sqrt[3]{64} = 4 \qquad \text{since} \qquad 4^3 = 4 \cdot 4 \cdot 4 = 64$$

The fourth root of number gives a number which when raised to the power four gives the number under the radical sign.

$$\sqrt[4]{81} = 3 \qquad \text{since} \qquad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

Even and odd roots

Radical expressions that have even indices are called **even roots** and radical expressions that have odd indices are called **odd roots**. There is a very important difference between even and odd roots in that they give drastically different results when the number inside the radical sign is negative.

Any real number raised to an even power results in a positive answer. Therefore, when the index of a radical is even, the number inside the radical sign must be non-negative in order to get a real answer.

On the other hand, a positive number raised to an odd power is positive and a negative number raised to an odd power is negative. Thus, a negative number inside the radical sign is not a problem. It just results in a negative answer.

Example 1

Evaluate each radical expression.

a) $\sqrt{121}$

b) $\sqrt[3]{125}$

c) $\sqrt[4]{-625}$

d) $\sqrt[5]{-32}$

Solution

a) $\sqrt{121} = 11$

b) $\sqrt[3]{125} = 5$

c) $\sqrt[4]{-625}$ is not a real number

d) $\sqrt[5]{-32} = -2$

Use the Product and Quotient Properties of Radicals

Radicals can be rewritten as exponent with rational powers. The radical $y = \sqrt[m]{a^n}$ is defined as $a^{\frac{n}{m}}$.

Example 2

Write each expression as an exponent with a rational value for the exponent.

a) $\sqrt{5}$

b) $\sqrt[4]{a}$

c) $\sqrt[3]{4xy}$

d) $\sqrt[6]{x^5}$

Solution

a) $\sqrt{5} = 5^{1/2}$

b) $\sqrt[4]{a} = a^{1/4}$

c) $\sqrt[3]{4xy} = (4xy)^{1/3}$

d) $\sqrt[6]{x^5} = x^{5/6}$

As a result of this property, for any non-negative number $\sqrt[n]{a^n} = a^{n/n} = a$.

Since roots of numbers can be treated as powers, we can use exponent rules to simplify and evaluate radical expressions. Let's review the product and quotient rule of exponents.

Raising a product to a power	$(x \cdot y)^n = x^n \cdot y^n$
Raising a quotient to a power	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

In radical notation, these properties are written as

Raising a product to a power	$\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$
Raising a quotient to a power	$\sqrt[m]{\frac{x}{y}} = \frac{\sqrt[m]{x}}{\sqrt[m]{y}}$

A very important application of these rules is reducing a radical expression to its simplest form. This means that we apply the root on all the factors of the number that are perfect roots and leave all factors that are not perfect roots inside the radical sign.

For example, in the expression $\sqrt{16}$, the number is a perfect square because $16 = 4^2$. This means that we can simplify.

$$\sqrt{16} = \sqrt{4^2} = 4$$

Thus, the square root disappears completely.

On the other hand, in the expression , the number $\sqrt{32}$ is not a perfect square so we cannot remove the square root. However, we notice that $32 = 16 \cdot 2$, so we can write 32 as the product of a perfect square and another number.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16 \cdot 2}$$

If we apply the "raising a product to a power" rule we obtain